# Elementary Differential Equations

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Subsection 1

## General Theory of $n$-th Order Linear Equations

# $n$-th Order Linear Differential Equations

- An ***n*-th order linear differential equation** is one of the form
$$P_0(t)\frac{d^n y}{dt^n} + P_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + P_{n-1}(t)\frac{dy}{dt} + P_n(t)y = G(t);$$
- $P_0, \ldots, P_n$, and $G$ are continuous real-valued functions on some interval $I : \alpha < t < \beta$, and $P_0$ is nowhere zero in this interval;
- By dividing by $P_0(t)$, we obtain
$L[y] = \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t);$
- The mathematical theory associated with this equation is completely analogous to that for the second order linear equation;
- Given $n$ conditions $y(t_0) = y_0$, $y'(t_0) = y_0'$, …, $y^{(n-1)}(t_0) = y_0^{(n-1)}$;

**Theorem (Existence and Uniqueness of Solutions)**

If the functions $p_1, p_2, \ldots, p_n$, and $g$ are continuous on the open interval $I$, then there exists exactly one solution $y = \phi(t)$ of the differential equation satisfying the given initial conditions; This solution exists throughout $I$.

# The Homogeneous Equation

- Consider the homogeneous equation
  $L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0;$
- If $y_1, y_2, \ldots, y_n$ are solutions, then
  $y = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$ is also a solution;
- Is every solution a linear combination of $y_1, \ldots, y_n$?
- This happens if, regardless of initial conditions, we can choose $c_1, \ldots, c_n$ so that the linear combination satisfies those conditions;
- That is, for any $t_0$ in $I$, and for any $y_0, y_0', \ldots, y_0^{(n-1)}$, we can determine $c_1, \ldots, c_n$ so that

$$\begin{array}{rcl} c_1 y_1(t_0) + \cdots + c_n y_n(t_0) & = & y_0 \\ c_1 y_1'(t_0) + \cdots + c_n y_n'(t_0) & = & y_0' \\ & \cdots & \\ c_1 y_1^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) & = & y_0^{(n-1)} \end{array}$$

- The system can be solved uniquely provided that the determinant of coefficients is not zero and conversely;

# The Homogeneous Equation (Cont'd)

- Hence, a necessary and sufficient condition for the existence of a solution for arbitrary values of $y_0, y_0', \ldots, y_0^{(n-1)}$ is that the Wronskian

$$W(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

is not zero at $t = t_0$;

**Theorem**

If the functions $p_1, p_2, \ldots, p_n$ are continuous on the open $I$, if the functions $y_1, y_2, \ldots, y_n$ are solutions, and if $W(y_1, y_2, \ldots, y_n)(t) \neq 0$ for at least one point $t$ in $I$, then every solution can be expressed as a linear combination of the solutions $y_1, y_2, \ldots, y_n$.

- A set of solutions $y_1, \ldots, y_n$ whose Wronskian is nonzero is called a **fundamental set of solutions**; A linear combination of such a set is called a **general solution**;

# Linear Dependence and Independence

- There is a close relationship between a fundamental set of solutions and the concept of linear independence studied in linear algebra;
- The functions $f_1$, $f_2, \ldots$, $f_n$ are said to be **linearly dependent** on an interval $I$ if there exists a set of constants $k_1, k_2, \ldots, k_n$, not all zero, such that

$$k_1 f_1(t) + k_2 f_2(t) + \cdots + k_n f_n(t) = 0,$$

  for all $t$ in $I$;
- The functions $f_1, \ldots, f_n$ are said to be **linearly independent** on $I$ if they are not linearly dependent there;

# Linear Dependence: An Example

- Are the functions $f_1(t) = 1$, $f_2(t) = t$, and $f_3(t) = t^2$ linearly independent or dependent on the interval $I : -\infty < t < \infty$?

  Form the linear combination
  $k_1 f_1(t) + k_2 f_2(t) + k_3 f_3(t) = k_1 + k_2 t + k_3 t^2$;
  Set it equal to zero to obtain $k_1 + k_2 t + k_3 t^2 = 0$; If the equation is to hold for all $t$ in $I$, then it must certainly be true at any three distinct points in $I$; Let us choose $t = 0$, $t = 1$, and $t = -1$; We obtain the system of equations

$$\left\{ \begin{array}{rcl} k_1 & = & 0 \\ k_1 + k_2 + k_3 & = & 0 \\ k_1 - k_2 + k_3 & = & 0 \end{array} \right\}$$

  From the first, $k_1 = 0$; From the other two equations it follows that $k_2 = k_3 = 0$ as well; Therefore there is no set of constants $k_1, k_2, k_3$, not all zero, for which the equation holds; Thus, the given functions are not linearly dependent on $I$, so they must be linearly independent;

# Linear Dependence: Another Example

- Are the functions $f_1(t) = 1$, $f_2(t) = 2 + t$, $f_3(t) = 3 - t^2$, and $f_4(t) = 4t + t^2$ linearly independent or dependent on any interval $I$?

  Form the linear combination

$$\begin{aligned} & k_1 f_1(t) + k_2 f_2(t) + k_3 f_3(t) + k_4 f_4(t) \\ & = k_1 + k_2(2 + t) + k_3(3 - t^2) + k_4(4t + t^2) \\ & = (k_1 + 2k_2 + 3k_3) + (k_2 + 4k_4)t + (-k_3 + k_4)t^2; \end{aligned}$$

  This expression is zero throughout an interval provided that $k_1 + 2k_2 + 3k_3 = 0$, $k_2 + 4k_4 = 0$, $-k_3 + k_4 = 0$; These three equations, with four unknowns, have many solutions; For instance, if $k_4 = 1$, then $k_3 = 1$, $k_2 = -4$ and $k_1 = 5$; Thus the given functions are linearly dependent on every interval;

# Fundamental Set of Solutions and Linear Independence

- Suppose that the functions $y_1, \ldots, y_n$ are solutions of $y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0$ on an interval $I$;
- Consider the equation $k_1 y_1(t) + \cdots + k_n y_n(t) = 0$;
- By differentiating repeatedly, we obtain the equations

$$\begin{array}{rcl} k_1 y_1'(t) + \cdots + k_n y_n'(t) & = & 0 \\ \cdots & & \\ k_1 y_1^{(n-1)}(t) + \cdots + k_n y_n^{(n-1)}(t) & = & 0 \end{array}$$

- The determinant of coefficients for the resulting system is the Wronskian $W(y_1, \ldots, y_n)(t)$ of $y_1, \ldots, y_n$;

**Theorem**

If $y_1(t), \ldots, y_n(t)$ is a fundamental set of solutions of
$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0$
on an interval $I$, then $y_1(t), \ldots, y_n(t)$ are linearly independent on $I$;
Conversely, if $y_1(t), \ldots, y_n(t)$ are linearly independent solutions, then they form a fundamental set of solutions on $I$.

# The Nonhomogeneous Equation

- Consider $L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = g(t)$;
- The difference of any two solutions of the nonhomogeneous is a solution of the homogeneous;
- So, if $y_1, \ldots, y_n$ is a fundamental set of solutions of the homogeneous, then any solution of the nonhomogeneous can be written as $y = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) + Y(t)$, where $Y$ is some particular solution of the nonhomogeneous;
- This is called the **general solution** of the nonhomogeneous;
- Thus, the primary problem is to determine a fundamental set of solutions $y_1, \ldots, y_n$ of the homogeneous;
- If the coefficients are constants, this is fairly simple;
- If the coefficients are not constants, it is usually necessary to use numerical methods;

Subsection 2

## Homogeneous Equations with Constant Coefficients

# Characteristic Polynomial and Roots

- Consider $L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$, where $a_0, a_1, \ldots, a_n$ are real constants;
- We anticipate that $y = e^{rt}$ is a solution for all $r$, for which $Z(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$;
- The polynomial $Z(r)$ is called the **characteristic polynomial** and $Z(r) = 0$ the **characteristic equation** of the differential equation;
- A polynomial of degree $n$ has $n$ zeros, say, $r_1, r_2, \ldots, r_n$, some of which may be equal;
- Thus, it can be written as $Z(r) = a_0(r - r_1)(r - r_2) \cdots (r - r_n)$;
- If the roots of the characteristic equation are real and no two are equal, then we have $n$ distinct solutions $e^{r_1 t}, e^{r_2 t}, \ldots, e^{r_n t}$;
- If these functions are linearly independent, then the general solution is $y = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}$;
- One way to establish the linear independence of $e^{r_1 t}$, $e^{r_2 t}, \ldots, e^{r_n t}$ is to evaluate their Wronskian determinant;

# Real and Unequal Roots: Example

- Find the general solution of $y^{(4)} + y''' - 7y'' - y' + 6y = 0$; Also find the solution that satisfies the initial conditions $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$, $y'''(0) = -1$;

  Let $y = e^{rt}$; To determine $r$ we solve $r^4 + r^3 - 7r^2 - r + 6 = 0$;

$$\begin{aligned}
& r^4 + r^3 - 7r^2 - r + 6 = 0 \\
\Rightarrow \quad & r^4 + r^3 - 7r^2 - 7r + 6r + 6 = 0 \\
\Rightarrow \quad & r^3(r+1) - 7r(r+1) + 6(r+1) = 0 \\
\Rightarrow \quad & (r+1)(r^3 - 7r + 6) = 0 \\
\Rightarrow \quad & (r+1)(r^3 - r - 6r + 6) = 0 \\
\Rightarrow \quad & (r+1)[(r-1)r(r+1) - 6(r-1)] = 0 \\
\Rightarrow \quad & (r+1)(r-1)(r^2 + r - 6) = 0 \\
\Rightarrow \quad & (r+1)(r-1)(r-2)(r+3) = 0.
\end{aligned}$$

  The roots of this equation are $r_1 = 1$, $r_2 = -1$, $r_3 = 2$, and $r_4 = -3$; Therefore the general solution of is $y = c_1e^t + c_2e^{-t} + c_3e^{2t} + c_4e^{-3t}$;

# Example (Cont'd)

- The general solution of is $y = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-3t}$;
- Recall the initial conditions $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$, $y'''(0) = -1$;
- These give

$$\left\{\begin{array}{rcl} c_1 + c_2 + c_3 + c_4 & = & 1 \\ c_1 - c_2 + 2c_3 - 3c_4 & = & 0 \\ c_1 + c_2 + 4c_3 + 9c_4 & = & -2 \\ c_1 - c_2 + 8c_3 - 27c_4 & = & -1 \end{array}\right\} \overset{\text{next}}{\Rightarrow} \left\{\begin{array}{rcl} c_1 & = & \frac{11}{8} \\ c_2 & = & \frac{5}{12} \\ c_3 & = & -\frac{2}{3} \\ c_4 & = & -\frac{1}{8} \end{array}\right.$$

Therefore the solution of the initial value problem is
$y = \frac{11}{8} e^t + \frac{5}{12} e^{-t} - \frac{2}{3} e^{2t} - \frac{1}{8} e^{-3t}$;

# Example (Cont'd)

- To solve the preceding system for $c_1, c_2, c_3$ and $c_4$ we reduce the augmented matrix in echelon form:

$$\left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -3 & 0 \\ 1 & 1 & 4 & 9 & -2 \\ 1 & -1 & 8 & -27 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -4 & -1 \\ 0 & 0 & 3 & 8 & -3 \\ 0 & -2 & 7 & -28 & -2 \end{array}\right]$$
$$\longrightarrow \left[\begin{array}{rrrr|r} 1 & 0 & \frac{3}{2} & -1 & \frac{1}{2} \\ 0 & -2 & 1 & -4 & -1 \\ 0 & 0 & 3 & 8 & -3 \\ 0 & 0 & 6 & -24 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{rrrr|r} 1 & 0 & \frac{3}{2} & -1 & \frac{1}{2} \\ 0 & -2 & 1 & -4 & -1 \\ 0 & 0 & 3 & 8 & -3 \\ 0 & 0 & 0 & -40 & 5 \end{array}\right].$$

Now we solve bottom-up: $c_4 = -\frac{1}{8}$;
$3c_3 = -3 - 8c_4 = -3 - 8(-\frac{1}{8}) = -2$; So $c_3 = -\frac{2}{3}$;
$-2c_2 = -1 - c_3 + 4c_4 = -1 + \frac{2}{3} - \frac{1}{2} - \frac{5}{6}$; So $c_2 = \frac{5}{12}$;
$c_1 = \frac{1}{2} - \frac{3}{2}c_3 + c_4 = \frac{1}{2} - \frac{3}{2}(-\frac{2}{3}) - \frac{1}{8} = \frac{11}{8}$;

# A Tip for Finding the Roots of the Characteristic Equation

- To factor the characteristic polynomial by hand, the following result is sometimes helpful:
- Suppose that the polynomial $a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$ has integer coefficients; If $r = \frac{p}{q}$ is a rational root, where $p$ and $q$ have no common factors, then $p$ must be a factor of $a_n$, and $q$ must be a factor of $a_0$;
- Example: In $r^4 + r^3 - 7r^2 - r + 6 = 0$ the factors of $a_0$ are $\pm 1$ and the factors of $a_n$ are $\pm 1, \pm 2, \pm 3$ and $\pm 6$; Thus the only possible rational roots of this equation are $\pm 1, \pm 2, \pm 3$ and $\pm 6$; By testing these possible roots, we find that $1, -1, 2$, and $-3$ are actual roots; In this case there are no other roots, since the polynomial is of fourth degree;
- If some of the roots are irrational or complex, as is usually the case, then this process will not find them, but at least the degree of the polynomial can be reduced by dividing the polynomial by the factors corresponding to the rational roots;

# Complex Roots

- If the characteristic equation has complex roots, they must occur in conjugate pairs, $\lambda \pm i\mu$, since the coefficients $a_0, \ldots, a_n$ are real numbers;
- Provided that none of the roots is repeated, the general solution of $L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$ is still of the form $y = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}$;
- However, just as for the second order equation, we can replace the complex-valued solutions $e^{(\lambda + i\mu)t}$ and $e^{(\lambda - i\mu)t}$ by the real-valued solutions $e^{\lambda t} \cos \mu t$, $e^{\lambda t} \sin \mu t$;
- Thus, even though some of the roots of the characteristic equation are complex, it is still possible to express the general solution of the differential equation as a linear combination of real-valued solutions;

# Complex Roots: An Example

- Find the general solution of $y^{(4)} - y = 0$; Also find the solution that satisfies the initial conditions $y(0) = \frac{7}{2}$, $y'(0) = -4$, $y''(0) = \frac{5}{2}$, $y'''(0) = -2$;

  Substituting $e^{rt}$ for $y$, we get $r^4 - 1 = (r^2 - 1)(r^2 + 1) = 0$;
  Therefore the roots are $r = 1, -1, i, -i$ and the general solution is
  $y = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$;
  If we impose the given initial conditions, we find

$$\left\{\begin{array}{l} c_1 + c_2 + c_3 = \frac{7}{2} \\ c_1 - c_2 + c_4 = -4 \\ c_1 + c_2 - c_3 = \frac{5}{2} \\ c_1 - c_2 - c_4 = -2 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} c_1 + c_2 + c_3 = \frac{7}{2} \\ c_1 - c_2 + c_4 = -4 \\ c_1 + c_2 = 3 \\ c_1 - c_2 = -3 \end{array}\right\} \Rightarrow \left\{\begin{array}{lcl} c_1 & = & 0 \\ c_2 & = & 3 \\ c_3 & = & \frac{1}{2} \\ c_4 & = & -1 \end{array}\right.$$

  Thus the solution of the given initial value problem is
  $y = 3e^{-t} + \frac{1}{2}\cos t - \sin t$;

# Repeated Roots

- If the roots of the characteristic equation $a_0r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n = 0$ are not distinct, then the solution $y = c_1e^{r_1t} + c_2e^{r_2t} + \cdots + c_ne^{r_nt}$ is not general;
- Recall that if $r_1$ is a repeated root for $a_0y'' + a_1y' + a_2y = 0$, then two linearly independent solutions are $e^{r_1t}$ and $te^{r_1t}$;
- For an equation of order $n$, if a root of $Z(r) = 0$, say $r = r_1$, has multiplicity $s$ (where $s \leq n$), then $e^{r_1t}, te^{r_1t}, t^2e^{r_1t}, \ldots, t^{s-1}e^{r_1t}$ are corresponding solutions;
- If a complex root $\lambda + i\mu$ is repeated $s$ times, the complex conjugate $\lambda - i\mu$ is also repeated $s$ times; Corresponding to these $2s$ solutions, we can find $2s$ real solutions by noting that the real and imaginary parts of $e^{(\lambda+i\mu)t}$, $te^{(\lambda+i\mu)t}, \ldots, t^{s-1}e^{(\lambda+i\mu)t}$ are also linearly independent solutions: $e^{\lambda t}\cos\mu t$, $e^{\lambda t}\sin\mu t$, $te^{\lambda t}\cos\mu t$, $te^{\lambda t}\sin\mu t$, $\ldots$, $t^{s-1}e^{\lambda t}\cos\mu t$, $t^{s-1}e^{\lambda t}\sin\mu t$; Hence the general solution can always be expressed as a linear combination of n real-valued solutions;

# Repeated Roots: An Example

- Find the general solution of $y^{(4)} + 2y'' + y = 0$;

  The characteristic equation is $r^4 + 2r^2 + 1 = (r^2 + 1)(r^2 + 1) = 0$; The roots are $r = i, i, -i, -i,$ and the general solution is

$$y = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t.$$

# Computing $n$-th Roots Using De Moivre's Formula

- Find the general solution of $y^{(4)} + y = 0$;

  The characteristic equation is $r^4 + 1 = 0$; To solve the equation, we must compute the fourth roots of $-1 = -1 + 0i = \cos\pi + i\sin\pi$; Thus, according to De Moivre's formula, its four complex fourth roots are

$$w_m = \cos\left(\frac{\pi}{4} + \frac{2m\pi}{4}\right) + i\sin\left(\frac{\pi}{4} + \frac{2m\pi}{4}\right), \quad m = 0, 1, 2, 3;$$

  So we have as roots $\frac{1+i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}$; The general solution of the differential equation, therefore, is

$$y = e^{t/\sqrt{2}}\left(c_1\cos\frac{t}{\sqrt{2}} + c_2\sin\frac{t}{\sqrt{2}}\right) + e^{-t/\sqrt{2}}\left(c_3\cos\frac{t}{\sqrt{2}} + c_4\sin\frac{t}{\sqrt{2}}\right);$$

Subsection 3

## The Method of Undetermined Coefficients

# The Method of Undetermined Coefficients

- A particular solution Y of
$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t)$$
can be obtained by the **method of undetermined coefficients** if $g(t)$ is of an appropriate form;
- When $L$ is applied to a polynomial, an exponential, a sine or a cosine function, the result is a polynomial, an exponential or a linear combination of sine and cosine functions, respectively;
- Hence, if $g(t)$ is a sum of such functions, we can find $Y(t)$ by choosing a suitable combination of such functions, multiplied by a number of undetermined constants;
- The constants are then determined by substituting the assumed expression into the differential equation;
- Some terms may need to be multiplied by powers of $t$ to make them different from terms in the solution of the homogeneous;

# Undetermined Coefficients: An Example

- Find the general solution of $y''' - 3y'' + 3y' - y = 4e^t$;

  The characteristic polynomial for the homogeneous is $r^3 - 3r^2 + 3r - 1 = (r-1)^3$, so the general solution of the homogeneous equation is $y_c(t) = c_1 e^t + c_2 te^t + c_3 t^2 e^t$;
  To find a particular solution $Y(t)$ of the nonhomogeneous, we start by $Y(t) = Ae^t$; Since $e^t, te^t, t^2 e^t$ are all solutions of the homogeneous, we must multiply by $t^3$; Assume $Y(t) = At^3 e^t$, where $A$ is an undetermined coefficient; Differentiate $Y(t)$ three times:

$$\begin{aligned}
Y(t) &= At^3 e^t;\\
Y'(t) &= 3At^2 e^t + At^3 e^t;\\
Y''(t) &= 6Ate^t + 3At^2 e^t + 3At^2 e^t + At^3 e^t\\
&= 6Ate^t + 6At^2 e^t + At^3 e^t;\\
Y'''(t) &= 6Ae^t + 6Ate^t + 12Ate^t + 6At^2 e^t + 3At^2 e^t + At^3 e^t\\
&= 6Ae^t + 18Ate^t + 9At^2 e^t + At^3 e^t.
\end{aligned}$$

# Undetermined Coefficients: An Example (Cont'd)

- Substitute for $y$ and its derivatives in the original equation $y''' - 3y'' + 3y' - y = 4e^t$:

$$\begin{aligned}(6Ae^t + 18Ate^t + 9At^2e^t + At^3e^t)\\ - 3(6Ate^t + 6At^2e^t + At^3e^t)\\ + 3(3At^2e^t + At^3e^t) - At^3e^t = 4e^t;\\ 6Ae^t + (18-18)Ate^t + (9-18+9)At^2e^t\\ + (1-3+3-1)At^3e^t = 4e^t;\end{aligned}$$

  So $6Ae^t = 4e^t$; Thus, $A = \frac{2}{3}$ and the particular solution is $Y(t) = \frac{2}{3}t^3e^t$;
  The general solution of the original equation is the sum of $y_c(t)$ and $Y(t)$, i.e., $y = c_1e^t + c_2te^t + c_3t^2e^t + \frac{2}{3}t^3e^t$;

# Undetermined Coefficients: Another Example

- Find a particular solution of the equation

$$y^{(4)} + 2y'' + y = 3\sin t - 5\cos t;$$

The characteristic equation of the homogeneous is $r^4 + 2r^2 + 1 = (r^2+1)^2 = 0$ and has roots $r = i, i, -i, -i$; The general solution of the homogeneous equation is $y_c(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t$;
The particular solution would be $Y(t) = A\sin t + B\cos t$, but we must multiply it by $t^2$ to make it different from the solutions of the homogeneous; Thus, $Y(t) = At^2 \sin t + Bt^2 \cos t$; Differentiate $Y(t)$ four times and substitute into the equation.

# Undetermined Coefficients: Example (Cont'd)

- We get

$$\begin{aligned}
Y(t) &= At^2 \sin t + Bt^2 \cos t; \\
Y'(t) &= 2At \sin t + At^2 \cos t + 2Bt \cos t - Bt^2 \sin t; \\
Y''(t) &= 2A \sin t + 2At \cos t + 2At \cos t - At^2 \sin t \\
&\quad 2B \cos t - 2Bt \sin t - 2Bt \sin t - Bt^2 \cos t \\
&= 2A \sin t + 4At \cos t - At^2 \sin t \\
&\quad + 2B \cos t - 4Bt \sin t - Bt^2 \cos t; \\
Y'''(t) &= 2A \cos t + 4A \cos t - 4At \sin t - 2At \sin t - At^2 \cos t \\
&\quad - 2B \sin t - 4B \sin t - 4Bt \cos t - 2Bt \cos t + Bt^2 \sin t \\
&= 6A \cos t - 6At \sin t - At^2 \cos t \\
&\quad - 6B \sin t - 6Bt \cos t + Bt^2 \sin t; \\
Y^{(4)}(t) &= -6A \sin t - 6A \sin t - 6At \cos t - 2At \cos t + At^2 \sin t \\
&\quad - 6B \cos t - 6B \cos t + 6Bt \sin t + 2Bt \sin t + Bt^2 \cos t \\
&= -12A \sin t - 8At \cos t + At^2 \sin t \\
&\quad - 12B \cos t + 8Bt \sin t + Bt^2 \cos t;
\end{aligned}$$

# Undetermined Coefficients: Example (Cont'd)

- Plugging into $y^{(4)} + 2y'' + y = 3\sin t - 5\cos t$, we get

$$\begin{aligned}
&(-12A\sin t - 8At\cos t + At^2\sin t - 12B\cos t + 8Bt\sin t + Bt^2\cos t)\\
&+2(2A\sin t + 4At\cos t - At^2\sin t + 2B\cos t - 4Bt\sin t - Bt^2\cos t)\\
&+At^2\sin t + Bt^2\cos t = 3\sin t - 5\cos t
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(-12A + 4A)\sin t + (-12B + 4B)\cos t\\
&(-8A + 8A)t\cos t + (8B - 8B)t\sin t\\
&(A - 2A + A)t^2\sin t + (B - 2B + B)t^2\cos t = 3\sin t - 5\cos t.
\end{aligned}$$

No we have $-8A\sin t - 8B\cos t = 3\sin t - 5\cos t$; Thus, $A = -\frac{3}{8}, B = \frac{5}{8}$, and the particular solution is

$$Y(t) = -\frac{3}{8}t^2\sin t + \frac{5}{8}t^2\cos t;$$

# Undetermined Coefficients: Separating Terms in $g(t)$

- Find a particular solution of

$$y''' - 4y' = t + 3\cos t + e^{-2t};$$

The characteristic equation of the homogeneous is $r^3 - 4r = r(r+2)(r-2) = 0$, and the roots are $r = 0, \pm 2$; So $y_c(t) = c_1 + c_2 e^{2t} + c_3 e^{-2t}$; The particular solution is the sum of the particular solutions of

$$y''' - 4y' = t,\ y''' - 4y' = 3\cos t,\ y''' - 4y' = e^{-2t};$$

  - For $Y_1(t)$ we would have $A_0 t + A_1$, but a constant is a solution of the homogeneous, so we multiply by $t$: $Y_1(t) = t(A_0 t + A_1)$;
  - For the second equation we choose $Y_2(t) = B\cos t + C\sin t$, and there is no need to modify this;
  - For the third equation, since $e^{-2t}$ is a solution of the homogeneous equation, we assume that $Y_3(t) = Ete^{-2t}$;

The constants are determined by substituting into the individual differential equations;

# Undetermined Coefficients (First Equation)

- We look at $y''' - 4y' = t$, with $Y_1(t) = A_0 t^2 + A_1 t$;

  We have
  $$\begin{array}{rcl} Y_1(t) & = & A_0 t^2 + A_1 t; \\ Y_1'(t) & = & 2A_0 t + A_1; \\ Y_1''(t) & = & 2A_0; \\ Y_1'''(t) & = & 0; \end{array}$$
  Therefore, we get
  $$\begin{array}{l} 0 - 4(2A_0 t + A_1) = t; \\ -8A_0 t - 4A_1 = t \\ A_0 = -\frac{1}{8}, \quad A_1 = 0. \end{array}$$

# Undetermined Coefficients (Second Equation)

- We look at $y''' - 4y' = 3\cos t$, with $Y_2(t) = B\cos t + C\sin t$;

  We have

$$\begin{array}{rcl} Y_2(t) & = & B\cos t + C\sin t; \\ Y_2'(t) & = & -B\sin t + C\cos t; \\ Y_2''(t) & = & -B\cos t - C\sin t; \\ Y_2'''(t) & = & B\sin t - C\cos t; \end{array}$$

  Therefore, we get

$$\begin{array}{l} (B\sin t - C\cos t) - 4(-B\sin t + C\cos t) = 3\cos t; \\ 5B\sin t - 5C\cos t = 3\cos t; \\ B = 0, \quad C = -\frac{3}{5}. \end{array}$$

# Undetermined Coefficients (Third Equation)

- We look at $y''' - 4y' = e^{-2t}$, with $Y_3(t) = Ete^{-2t}$;

  We have

$$\begin{array}{rcl} Y_3(t) & = & Ete^{-2t}; \\ Y_3'(t) & = & Ee^{-2t} - 2Ete^{-2t}; \\ Y_3''(t) & = & -2Ee^{-2t} - 2Ee^{-2t} + 4Ete^{-2t} \\ & = & -4Ee^{-2t} + 4Ete^{-2t}; \\ Y_3'''(t) & = & 8Ee^{-2t} + 4Ee^{-2t} - 8Ete^{-2t} \\ & = & 12Ee^{-2t} - 8Ete^{-2t}; \end{array}$$

  Therefore, we get

$$(12Ee^{-2t} - 8Ete^{-2t}) - 4(Ee^{-2t} - 2Ete^{-2t}) = e^{-2t};$$
$$8Ee^{-2t} = e^{-2t};$$
$$E = \frac{1}{8}$$

# Undetermined Coefficients (Conclusion)

- We had

$$Y_1(t) = A_0 t^2 + A_1 t, \quad Y_2(t) = B\cos t + C\sin t, \quad Y_3(t) = Ee^{-2t}$$

and we found

$$A_0 = -\frac{1}{8}, \quad A_1 = 0, \quad B = 0, \quad C = -\frac{3}{5}, \quad E = \frac{1}{8}.$$

We conclude that A particular solution of

$$y''' - 4y' = t + 3\cos t + e^{-2t}$$

is

$$Y(t) = -\frac{1}{8}t^2 - \frac{3}{5}\sin t + \frac{1}{8}te^{-2t};$$

Subsection 4

## The Method of Variation of Parameters

# The Method of Variation of Parameters I

- The **method of variation of parameters** for determining a particular solution of the nonhomogeneous $n$-th order linear differential equation
$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t)$$
is a direct extension of the method for the second order case;
- Suppose then that we know a fundamental set of solutions $y_1$, $y_2$, ..., $y_n$ of the homogeneous;
- Then, the general solution of the homogeneous equation is $y_c(t) = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t)$;
- The method of variation of parameters for determining a particular solution of the nonhomogeneous rests on the possibility of determining $n$ functions $u_1, u_2, \ldots, u_n$ such that $Y(t)$ is of the form
$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + \cdots + u_n(t)y_n(t);$$

# The Method of Variation of Parameters II

- We specify $n$ conditions;
- One is that $Y$ satisfy $L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t)$;
- The other $n-1$ conditions are chosen so as to make the calculations as simple as possible;
- From $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + \cdots + u_n(t)y_n(t)$ we get $Y' = (u_1y_1' + u_2y_2' + \cdots + u_ny_n') + (u_1'y_1 + u_2'y_2 + \cdots + u_n'y_n)$;
- We impose
$$u_1'y_1 + u_2'y_2 + \cdots + u_n'y_n = 0;$$
- Therefore, $Y' = u_1y_1' + u_2y_2' + \cdots + u_ny_n'$;
- Now we get $Y'' = (u_1y_1'' + u_2y_2'' + \cdots + u_ny_n'') + (u_1'y_1' + u_2'y_2' + \cdots + u_n'y_n')$;
- We impose
$$u_1'y_1' + u_2'y_2' + \cdots + u_n'y_n' = 0;$$
- Therefore, $Y'' = u_1y_1'' + u_2y_2'' + \cdots + u_ny_n''$;

# The Method of Variation of Parameters III

- Similarly, going through $n-1$ derivatives of $Y$, we impose the conditions

$$u_1'y_1^{(m)}+u_2'y_2^{(m)}+\cdots+u_n'y_n^{(m)}=0, \quad m=1,\ldots,n-2;$$

- And we obtain

$$Y^{(m)}=u_1y_1^{(m)}+u_2y_2^{(m)}+\cdots+u_ny_n^{(m)}, \quad m=0,1,\ldots,n-1;$$

- The $n$-th derivative of Y is

$$Y^{(n)}=(u_1y_1^{(n)}+\cdots+u_ny_n^{(n)})+(u_1'y_1^{(n-1)}+\cdots+u_n'y_n^{(n-1)});$$

# The Method of Variation of Parameters IV

- On substituting for the derivatives of $Y$, we get
$$\begin{aligned}&(u_1y_1^{(n)}+\cdots+u_ny_n^{(n)})+(u_1'y_1^{(n-1)}+\cdots+u_n'y_n^{(n-1)})\\&+p_1(u_1y_1^{(n-1)}+\cdots+u_ny_n^{(n-1)})\\&+\cdots\\&+p_{n-1}(u_1y_1'+\cdots+u_ny_n')\\&+p_n(u_1y_1+\cdots+u_ny_n)=g(t).\end{aligned}$$
- Rearranging and collecting terms, we get
$$\begin{aligned}&u_1'y_1^{(n-1)}+\cdots+u_n'y_n^{(n-1)}\\&+u_1(y_1^{(n)}+p_1y_1^{(n-1)}+\cdots+p_{n-1}y_1'+p_ny_1)\\&+\cdots\\&+u_n(y_n^{(n)}+p_1y_n^{(n-1)}+\cdots+p_{n-1}y_n'+p_ny_n)=g(t).\end{aligned}$$
- Since $L[y_i]=0$, $i=1,2,\ldots,n$, all parentheses vanish:
$$u_1'y_1^{(n-1)}+u_2'y_2^{(n-1)}+\ldots+u_n'y_n^{(n-1)}=g;$$

# The Method of Variation of Parameters V

- Thus, we obtain the following system for $u_1', u_2', \ldots, u_n'$:

$$\begin{array}{rcl} y_1 u_1' + y_2 u_2' + \cdots + y_n u_n' & = & 0 \\ y_1' u_1' + y_2' u_2' + \cdots + y_n' u_n' & = & 0 \\ & \vdots & \\ y_1^{(n-1)} u_1' + y_2^{(n-1)} u_2' + \cdots + y_n^{(n-1)} u_n' & = & g \end{array}$$

- By solving this system and integrating the resulting expressions, we can obtain the coefficients $u_1, \ldots, u_n$;
- The determinant of coefficients is $W(y_1, y_2, \ldots, y_n) \neq 0$, so it is possible to determine $u_1', \ldots, u_n'$;

# The Method of Variation of Parameters VI

- By Cramers rule,
$$u_m'(t) = \frac{g(t)W_m(t)}{W(t)}, \quad m = 1, 2, \ldots, n,$$
were $W(t) = W(y_1, y_2, \ldots, y_n)(t)$, and $W_m$ is the determinant obtained from $W$ by replacing the $m$-th column by the column $(0, 0, ..., 0, 1)$;
- So, a particular solution is given by
$$Y(t) = \sum_{m=1}^{n} y_m(t) \int_{t_0}^{t} \frac{g(s)W_m(s)}{W(s)} ds,$$ where $t_0$ is arbitrary;
- Determining $Y(t)$ may involve very difficult algebraic computations as $n$ increases;
- In some cases the calculations may be simplified to some extent by using Abels identity $W(t) = W(y_1, \ldots, y_n)(t) = ce^{-\int p_1(t)dt}$;

# Example

- If $y_1(t) = e^t$, $y_2(t) = te^t$, $y_3(t) = e^{-t}$ are solutions of the homogeneous corresponding to $y''' - y'' - y' + y = g(t)$, determine a particular solution of the nonhomogeneous in terms of an integral;

$$W(t) = W(e^t, te^t, e^{-t})(t) = \begin{vmatrix} e^t & te^t & e^{-t} \\ e^t & (t+1)e^t & -e^{-t} \\ e^t & (t+2)e^t & e^{-t} \end{vmatrix} =$$

$$e^t \begin{vmatrix} 1 & t & 1 \\ 1 & t+1 & -1 \\ 1 & t+2 & 1 \end{vmatrix} = e^t \begin{vmatrix} 1 & t & 1 \\ 0 & 1 & -2 \\ 0 & 2 & 0 \end{vmatrix} = 4e^t;$$

$$W_1(t) = \begin{vmatrix} 0 & te^t & e^{-t} \\ 0 & (t+1)e^t & -e^{-t} \\ 1 & (t+2)e^t & e^{-t} \end{vmatrix} = \begin{vmatrix} te^t & e^{-t} \\ (t+1)e^t & -e^{-t} \end{vmatrix} = -2t - 1;$$

# Example (Cont'd)

$$W_2(t) = \begin{vmatrix} e^t & 0 & e^{-t} \\ e^t & 0 & -e^{-t} \\ e^t & 1 & e^{-t} \end{vmatrix} = -\begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} = 2;$$

$$W_3(t) = \begin{vmatrix} e^t & te^t & 0 \\ e^t & (t+1)e^t & 0 \\ e^t & (t+2)e^t & 1 \end{vmatrix} = \begin{vmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{vmatrix} = e^{2t};$$

Substituting these results in $Y(t) = \sum_{m=1}^{n} y_m(t) \int_{t_0}^{t} \frac{g(s)W_m(s)}{W(s)} ds$, we get

$$\begin{aligned} Y(t) &= e^t \int_{t_0}^{t} \frac{g(s)(-1-2s)}{4e^s} ds + te^t \int_{t_0}^{t} \frac{g(s)(2)}{4e^s} ds + e^{-t} \int_{t_0}^{t} \frac{g(s)e^{2s}}{4e^s} ds \\ &= \frac{1}{4} \int_{t_0}^{t} [e^{t-s}[-1+2(t-s)] + e^{-(t-s)}] g(s) ds; \end{aligned}$$