HOMEWORK 4 SOLUTIONS - MATH 300
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Problem 1 *Consider the language of algebras $\mathcal{L}$ consisting of the set of function symbols $\mathcal{F} = \{\bowtie, \rhd\}$, where $\bowtie$ is binary and $\rhd$ is unary, and the set of constants $\mathcal{C} = \emptyset$. Provide a list of all possible $\mathcal{L}$-terms in prefix notation (without using parentheses) that contain only the variable $x \in X$ and are of length at most 5 (i.e., contain at most five symbols).*

Solution: We classify the terms according to their length:

Lenth	Terms
1	x
2	$\rhd x$
3	$\rhd \rhd x,\ \bowtie xx$
4	$\rhd \rhd \rhd x,\ \rhd \bowtie xx,\ \bowtie \rhd xx,\ \bowtie x \rhd x$
5	$\rhd \rhd \rhd \rhd x,\ \rhd\rhd \bowtie xx,\ \rhd \bowtie \rhd xx,\ \rhd \bowtie x \rhd x,$ $\bowtie \rhd x \rhd x,\ \bowtie\bowtie xxx,\ \bowtie x \bowtie xx$

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Problem 2 *How many $\mathcal{L}$-structures can one find on the set $A = \{a, b\}$ when $\mathcal{L} = \{+, \cdot, -, 0, 1\}$, with $+, \cdot$ binary function symbols, $-$ a unary function symbol and $0, 1$ constant symbols?*
(**Hint:** *See Exercise 3.1.4 of our textbook.*)

Solution: The "addition" table

$+$	a	b
a		
b		

may be filled-in in $2^4 = 16$ ways. The "multiplication" table

$\cdot$	a	b
a		
b		

may be filled-in in $2^4 = 16$ ways. The "negation" table

	$-$
a	
b	

may be filled-in in $2^2 = 4$ ways. Finally, for each of the constants in A, we have 2 choices. Thus, by the multiplication principle, the total number of $\mathcal{L}$-structures on A is $16 \cdot 16 \cdot 4 \cdot 2 \cdot 2 = 4,096$. ■

Problem 3 *Note that*

> *A binary operation $g : A^2 \to A$ on a set A is* **idempotent** *if $g(a, a) = a$, for all $a \in A$.*
> *An element $a \in A$ is a* **fixed-point** *of a unary operation $f : A \to A$ if $f(a) = a$.*

Find the number of $\mathcal{L}$-structures on the set $A = \{a, b, c\}$ when $\mathcal{L} = \{\vee, \wedge, ', 0, 1\}$, with $\vee, \wedge$ binary and $'$ a unary function symbol and $0, 1$ constant symbols, if one insists that

- *$\vee$ and $\wedge$ are interpreted as idempotent operations in A;*
- *no element of A is a fixed-point under the interpretation of $'$ in A.*

Solution: Since $\vee$ is supposed to be idempotent, the "join" table

$\vee$	a	b	c
a	a		
b		b	
c			c

may be filled-in in $3^6 = 629$ ways. Similarly, since $\wedge$ is supposed to be idempotent, the "meet" table

$\wedge$	a	b	c
a	a		
b		b	
c			c

may also be filled-in in $3^6 = 629$ ways. The fact that the "complementation" operation is supposed to not have any fixed-points, means that its table

	$'$
a	
b	
c	

may be filled-in in $2^3 = 8$ ways. Finally, for each of the constants in A, we have 3 choices. Thus, by the multiplication principle, the total number of $\mathcal{L}$-structures on A is $629 \cdot 629 \cdot 8 \cdot 3 \cdot 3 = 28,486,152$. ■

Problem 4 (**Syntax**) *Consider the language of algebras $\mathcal{L}$ consisting of the set of function symbols $\mathcal{F} = \{f, g, h\}$, with f unary, g binary and h ternary, and the set of constant symbols $\mathcal{C} = \emptyset$. Consider, also*

$$t(x, y, z) = gfgxzhygzfxfy.$$

(a) *Show the run on t of our γ-algorithm for determining syntactic validity and state clearly the conclusions of the algorithm.*

(b) *Create the syntax tree for t.*

(c) *Use the recursive definition of subterms to find all subterms of t; show carefully how each step of the recursive procedure is applied in finding the set of all subterms.*

Solution:

(a) The following gives the values of γ as it scans the input string:

Input Symbol	g	f	g	x	z	h	y	g	z	f	x	f	y
γ	0	0	-1	0	1	-1	0	-1	0	0	1	1	2

Since all values of γ before the last are < 2 and the last is equal to 2, the input string represents a valid $\mathcal{L}$-term;

Moreover, the first argument of the initial g is $fgxz$ and the second argument is the subterm $hygzfxfy$.

(b) The following is the syntax tree for t:

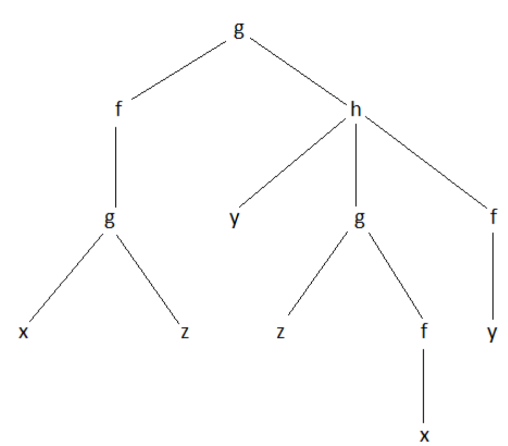

Figure 1: Syntax Tree for $t(x, y, z) = gfgxzhygzfxfy$.

(c) We show by increasing indentations how the algorithms proceeds recursively into exploring deeper and deeper subterms:

$$
\begin{array}{lllll}
gfgxzhygzfxfy & & & & \\
 & fgxz & & & \\
 & & gxz & & \\
 & & & x & \\
 & & & z & \\
 & hygzfxfy & & & \\
 & & y & & \\
 & & gzfx & & \\
 & & & z & \\
 & & & fx & \\
 & & & & x \\
 & & fy & & \\
 & & & y &
\end{array}
$$

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Problem 5 (**Semantics**) *Consider the language of algebras $\mathcal{L}$ consisting of the set of function symbols $\mathcal{F} = \{f, g, h\}$, with f unary, g binary and h ternary, and the set of constant symbols $\mathcal{C} = \emptyset$. Moreover, let $A = \{a, b, c\}$ and interpret the three functions symbols as the functions $f^{\mathbf{A}} : A \to A$, $g^{\mathbf{A}} : A^2 \to A$ and $h^{\mathbf{A}} : A^3 \to A$, given by the following:*

	$f^{\mathbf{A}}$
a	b
b	c
c	a

$g^{\mathbf{A}}$	a	b	c
a	a	b	c
b	c	c	a
c	c	b	b

$$h^{\mathbf{A}}(x, y, z) = \begin{cases} w, & \textit{if at least two of } x, y, z \textit{ are equal to } w \\ y, & \textit{otherwise} \end{cases}$$

(a) *Consider again the term $t(x, y, z) = gfgxzhygzfxfy$ and compute the values*

(i) $t^{\mathbf{A}}(a, a, c)$;

(ii) $t^{\mathbf{A}}(a, b, c)$;

(iii) $t^{\mathbf{A}}(b, a, b)$;

(iv) $t^{\mathbf{A}}(b, c, a)$;

(v) $t^{\mathbf{A}}(c, c, c)$.

(b) *Create the full Cayley table for $s^{\mathbf{A}}(x, y)$, where $s(x, y) = gxfgyx$.*

Solution:

(a) Let $t(x, y, z) = gfgxzhygzfxfy$. We compute the values

(i) $t^{\mathbf{A}}(a, a, c) = g^{\mathbf{A}} f^{\mathbf{A}} g^{\mathbf{A}} ach^{\mathbf{A}} ag^{\mathbf{A}} cf^{\mathbf{A}} af^{\mathbf{A}} a = g^{\mathbf{A}} f^{\mathbf{A}} ch^{\mathbf{A}} ag^{\mathbf{A}} cbb = g^{\mathbf{A}} ah^{\mathbf{A}} abb = g^{\mathbf{A}} ab = b$;

(ii) $t^{\mathbf{A}}(a, b, c) = g^{\mathbf{A}} f^{\mathbf{A}} g^{\mathbf{A}} ach^{\mathbf{A}} bg^{\mathbf{A}} cf^{\mathbf{A}} af^{\mathbf{A}} b = g^{\mathbf{A}} f^{\mathbf{A}} ch^{\mathbf{A}} bg^{\mathbf{A}} cbc = g^{\mathbf{A}} ah^{\mathbf{A}} bbc = g^{\mathbf{A}} ab = b$;

(iii) $t^{\mathbf{A}}(b, a, b) = g^{\mathbf{A}} f^{\mathbf{A}} g^{\mathbf{A}} bbh^{\mathbf{A}} ag^{\mathbf{A}} bf^{\mathbf{A}} bf^{\mathbf{A}} a = g^{\mathbf{A}} f^{\mathbf{A}} ch^{\mathbf{A}} ag^{\mathbf{A}} bcb = g^{\mathbf{A}} ah^{\mathbf{A}} aab = g^{\mathbf{A}} aa = a$;

(iv) $t^{\mathbf{A}}(b, c, a) = g^{\mathbf{A}} f^{\mathbf{A}} g^{\mathbf{A}} bah^{\mathbf{A}} cg^{\mathbf{A}} af^{\mathbf{A}} bf^{\mathbf{A}} c = g^{\mathbf{A}} f^{\mathbf{A}} ch^{\mathbf{A}} cg^{\mathbf{A}} aca = g^{\mathbf{A}} ah^{\mathbf{A}} cca = g^{\mathbf{A}} ac = c$;

(v) $t^{\mathbf{A}}(c, c, c) = g^{\mathbf{A}} f^{\mathbf{A}} g^{\mathbf{A}} cch^{\mathbf{A}} cg^{\mathbf{A}} cf^{\mathbf{A}} cf^{\mathbf{A}} c = g^{\mathbf{A}} f^{\mathbf{A}} bh^{\mathbf{A}} cg^{\mathbf{A}} caa = g^{\mathbf{A}} ch^{\mathbf{A}} cca = g^{\mathbf{A}} cc = b$.

(b) The operation table for $s^{\mathbf{A}}(x,y) = g^{\mathbf{A}} x f^{\mathbf{A}} g^{\mathbf{A}} yx$ is shown below:

x	y	$g^{\mathbf{A}}yx$	$f^{\mathbf{A}}g^{\mathbf{A}}yx$	$g^{\mathbf{A}}xf^{\mathbf{A}}g^{\mathbf{A}}yx$
a	a	a	b	b
a	b	c	a	a
a	c	c	a	a
b	a	b	c	a
b	b	c	a	c
b	c	b	c	a
c	a	c	a	c
c	b	a	b	b
c	c	b	c	b

$S^{\mathbf{A}}$	a	b	c
a	b	a	a
b	a	c	a
c	c	b	b

Therefore, the Cayley table is the one shown on the right above.

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Problem 6 *Suppose that our language of algebras has a ternary function symbol h. Moreover, let $A = \{0,1,2\}$ and suppose that $h^{\mathbf{A}} : A^3 \to A$ is the function*

$$h^{\mathbf{A}}(a,b,c) = a - b + c \pmod 3.$$

Find the values of the term functions associated with the indicated term at the arguments shown:

(a) $t(x) = hhxxxxhxxx$ *at* 1;

(b) $t(x,y) = hxhxyxy$ *at* $(0,1)$;

(c) $t(x,y,z) = hxhyzhxyzz$ *at* $(0,1,0)$;

(d) $t(x,y,z,w) = hhyzhwxyzhwxw$ *at* $(0,1,0,1)$.

Solution: We evaluate the corresponding term functions:

(a) $t^{\mathbf{A}}(1) = h^{\mathbf{A}}h^{\mathbf{A}}1111h^{\mathbf{A}}111 = h^{\mathbf{A}}111 = 1$;

(b) $t^{\mathbf{A}}(0,1) = h^{\mathbf{A}}0h^{\mathbf{A}}0101 = h^{\mathbf{A}}021 = 2$;

(c) $t^{\mathbf{A}}(0,1,0) = h^{\mathbf{A}}0h^{\mathbf{A}}10h^{\mathbf{A}}0100 = h^{\mathbf{A}}0h^{\mathbf{A}}1020 = h^{\mathbf{A}}000 = 0$;

(d) $t^{\mathbf{A}}(0,1,0,1) = h^{\mathbf{A}}h^{\mathbf{A}}10h^{\mathbf{A}}1010h^{\mathbf{A}}101 = h^{\mathbf{A}}h^{\mathbf{A}}10202 = h^{\mathbf{A}}002 = 2$.

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