George Voutsadakis

Categorical Abstract Algebraic Logic Metalogical Properties*

Abstract. Metalogical properties that have traditionally been studied in the deductive system context (see, e.g., [21]) and transferred later to the institution context [33], are here formulated in the π-institution context. Preservation under deductive equivalence of π-institutions is investigated. If a property is known to hold in all algebraic π-institutions and is preserved under deductive equivalence, then it follows that it holds in all algebraizable π-institutions in the sense of [36].

Keywords: algebraic logic, equivalent deductive systems, algebraizable logics, institutions, equivalent institutions, adjunctions, deduction-detachment property, conjunction property, disjunction property, negation, Craig interpolation, Robinson consistency, Lindenbaum property.
1991 AMS Subject Classification: Primary: 03Gxx, 18Cxx, *Secondary:* 08Bxx, 08Cxx, 68N05.

1. Introduction

In their seminal "Memoirs monograph" [8], Blok and Pigozzi, following in the footsteps of Czelakowski [13] and their own previous work [7], made for the first time precise the notion of an algebraizable sentential logic. Since then, a bulk of work extending theirs has appeared [3, 9, 10, 17, 19, 21, 22, 25, 26, 27, 32] that has come to be collectively known under the term *abstract algebraic logic*. For an overview of this area of algebraic logic the reader is referred to [18]. Two have been the main directions of development of abstract algebraic logic. One is the study of the *algebraization process* itself and the other is the extent to which *metalogical properties* are related to algebraic properties via algebraizability, or, more generally, whether they are preserved or not under equivalence of deductive systems. In [10, 16, 21], e.g., a detailed study of the deduction-detachment property for deductive systems is undertaken. Some other examples include [14] that studies the amalgamation property and [15] on the Maehara interpolation property. Very recently, in [34, 36] the notion of algebraizability for deductive systems has been extended to institutions [23, 24] and π-institutions [20]. To cover this more general framework, equiv-

* The research that led to the results presented in this paper began with the author's doctoral dissertation, written under the supervision of Professor *Don Pigozzi* of *Iowa State University*.
The author was partially supported by National Science Foundation grant CCR - 9593168.

Presented by **Daniele Mundici**; *Received* June 10, 2002

Studia Logica **74**: 369–398, 2003.

alence of deductive systems, as defined in [10], has been extended in [34, 35] to the notion of deductive equivalence for π-institutions. This more abstract framework handles more effectively multi-signature logics with quantifiers, some of which could also be handled in the deductive system framework but in a rather unsatisfactory ad-hoc way (see the discussion in [36]). The generalized algebraizability framework of [34, 35, 36] is referred to as *categorical abstract algebraic logic* because of its extensive use of categorical algebraic rather than universal algebraic techniques. It is only natural that the two main directions of research in abstract algebraic logic, mentioned above, will be in the main focus of categorical abstract algebraic logic as well, the starting point being relations between π-institutions or institutions like the ones introduced in [35]. The first direction of research, i.e., the study and the analysis of the algebraization process itself in the context of π-institutions, has been pursued further in [35, 36]. Here, some aspects and properties pertaining to the second direction of research are developed.

Various metalogical properties of institutions have already been defined in [33]. Some of those are revisited in this paper and reformulated, in a somewhat nonstandard way, in the π-institution framework and some new ones are defined. Then the effect that deductive equivalence has on these properties is explored.

Some of the basic notions that are necessary to understand the proofs in the paper are now recalled. The reader is referred to [4] and [30] for all unexplained categorical notation.

DEFINITION 1.1. [23, 24] An **institution** $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models \rangle$ consists of

(i) a category **Sign** whose objects are called **signatures**,

(ii) a functor SEN : $\mathbf{Sign} \to \mathbf{Set}$, from the category **Sign** of signatures into the category **Set** of sets, called the **sentence functor** and giving, for each signature Σ, a set whose elements are called **sentences over** that signature Σ or Σ-**sentences**,

(iii) a functor MOD : $\mathbf{Sign} \to \mathbf{CAT}^{op}$ from the category of signatures into the opposite of the category of categories, called the **model functor** and giving, for each signature Σ, a category whose objects are called Σ-**models** and whose morphisms are called Σ-**morphisms** and

(iv) a relation $\models_\Sigma \subseteq |\mathrm{MOD}(\Sigma)| \times \mathrm{SEN}(\Sigma)$, for each $\Sigma \in |\mathbf{Sign}|$, called Σ-**satisfaction,** such that for every morphism $f : \Sigma_1 \to \Sigma_2$ in **Sign** the **satisfaction condition**

$$m_2 \models_{\Sigma_2} \mathrm{SEN}(f)(\phi_1) \quad \text{if and only if} \quad \mathrm{MOD}(f)(m_2) \models_{\Sigma_1} \phi_1$$

holds, for every $m_2 \in |\mathrm{MOD}(\Sigma_2)|$ and every $\phi_1 \in \mathrm{SEN}(\Sigma_1)$.

The defining categories and functors of an institution together with their interconnections are illustrated by the following diagram:

$$\begin{tikzcd} & \mathbf{Set} \\ \mathbf{Sign} \arrow[ur, "\mathrm{SEN}"] \arrow[dr, "\mathrm{MOD}"'] & \models \\ & \mathbf{CAT}^{\mathrm{op}} \end{tikzcd}$$

Furthermore, the satisfaction condition can be given pictorially as follows: If $f : \Sigma_1 \to \Sigma_2$ is a morphism in **Sign**, then,

$$\begin{tikzcd} \mathrm{MOD}(\Sigma_1) \arrow[r, phantom, "\models_{\Sigma_1}"] & \mathrm{SEN}(\Sigma_1) \arrow[d, "\mathrm{SEN}(f)"] \\ \mathrm{MOD}(\Sigma_2) \arrow[u, "\mathrm{MOD}(f)"] \arrow[r, phantom, "\models_{\Sigma_2}"'] & \mathrm{SEN}(\Sigma_2) \end{tikzcd}$$

Given an institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models \rangle, \Sigma \in |\mathbf{Sign}|, \Phi \subseteq \mathrm{SEN}(\Sigma)$ and $M \subseteq |\mathrm{MOD}(\Sigma)|$, we define

$$\Phi^* = \{m \in |\mathrm{MOD}(\Sigma)| : m \models_\Sigma \phi \quad \text{for every} \quad \phi \in \Phi\}$$

and

$$M^* = \{\phi \in \mathrm{SEN}(\Sigma) : m \models_\Sigma \phi \quad \text{for every} \quad m \in M\}.$$

Moreover we set $\Phi^c = \Phi^{**}$ and $M^c = M^{**}$.

From now on when the "c" symbol is used, its scope will be the largest possible well-formed expression to its left. For instance, in $\mathrm{SEN}(f)(\Phi)^c$ the scope of "c" is $\mathrm{SEN}(f)(\Phi)$ and not just (Φ), and in $\mathrm{SEN}(f)(\mathrm{SEN}(f)^{-1}(\Phi^c))^c$ the scope of the second "c" is $\mathrm{SEN}(f)(\mathrm{SEN}(f)^{-1}(\Phi^c))$ and not just $\mathrm{SEN}(f)^{-1}$ (Φ^c).

Goguen and Burstall [24], prove the following very useful lemma that is used below to obtain the π-institution associated with a given institution $\mathcal{I}$.

LEMMA 1.2. [Closure Lemma] *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models \rangle$ *be an institution,* $f : \Sigma_1 \to \Sigma_2 \in \mathrm{Mor}(\mathbf{Sign})$ *and* $\Phi \subseteq \mathrm{SEN}(\Sigma_1)$. *Then* $\mathrm{SEN}(f)(\Phi^c) \subseteq \mathrm{SEN}(f)(\Phi)^c$.

Prototypical examples of institutions are provided by the well-known institutions of equational and of first-order logic. Several versions of these institutions are available and described in some detail in the literature. In [23], Section 2, an institution for multi-sorted equational logic is presented. Several other examples, including an institution for first-order logic, are given in [23], Section 3. These examples are further elaborated on in Appendix A of [24]. Variants of these two institutions, that are more suitable for algebraization purposes in the context of categorical abstract algebraic logic, are outlined in [36]. On the one hand, these versions are dealing only with the single-sorted case but, on the other, allow for substitutions of terms for basic operation symbols and of formulas for basic relation symbols, respectively, whereas, the versions of [23] only allow basic operation symbols to be substituted by basic operation symbols and the same for relation symbols. The versions of [36] are described in more detail in [37] and [38], respectively.

Two other, not so well-known, institutions will be briefly described here to provide additional examples to the ones mentioned above to illustrate the definition. The first is borrowed from the categorical theory of sketches. For definitions and other information pertaining to sketches the reader is referred to the following rich introductions and references [4, 5, 11, 12, 6, 1, 2]. The second is borrowed from the theory of computation and, more specifically, describes an institution formalizing changes of alphabets and states in finite state automata that preserve acceptance of strings. For relevant definitions in the theory of finite state automata the reader is referred to [28, 29, 31].

First Example: Sketch Logic

A **sketch** $\mathcal{S} = \langle G(\mathcal{S}), D(\mathcal{S}), L(\mathcal{S}), C(\mathcal{S})\rangle$ consists of a graph $G(\mathcal{S})$, a set $D(\mathcal{S})$ of diagrams in $G(\mathcal{S})$, a set $L(\mathcal{S})$ of cones in $G(\mathcal{S})$ and a set $C(\mathcal{S})$ of cocones in $G(\mathcal{S})$. A **sketch morphism** $f : \mathcal{S}_1 \to \mathcal{S}_2$ from a sketch $\mathcal{S}_1$ to a sketch $\mathcal{S}_2$ is a morphism $f : G(\mathcal{S}_1) \to G(\mathcal{S}_2)$ of graphs that takes diagrams in $D(\mathcal{S}_1)$ to diagrams in $D(\mathcal{S}_2)$, cones in $L(\mathcal{S}_1)$ to cones in $L(\mathcal{S}_2)$ and cocones in $C(\mathcal{S}_1)$ to cocones in $C(\mathcal{S}_2)$.

Obviously, identity graph morphisms act as identity sketch morphisms and the composition of two sketch morphisms is again a sketch morphism. Therefore, sketches with sketch morphisms between them form a category, called the **category of sketches** and denoted by **Skt**.

Given a category $\mathbf{C}$ one may construct a sketch $\mathrm{SK}(\mathbf{C})$ by taking the underlying graph of the category as the graph of this sketch and stipulating that its diagrams be all the commutative diagrams in $\mathbf{C}$, its cones be all the limit cones in $\mathbf{C}$ and its cocones be all colimit cocones in $\mathbf{C}$. Moreover, given a limit and colimit preserving functor $F : \mathbf{C}_1 \to \mathbf{C}_2$, one obtains a sketch morphism $\mathrm{SK}(F) : \mathrm{SK}(\mathbf{C}_1) \to \mathrm{SK}(\mathbf{C}_2)$, since the diagram condition is automatically satisfied and the cone and cocone conditions may be derived easily by the continuity and cocontinuity of F. This defines a functor $\mathrm{SK} : \mathbf{Cat}_{\rightarrow} \to \mathbf{Skt}$ from the category of categories with limit and colimit preserving functors into the category of sketches.

The category of signatures, denoted **SLSig**, of the institution of Sketch Logic will now be defined. In order to do this, we state first a well-known theorem [4].

THEOREM 1.3. *Let $\mathcal{S}$ be a sketch. Then there is a category $\mathbf{Th}\mathcal{S}$ and a sketch morphism $\eta_{\mathcal{S}} : \mathcal{S} \to \mathrm{SK}(\mathbf{Th}\mathcal{S})$ such that, for any sketch morphism $f : \mathcal{S} \to \mathrm{SK}(\mathbf{C})$, there is a unique up to natural isomorphism limit and colimit preserving functor $f^* : \mathbf{Th}\mathcal{S} \to \mathbf{C}$, such that the following diagram commutes*

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\eta_{\mathcal{S}}"] \arrow[dr, "f"'] & \mathrm{SK}(\mathbf{Th}\mathcal{S}) \arrow[d, "\mathrm{SK}(f^*)"] \\ & \mathrm{SK}(\mathbf{C}) \end{tikzcd}$$

Motivated by this theorem, we will identify functors in $\mathbf{Cat}_{\rightarrow}$ that are naturally isomorphic. Furthermore, from now on we will write $f : \mathcal{S}_1 \rightharpoondown \mathcal{S}_2$ to denote a sketch morphism $f : \mathcal{S}_1 \to \mathrm{SK}(\mathbf{Th}\mathcal{S}_2)$. Given two such morphisms $f : \mathcal{S}_1 \rightharpoondown \mathcal{S}_2$ and $g : \mathcal{S}_2 \rightharpoondown \mathcal{S}_3$ their composition $g \circ f : \mathcal{S}_1 \rightharpoondown \mathcal{S}_3$ is defined to be the sketch morphism $g \circ f = \mathrm{SK}(g^*)f$.

$$\begin{tikzcd} \mathcal{S}_2 \arrow[r, "\eta_{\mathcal{S}_2}"] \arrow[dr, "g"'] & \mathrm{SK}(\mathbf{Th}\mathcal{S}_2) \arrow[d, "\mathrm{SK}(g^*)"] \\ & \mathrm{SK}(\mathbf{Th}\mathcal{S}_3) \end{tikzcd}$$

Then it is not hard to prove that if $f : \mathcal{S}_1 \rightharpoondown \mathcal{S}_2$ and $g : \mathcal{S}_2 \rightharpoondown \mathcal{S}_3$ are sketch morphisms, we have $\mathrm{SK}(g^*)\mathrm{SK}(f^*) \cong \mathrm{SK}((\mathrm{SK}(g^*)f)^*$, since both of these make the following diagram commute:

$$\begin{tikzcd} \mathcal{S}_1 \arrow[r, "\eta_{\mathcal{S}_1}"] \arrow[dr, "f"'] & \mathrm{SK}(\mathbf{Th}\mathcal{S}_1) \arrow[d, "\mathrm{SK}(f^*)"] \\ \mathcal{S}_2 \arrow[r, "\eta_{\mathcal{S}_2}"] \arrow[dr, "g"'] & \mathrm{SK}(\mathbf{Th}\mathcal{S}_2) \arrow[d, "\mathrm{SK}(g^*)"] \\ \mathcal{S}_3 \arrow[r, "\eta_{\mathcal{S}_3}"'] & \mathrm{SK}(\mathbf{Th}\mathcal{S}_3) \end{tikzcd}$$

Having this property at hand, it is not hard to show that the composition $\circ$ is associative, i.e., given three morphisms $f : \mathcal{S}_1 \rightharpoondown \mathcal{S}_2, g : \mathcal{S}_2 \rightharpoondown \mathcal{S}_3$ and

$h : \mathcal{S}_3 \rightharpoonup \mathcal{S}_4$, we have $(h \circ g) \circ f \cong h \circ (g \circ f)$, and that, moreover, $\eta_{\mathcal{S}} : \mathcal{S} \rightharpoonup \mathcal{S}$ acts as an identity. Thus, **SLSig** having as collection of objects $|\mathbf{Skt}|$ and as collections of morphisms

$$\mathbf{SLSig}(\mathcal{S}_1, \mathcal{S}_2) = \{f : \mathcal{S}_1 \rightharpoonup \mathcal{S}_2 : f \in \mathbf{Skt}(\mathcal{S}_1, \mathrm{SK}(\mathbf{Th}\mathcal{S}_2))\}$$

for all $\mathcal{S}_1, \mathcal{S}_2 \in |\mathbf{Skt}|$, with composition $\circ$ and $\mathcal{S}$-identity $\eta_{\mathcal{S}}$ (identifying morphisms that are naturally isomorphic) is a category. It will serve as the signature category of the institution $\mathcal{SKL}$ of sketch logic.

Next, define the sentence functor SLSEN : $\mathbf{SLSig} \to \mathbf{Set}$ as follows. At the object level, for every $\mathcal{S} \in |\mathbf{Skt}|$, we define $\mathrm{SLSEN}(\mathcal{S})$ to be the set of all diagrams, cones and cocones in $\mathbf{Th}\mathcal{S}$. Thus $\mathrm{SLSEN}(\mathcal{S}) = \mathrm{DCC}(\mathbf{Th}\mathcal{S})$, where $\mathrm{DCC}(\mathbf{C})$ denotes the collection of all diagrams, cones and cocones in the underlying graph of the category $\mathbf{C}$ and, for a sketch $\mathcal{S}$, $\mathrm{DCC}(\mathcal{S}) = D(\mathcal{S}) \cup L(\mathcal{S}) \cup C(\mathcal{S})$. We call a $\delta \in \mathrm{SLSEN}(\mathcal{S})$ an $\mathcal{S}$-**sentence.** Note that, for a category $\mathbf{C}$, $\mathrm{DCC}(\mathrm{SK}(\mathbf{C})) \subseteq \mathrm{DCC}(\mathbf{C})$, since the first contains only commutative diagrams, limit cones and colimit cocones in $\mathbf{C}$ whereas the second contains all diagrams, cones and cocones in $\mathbf{C}$, regardless of whether they are commutative, limiting or colimiting, respectively. At the morphism level, given $f : \mathcal{S}_1 \rightharpoonup \mathcal{S}_2 \in \mathrm{Mor}(\mathbf{SLSig})$, we define $\mathrm{SLSEN}(f) : \mathrm{SLSEN}(\mathcal{S}_1) \to \mathrm{SLSEN}(\mathcal{S}_2)$ by letting

$$\mathrm{SLSEN}(f)(\delta) = f^*(\delta).$$

$\mathrm{SEN}(f)$ is well-defined and it is not difficult to see that it is a functor.

The model functor SLMOD : $\mathbf{SLSig} \to \mathbf{CAT}^{\mathrm{op}}$ of the institution of sketch logic is described next. At the object level, given $\mathcal{S} \in |\mathbf{Skt}|$, the category $\mathrm{SLMOD}(\mathcal{S})$ has as collection of objects the collection of all models $M : \mathcal{S} \to \mathrm{SK}(\mathbf{C})$ of the sketch $\mathcal{S}$ in a category $\mathbf{C}$ and as collections of morphisms $\mathrm{SLMOD}(\mathcal{S})(M_1, M_2)$ all natural transformations $\alpha : M_1 \to M_2$, for all $M_1, M_2 : \mathcal{S} \to \mathrm{SK}(\mathbf{C})$. These are $G(\mathcal{S})$-indexed collections of $\mathbf{C}$-morphisms $\{\alpha_n : n \in G(\mathcal{S})\}$ such that, for all $f : n_0 \to n_1 \in G(\mathcal{S})$,

$$\begin{tikzcd} M_1(n_0) \arrow[r, "M_1(f)"] \arrow[d, "\alpha_{n_0}"'] & M_1(n_1) \arrow[d, "\alpha_{n_1}"] \\ M_2(n_0) \arrow[r, "M_2(f)"'] & M_2(n_1) \end{tikzcd}$$

commutes in $\mathbf{C}$. In $\mathrm{SLMOD}(\mathcal{S})$ composition of $\alpha : M_1 \to M_2, \beta : M_2 \to M_3$ is defined as usual by $(\beta\alpha)_n = \beta_n \alpha_n$, for all $n \in G(\mathcal{S})$, and identity natural transformations act as identities. For SLMOD at the morphism level, let

$f : \mathcal{S}_1 \rightharpoondown \mathcal{S}_2 \in \mathrm{Mor}(\mathbf{SLSig})$. $\mathrm{SLMOD}(f) : \mathrm{SLMOD}(\mathcal{S}_2) \to \mathrm{SLMOD}(\mathcal{S}_1)$ is the functor defined as follows: Given $M : \mathcal{S}_2 \to \mathrm{SK}(\mathbf{C}) \in |\mathrm{SLMOD}(\mathcal{S}_2)|$,

$$\mathrm{SLMOD}(f)(M) = \mathrm{SK}(M^*)f$$

$$\begin{tikzcd}
\mathcal{S}_1 \arrow[r, "f"] \arrow[dr, "\mathrm{SK}(M^*)f"'] & \mathrm{SK}(\mathbf{Th}\mathcal{S}_2) \arrow[d, "\mathrm{SK}(M^*)"] & \mathcal{S}_2 \arrow[l, "\eta_{\mathcal{S}_2}"'] \arrow[dl, "M"] \\
& \mathrm{SK}(\mathbf{C}) &
\end{tikzcd}$$

and, given $\alpha : M_1 \to M_2 \in \mathrm{Mor}(\mathrm{SLMOD}(\mathcal{S}_2)), \mathrm{SLMOD}(f)(\alpha) : \mathrm{SK}(M_1^*)f \to \mathrm{SK}(M_2^*)f$ is given by

$$\mathrm{SLMOD}(f)(\alpha) = \mathrm{SK}(\alpha^*)f,$$

$$\begin{tikzcd}
\mathcal{S}_2 \arrow[r, "\eta_{\mathcal{S}_2}"] \arrow[dr, shift right, "M_1"'] \arrow[dr, shift left, "M_2"] & \mathrm{SK}(\mathbf{Th}\mathcal{S}_2) \arrow[d, shift right, "M_1^*"'] \arrow[d, shift left, "M_2^*"] \\
& \mathrm{SK}(\mathbf{C})
\end{tikzcd}$$

where $\alpha^* : M_1^* \to M_2^*$ is the unique up to a natural isomorphism natural transformation, such that $\mathrm{SK}(\alpha^*)\eta_{\mathcal{S}_2} = \alpha$.

SLMOD : $\mathbf{SLSig} \to \mathbf{CAT}^{\mathrm{op}}$, defined as above is a functor.

Finally, define $\mathcal{SKL} = \langle \mathbf{SLSig}, \mathrm{SLSEN}, \mathrm{SLMOD}, \models \rangle$ by letting , for every $\mathcal{S} \in |\mathbf{SLSig}|, \models_{\mathcal{S}} \subseteq |\mathrm{SLMOD}(\mathcal{S})| \times \mathrm{SLSEN}(\mathcal{S})$ be defined by

$$M \models_{\mathcal{S}} \delta \quad \text{iff} \quad M^*(\delta) \in \mathrm{DCC}(\mathrm{SK}(\mathbf{C})),$$

for all $M \in |\mathrm{SLMOD}(\mathcal{S})|$ and $\delta \in \mathrm{SLSEN}(\mathcal{S})$. Given $f : \mathcal{S}_1 \rightharpoondown \mathcal{S}_2 \in \mathrm{Mor}(\mathbf{SLSig}), M \in |\mathrm{SLMOD}(\mathcal{S}_2)|$ and $\delta \in \mathrm{SLSEN}(\mathcal{S}_1)$, it may be shown that

$$\mathrm{SLMOD}(f)(M) \models_{\mathcal{S}_1} \delta \quad \text{iff} \quad M \models_{\mathcal{S}_2} \mathrm{SLSEN}(f)(\delta).$$

Hence $\mathcal{SKL}$ is an institution, called the **institution of sketch logic**.

Second Example: Finite State Automata

In this section λ denotes the empty string. Given a set X, let X^* be the set of all finite strings in the alphabet X, including the empty string λ. Concatenation $\cdot : X^* \times X^* \to X^*$ on X^* is defined as usual. Moreover, given two sets X, Y and a map $f : X \to Y^*$, define $f^* : X^* \to Y^*$ as the map extending f on strings.

The category of signatures **FASig** of the institution of finite state automata is the category with collection of objects $|\mathbf{Set}|$ and morphisms $f : X \rightharpoondown Y$ all set maps $f : X \to Y^*$. Given $f : X \rightharpoondown Y, g : Y \rightharpoondown Z \in \mathrm{Mor}(\mathbf{FASig})$, composition is defined by $g \circ f = g^* f$.

The sentence functor FASEN : $\mathbf{FASig} \to \mathbf{Set}$ sends an object $X \in |\mathbf{FASig}|$ to the set X^* and a morphism $f : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{FASig})$ to the set map $\mathrm{FASEN}(f) = f^* : \mathrm{FASEN}(X) \to \mathrm{FASEN}(Y)$.

To define the model functor FAMOD : $\mathbf{FASig} \to \mathbf{CAT}^{\mathrm{op}}$ the following preliminary definitions are needed.

Let $X \in |\mathbf{Set}|$. By an $X-$**automaton** we mean a pair $\langle\langle Q, Y, q_0, \delta, A\rangle, f\rangle$, where $Q, Y \in |\mathbf{Set}|, q_0 \in Q$ is the **initial state** of the automaton, $\delta : Q \times Y \to Q$ is a function, called the **transition function** of the automaton, $A \subseteq Q$ is the set of **accepting states** of the automaton and $f : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{FASig})$.

Given an $X-$automaton $\langle\langle Q, Y, q_0, \delta, A\rangle, f\rangle$, let $\delta^* : Q \times Y^* \to Q$ be the function that uniquely extends the transition function from letters to strings. This function gives the transition "in many steps" of the automaton and may be defined formally by induction on the length of a string.

Let $X \in |\mathbf{Set}|$ and $\langle\langle P, Y, p_0, \gamma, A\rangle, f\rangle, \langle\langle Q, Z, q_0, \delta, B\rangle, g\rangle$ be two $X-$automata. By an $X-$**automaton morphism** $h : \langle\langle P, Y, p_0, \gamma, A\rangle, f\rangle \to \langle\langle Q, Z, q_0, \delta, B\rangle, g\rangle$ we understand a pair $h = (h_1, h_2)$ of two set maps $h_1 : P \to Q$ and $h_2 : Y \rightharpoondown Z$, such that

(i) $h_1(p_0) = q_0$,

(ii) $h_1(A) \subseteq B$,

(iii) $\delta^*(q_0, h_2(y)) = h_1(\gamma(p_0, y))$, for every $y \in Y$, and

(iv) $g = h_2^* f$.

If composition of two $X-$automaton morphisms $k = (k_1, k_2) : \langle\langle P, Y, p_0, \gamma, A\rangle, f\rangle \to \langle\langle Q, Z, q_0, \delta, B\rangle, g\rangle$ and $l = (l_1, l_2) : \langle\langle Q, Z, q_0, \delta, B\rangle, g\rangle \to \langle\langle R, W, r_0, \epsilon, C\rangle, h\rangle$ is defined by $l \circ k = (l_1 k_1, l_2 \circ k_2)$, then $X-$automata together with $X-$automaton morphisms between them form a category $\mathbf{AUT}_X$, the **category of** $X-$**automata.**

FAMOD : $\mathbf{FASig} \to \mathbf{CAT}^{\mathrm{op}}$ sends an object $X \in |\mathbf{FASig}|$ to the category $\mathbf{AUT}_X$ of $X-$automata and a morphism $f : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{FASig})$, to the functor $\mathrm{FAMOD}(f) : \mathrm{FAMOD}(Y) \to \mathrm{FAMOD}(X)$ sending $\langle\langle Q, Z, q_0, \delta, A\rangle, g\rangle$ to $\langle\langle Q, Z, q_0, \delta, A\rangle, g \circ f\rangle$, and a morphism $k : \langle\langle P, Z, p_0, \gamma, A\rangle, g\rangle \to \langle\langle Q, W, q_0, \delta, B\rangle, h\rangle$ to $\mathrm{FAMOD}(f)(k) : \langle\langle P, Z, p_0, \gamma, A\rangle, g \circ f\rangle \to \langle\langle Q, W, q_0, \delta, B\rangle, h \circ f\rangle$ defined by $\mathrm{FAMOD}(f)(k) = k$.

Finally, the institution $\mathcal{FSA} = \langle \mathbf{FASig}, \mathrm{FASEN}, \mathrm{FAMOD}, \models\rangle$ of finite state automata is fully defined by stipulating that, for every $X \in |\mathbf{FASig}|$, $\models_X \subseteq |\mathrm{FAMOD}(X)| \times \mathrm{FASEN}(X)$ is given by

$$\langle\langle Q, Y, q_0, \delta, A\rangle, g\rangle \models_X w \text{ if and only if } \delta^*(q_0, f^*(w)) \in A,$$

for every $\langle\langle Q, Y, q_0, \delta, A\rangle, g\rangle \in |\mathrm{FAMOD}(X)|, w \in X^*$.

$\mathcal{FSA} = \langle \mathbf{FASig}, \mathrm{FASEN}, \mathrm{FAMOD}, \models\rangle$ is an institution, called the **institution of finite state automata**.

Fiadeiro and Sernadas [20], modified the notion of an institution to free the structure from the model theoretic satisfaction relations and bring it closer in spirit to the deductive system framework. The model theoretic deductions were replaced by logical closure operators. The emerging structures were termed π-institutions.

DEFINITION 1.4. [20] A **π-institution** $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ consists of

(i) a category **Sign** whose objects are called **signatures**,

(ii) a functor SEN : **Sign** $\rightarrow$ **Set**, from the category **Sign** of signatures into the category **Set** of sets, called the **sentence functor** and giving, for each signature Σ, a set whose elements are called **sentences over** that signature Σ or **Σ-sentences** and

(iii) a mapping $C_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \rightarrow \mathcal{P}(\mathrm{SEN}(\Sigma))$, for each $\Sigma \in |\mathbf{Sign}|$, called **$\Sigma$-closure,** such that

 (a) $A \subseteq C_\Sigma(A)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq \mathrm{SEN}(\Sigma)$,

 (b) $C_\Sigma(C_\Sigma(A)) = C_\Sigma(A)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq \mathrm{SEN}(\Sigma)$,

 (c) $C_\Sigma(A) \subseteq C_\Sigma(B)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq B \subseteq \mathrm{SEN}(\Sigma)$,

 (d) $\mathrm{SEN}(f)(C_{\Sigma_1}(A)) \subseteq C_{\Sigma_2}(\mathrm{SEN}(f)(A))$, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, $A \subseteq \mathrm{SEN}(\Sigma_1)$.

Given an institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models\rangle$, define

$$\pi(\mathcal{I}) = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle,$$

by setting

$$C_\Sigma(\Phi) = \Phi^c, \text{ for all } \Sigma \in |\mathbf{Sign}|, \Phi \subseteq \mathrm{SEN}(\Sigma).$$

It is easy to verify, using Lemma 1.2, that $\pi(\mathcal{I})$ is a π-institution. We will refer to $\pi(\mathcal{I})$ as to the **π-institution associated with** the institution $\mathcal{I}$.

From now on, given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$, a signature Σ and $\Phi \subseteq \mathrm{SEN}(\Sigma)$, we will use the simplified notation Φ^c to denote $C_\Sigma(\Phi)$. Usually the signature Σ is clear from context and therefore this simplified notation does not cause any confusion.

The following lemmas and corollaries were proven in [34] and will be used in the proofs of the main theorems concerning the preservation of the metalogical properties under deductive equivalence in the following sections. The statements are included here for the convenience of the reader and for the sake of completeness.

LEMMA 1.5. *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}\rangle$ *be a* π*-institution. Then, for all* $f : \Sigma_1 \to \Sigma_2 \in \mathrm{Mor}(\mathbf{Sign}), \Phi \subseteq \mathrm{SEN}(\Sigma_1)$,

$$\mathrm{SEN}(f)(\Phi^c)^c = \mathrm{SEN}(f)(\Phi)^c.$$

LEMMA 1.6. *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}\rangle$ *be a* π*-institution,* $f : \Sigma_1 \to \Sigma_2$ *a morphism in* $\mathbf{Sign}$ *and* $\Phi \subseteq \mathrm{SEN}(\Sigma_2)$*. Then*

$$\mathrm{SEN}(f)^{-1}(\Phi^c)^c = \mathrm{SEN}(f)^{-1}(\Phi^c).$$

COROLLARY 1.7. *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}\rangle$ *be a* π*-institution,* $f : \Sigma_1 \to \Sigma_2$ *an isomorphism in* $\mathbf{Sign}$ *and* $\Phi \subseteq \mathrm{SEN}(\Sigma_1)$*. Then* $\mathrm{SEN}(f)(\Phi^c)^c = \mathrm{SEN}(f)(\Phi^c)$.

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}\rangle$ be a π-institution. Following [20] we define its **category of theories** $\mathbf{TH}(\mathcal{I})$, as follows:

The objects of $\mathbf{TH}(\mathcal{I})$ are pairs $\langle \Sigma, T\rangle$, where $\Sigma \in |\mathbf{Sign}|$ and $T \subseteq \mathrm{SEN}(\Sigma)$ with $T^c = T$. The morphisms $f : \langle \Sigma_1, T_1\rangle \to \langle \Sigma_2, T_2\rangle$ are $\mathbf{Sign}$−morphisms $f : \Sigma_1 \to \Sigma_2$, such that $\mathrm{SEN}(f)(T_1) \subseteq T_2$. Let $\pi_2 : |\mathbf{TH}(\mathcal{I})| \to \mathbf{Set}$ denote the projection onto the second coordinate.

Now, define a functor $\mathrm{SIG} : \mathbf{TH}(\mathcal{I}) \to \mathbf{Sign}$ by

$$\mathrm{SIG}(\langle \Sigma, T\rangle) = \Sigma, \quad \text{for every} \quad \langle \Sigma, T\rangle \in |\mathbf{TH}(\mathcal{I})|,$$

and

$$\mathrm{SIG}(f) = f, \quad \text{for every} \quad f : \langle \Sigma_1, T_1\rangle \to \langle \Sigma_2, T_2\rangle \in \mathrm{Mor}(\mathbf{TH}(\mathcal{I})).$$

The following relations between the categories of theories of two π-institutions will be useful in what follows.

DEFINITION 1.8. Let $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_1|}\rangle, \mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_2|}\rangle$ be two π-institutions. A functor $F : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ will be called **signature-respecting** if there exists a functor $F' : \mathbf{Sign}_1 \to \mathbf{Sign}_2$, such that the following rectangle commutes

$$\begin{tikzcd} \mathbf{TH}(\mathcal{I}_1) \arrow[r, "F"] \arrow[d, "\mathrm{SIG}"'] & \mathbf{TH}(\mathcal{I}_2) \arrow[d, "\mathrm{SIG}"] \\ \mathbf{Sign}_1 \arrow[r, "F'"'] & \mathbf{Sign}_2 \end{tikzcd} \tag{i}$$

If this is the case, it is easy to verify that F' is necessarily unique. F is said to be **monotonic** if, for all $\langle \Sigma_1, T_1\rangle, \langle \Sigma_1, T_1'\rangle \in |\mathbf{TH}(\mathcal{I}_1)|$,

$$T_1 \subseteq T_1' \quad \text{implies} \quad \pi_2(F(\langle \Sigma_1, T_1\rangle)) \subseteq \pi_2(F(\langle \Sigma_1, T_1'\rangle)).$$

A signature-respecting functor $F : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ will be said to **commute with substitutions** if, for every $f : \Sigma_1 \to \Sigma_1' \in \mathrm{Mor}(\mathbf{Sign}_1)$,

$$\mathrm{SEN}_2(F'(f))(\pi_2(F(\langle \Sigma_1, T_1\rangle)))^c = \pi_2(F(\langle \Sigma_1', \mathrm{SEN}_1(f)(T_1)^c\rangle)),$$

for every $\langle \Sigma_1, T_1\rangle \in |\mathbf{TH}(\mathcal{I}_1)|$, where $F' : \mathbf{Sign}_1 \to \mathbf{Sign}_2$ is the (necessarily unique) functor of diagram (i).

The properties above may be extended to the case where the two categories of theories $\mathbf{TH}(\mathcal{I}_1)$ and $\mathbf{TH}(\mathcal{I}_2)$ are related via an adjunction. The following definition then applies

DEFINITION 1.9. An adjunction $\langle F, G, \eta, \epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ will be called **signature-respecting** if both F and G are signature-respecting. It is said to be **monotonic** if both F and G are monotonic. A signature-respecting adjunction will be said to **commute with substitutions** if both F and G commute with substitutions.

LEMMA 1.10. *Let* $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_1|}\rangle, \mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_2|}\rangle$ *be two* π*-institutions. A signature-respecting adjoint equivalence* $\langle F, G, \eta, \epsilon\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ *is monotonic and injective on* Σ_1*-theories, i.e., for all* $\Sigma_1 \in |\mathbf{Sign}_1|$, $\langle \Sigma_1, T_1\rangle$, $\langle \Sigma_1, T_1'\rangle \in |\mathbf{TH}(\mathcal{I}_1)|$,

$$\langle \Sigma_1, T_1\rangle \neq \langle \Sigma_1, T_1'\rangle \quad \textit{implies} \quad F(\langle \Sigma_1, T_1\rangle) \neq F(\langle \Sigma_1, T_1'\rangle),$$

and the same holds for Σ_2*-theories, for every* $\Sigma_2 \in |\mathbf{Sign}_2|$.

Next, relations between π-institutions are reviewed with the goal of transferring properties of related π-institutions to their categories of theories.

DEFINITION 1.11. Let $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_1|}\rangle, \mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_2|}\rangle$ be two π-institutions. A **translation of** $\mathcal{I}_1$ **in** $\mathcal{I}_2$ is a pair $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ consisting of a functor $F : \mathbf{Sign}_1 \to \mathbf{Sign}_2$ and a natural transformation $\alpha : \mathrm{SEN}_1 \to \mathcal{P}\mathrm{SEN}_2\ F$. A translation $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ is an **interpretation of** $\mathcal{I}_1$ **in** $\mathcal{I}_2$ if, for all $\Sigma_1 \in |\mathbf{Sign}_1|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}_1(\Sigma_1)$,

$$\phi \in \Phi^c \quad \text{if and only if} \quad \alpha_{\Sigma_1}(\phi) \subseteq \alpha_{\Sigma_1}(\Phi)^c. \tag{ii}$$

Using these notions the relation of deductive equivalence on π-institutions can be defined.

Definition 1.12. Let $\mathcal{I}_1, \mathcal{I}_2$ be two π-institutions, as above. $\mathcal{I}_1$ and $\mathcal{I}_2$ will be said to be **deductively equivalent** if there exist interpretations $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ and $\langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$, such that

1. $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$ is an adjoint equivalence and
2. for all $\Sigma_1 \in |\mathbf{Sign}_1|, \phi \in \mathrm{SEN}_1(\Sigma_1)$,

$$\mathrm{SEN}_1(\eta_{\Sigma_1})(\phi)^c = \beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\phi))^c \tag{iii}$$

and, for all $\Sigma_2 \in |\mathbf{Sign}_2|, \psi \in \mathrm{SEN}_2(\Sigma_2)$,

$$\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\psi)))^c = \{\psi\}^c. \tag{iv}$$

Note that, if $\mathcal{I}_1$ and $\mathcal{I}_2$ are deductively equivalent via the interpretations $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ and $\langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ and the adjoint equivalence $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$, then, for all $\Sigma_2 \in |\mathbf{Sign}_2|$ and $\psi \in \mathrm{SEN}_2(\Sigma_2)$,

$$\{\psi\}^c = \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\psi))^c), \tag{v}$$

and, for all $\Sigma_1 \in |\mathbf{Sign}_1|$ and $\phi \in \mathrm{SEN}_1(\Sigma_1)$,

$$\{\phi\}^c = \mathrm{SEN}_1(\eta_{\Sigma_1})^{-1}(\beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\phi))^c). \tag{vi}$$

In this case (v) and (vi) are equivalent to (iv) and (iii), respectively, in view of Lemma 1.5 and Corollary 1.7 and the fact that η_{Σ_1} and ϵ_{Σ_2} are isomorphisms.

Several examples of deductive equivalences are provided in [36]. They are all borrowed from considerations in abstract algebraic logic and, therefore, deal with the deductive equivalence of institutions representing logical systems and ones representing algebraic counterparts. The two prototypical examples of equational and first-order logics together with detailed accounts of their deductive equivalence with the institutions of equational algebras and first-order algebras, respectively, are presented in [37] and [38], respectively.

Lemma 1.13. *Let* $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_1|}\rangle, \mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_2|}\rangle$ *be two* π*-institutions and* $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ *an interpretation. Then*

$$\alpha_{\Sigma_1}(\Phi^c)^c = \alpha_{\Sigma_1}(\Phi)^c, \quad \textit{for all} \quad \Sigma_1 \in |\mathbf{Sign}_1|, \Phi \subseteq \mathrm{SEN}_1(\Sigma_1). \tag{vii}$$

LEMMA 1.14. *Let* $\mathcal{I}_1 = \langle \mathbf{Sign_1}, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign_1}|}\rangle, \mathcal{I}_2 = \langle \mathbf{Sign_2}, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign_2}|}\rangle$ *be two* π*-institutions such that there exist translations* $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ *and an adjunction* $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign_1} \to \mathbf{Sign_2}$*, such that, for all* $\Sigma_1 \in |\mathbf{Sign_1}|$ *and all* $\phi \in \mathrm{SEN}_1(\Sigma_1)$*,* (iii) *holds. Then*

$$\mathrm{SEN}_1(\eta_{\Sigma_1})(\Phi)^c = \beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\Phi))^c \text{ for all } \Sigma_1 \in |\mathbf{Sign_1}|, \Phi \subseteq \mathrm{SEN}_1(\Sigma_1). \tag{viii}$$

Similarly, if, for all $\Sigma_2 \in |\mathbf{Sign_2}|$ *and all* $\psi \in \mathrm{SEN}_2(\Sigma_2)$*,* (iv) *holds, then*

$$\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Psi)))^c = \Psi^c \text{ for all } \Sigma_2 \in |\mathbf{Sign_2}|, \Psi \subseteq \mathrm{SEN}_2(\Sigma_2). \tag{ix}$$

If, in the hypothesis above, instead of the equalities in (iii) *and* (iv) *only left-to-right inclusions hold, then the equalities in* (viii) *and* (ix) *should be replaced in the conclusion by left-to-right inclusions as well.*

The following constitutes one of the main theorems of [34, 35].

THEOREM 1.15. *Let* $\mathcal{I}_1 = \langle \mathbf{Sign_1}, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign_1}|}\rangle, \mathcal{I}_2 = \langle \mathbf{Sign_2}, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign_2}|}\rangle$ *be two* π*-institutions. If* $\mathcal{I}_1$ *and* $\mathcal{I}_2$ *are deductively equivalent then there exists a signature-respecting adjoint equivalence* $\langle F', G', \eta', \epsilon'\rangle : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ *that commutes with substitutions.*

For what follows it must be observed that, if $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ and $\langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ are the interpretations and $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign_1} \to \mathbf{Sign_2}$ the adjoint equivalence witnessing the deductive equivalence in the hypothesis of Theorem 1.15, then the functors $F' : \mathbf{TH}(\mathcal{I}_1) \to \mathbf{TH}(\mathcal{I}_2)$ and $G' : \mathbf{TH}(\mathcal{I}_2) \to \mathbf{TH}(\mathcal{I}_1)$ are given by

$$F'(\langle \Sigma_1, T_1\rangle) = \langle F(\Sigma_1), \alpha_{\Sigma_1}(T_1)^c\rangle \quad \text{and} \quad G'(\langle \Sigma_2, T_2\rangle) = \langle G(\Sigma_2), \beta_{\Sigma_2}(T_2)^c\rangle,$$

for all $\langle \Sigma_1, T_1\rangle \in |\mathbf{TH}(\mathcal{I}_1)|, \langle \Sigma_2, T_2\rangle \in |\mathbf{TH}(\mathcal{I}_2)|$.

A brief outline of the contents of the paper is now presented. In Section 2, the deduction-detachment property for π-institutions is introduced and it is shown that it is preserved under deductive equivalence. The disjunction property is introduced in Section 3 and is also shown to be invariant under deductive equivalence. Section 4 deals briefly with conjunction. It is defined in such a way that, owing to the fact that the sentence functors of π-institutions map into **Set**, every π-institution has conjunction. Negation is explored in Section 5. It is introduced at the π-institution level and its invariance under deductive equivalence demonstrated. Section 6 focuses on the Craig interpolation property, which was first translated to the institution framework by Tarlecki [33]. It is also preserved under deductive equivalence of π-institutions. In Sections 7 and 8 this invariance property is proved for the Robinson consistency property and the Lindenbaum property, respectively.

In a nutshell, the proofs given justify the claim that a wide variety of interesting metalogical properties, when translated from the deductive system to the institutional framework, are invariant under deductive equivalence. Thus, any of these properties that holds in all algebraic π-institutions will hold automatically for all algebraizable π-institutions in the sense of [36]. Such nicely behaved properties have been historically of central interest in algebraic logic.

2. Deduction-Detachment Property

The Deduction-Detachment property for a π-institution is now introduced and it is shown that it is invariant under deductive equivalence of π-institutions.

DEFINITION 2.16. Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|} \rangle$ be a π-institution. A natural transformation $E : \mathcal{P}\mathrm{SEN}^2 \to \mathcal{P}\mathrm{SEN}$ will be called a **Deduction-Detachment transformation** (**DDT**) for $\mathcal{I}$ if, for all $\Sigma \in |\mathbf{Sign}|, \Gamma \cup \Delta \cup \Phi \subseteq \mathrm{SEN}(\Sigma)$,

$$\Phi \subseteq (\Gamma \cup \Delta)^c \quad \text{iff} \quad E_\Sigma(\Delta, \Phi) \subseteq \Gamma^c.$$

$\mathcal{I}$ will be said to **have the Deduction-Detachment property** (**DDP**) if there exists a Deduction-Detachment transformation for $\mathcal{I}$.

THEOREM 2.17. *Let* $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_1|} \rangle, \mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_2|} \rangle$ *be two deductively equivalent* π*-institutions. Then* $\mathcal{I}_1$ *has the* DDP *if and only if* $\mathcal{I}_2$ *has the* DDP.

PROOF. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be deductively equivalent π-institutions via the interpretations $\langle F, \alpha \rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G, \beta \rangle : \mathcal{I}_2 \to \mathcal{I}_1$ and the adjoint equivalence $\langle F, G, \eta, \epsilon \rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$. Suppose $\mathcal{I}_1$ has the DDP with DDT $E : \mathcal{P}\mathrm{SEN}_1^2 \to \mathcal{P}\mathrm{SEN}_1$. Then, for all $\Sigma_2 \in |\mathbf{Sign}_2|, \Gamma \cup \Delta \cup \Phi \subseteq \mathrm{SEN}_2(\Sigma_2)$,

$$\Phi \subseteq (\Gamma \cup \Delta)^c \quad \text{iff, since } \langle G, \beta \rangle \text{ is an intrepretation,}$$

$$\beta_{\Sigma_2}(\Phi) \subseteq \beta_{\Sigma_2}(\Gamma \cup \Delta)^c \quad \text{iff}$$

$$\beta_{\Sigma_2}(\Phi) \subseteq (\beta_{\Sigma_2}(\Gamma) \cup \beta_{\Sigma_2}(\Delta))^c \quad \text{iff, since } E \text{ is a DDT for } \mathcal{I}_1,$$

$$E_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Delta), \beta_{\Sigma_2}(\Phi)) \subseteq \beta_{\Sigma_2}(\Gamma)^c \quad \text{iff, since } \langle F, \alpha\rangle \text{ is an interpretation,}$$
$$\alpha_{G(\Sigma_2)}(E_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Delta), \beta_{\Sigma_2}(\Phi))) \subseteq \alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Gamma))^c$$
$$\text{iff, since } \epsilon_{\Sigma_2} \text{ is an isomorphism,}$$
$$\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(E_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Delta), \beta_{\Sigma_2}(\Phi)))) \subseteq$$
$$\subseteq \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Gamma))^c)$$
$$\text{iff, by Lemma 1.14,} \quad \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(E_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Delta), \beta_{\Sigma_2}(\Phi)))) \subseteq \Gamma^c.$$

Let $E' : \mathcal{P}\mathrm{SEN}_2^2 \to \mathcal{P}\mathrm{SEN}_2$ be defined by

$$E'_{\Sigma_2}(\Delta, \Phi) = \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(E_{G(\Sigma_2)}(\beta^2_{\Sigma_2}(\Delta, \Phi)))),$$

for all $\Sigma_2 \in |\mathbf{Sign}_2|, \Delta, \Phi \subseteq \mathrm{SEN}_2(\Sigma_2)$. Note that $E' : \mathcal{P}\mathrm{SEN}_2^2 \to \mathcal{P}\mathrm{SEN}_2$ is a natural transformation since it is the composition of the natural transformations $\beta^2 : \mathrm{SEN}_2^2 \to \mathcal{P}\mathrm{SEN}_1^2 G, E_G : \mathcal{P}\mathrm{SEN}_1^2 G \to \mathcal{P}\mathrm{SEN}_1 G, \alpha_G : \mathrm{SEN}_1 G \to \mathcal{P}\mathrm{SEN}_2 FG$ and $\mathrm{SEN}_2(\epsilon) : \mathrm{SEN}_2 FG \to \mathrm{SEN}_2$. Thus, from what was just shown, it follows that E' is a DDT for $\mathcal{I}_2$ and, therefore, $\mathcal{I}_2$ has the DDP. The converse follows by symmetry. ■

3. Disjunction Property

The abstract property of disjunction for deductive systems in the context of abstract algebraic logic has been studied in [22] and taken up again in [21]. The property of conjunction for institutions has been introduced in [33]. Modifying this definition appropriately, an institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models\rangle$ is said to *have disjunction* if, for every signature Σ and finite set $\Phi \subseteq \mathrm{SEN}(\Sigma)$, there exists $\bigvee \Phi \in \mathrm{SEN}(\Sigma)$, such that, for every $M \in |\mathrm{MOD}(\Sigma)|, M \models_\Sigma \bigvee \Phi$ if and only if $M \models_\Sigma \phi$, for some $\phi \in \Phi$.

A somewhat nonstandard formulation of the conjunction property for a π-institution will now be given and it will be shown that it is preserved under deductive equivalence of π-institutions.

Definition 3.18. Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ be a π-institution. A natural transformation $\bigvee : \mathcal{P}\mathrm{SEN}^2 \to \mathcal{P}\mathrm{SEN}$ will be called a **disjunction for** $\mathcal{I}$ if, for all $\Sigma \in |\mathbf{Sign}|, \Phi, \Gamma, \Delta \subseteq \mathrm{SEN}(\Sigma)$,

$$(\Phi \cup \bigvee_\Sigma(\Gamma, \Delta))^c = (\Phi \cup \Gamma)^c \cap (\Phi \cup \Delta)^c.$$

$\mathcal{I}$ will be said to **have disjunction** if there exists a disjunction for $\mathcal{I}$.

A lemma is needed for the proof of our main result.

LEMMA 3.19. *Let* $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_1|}\rangle, \mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_2|}\rangle$ *be two deductively equivalent* π*-institutions via the interpretations* $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ *and* $\langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ *and the adjoint equivalence* $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$*. Then, for all* $\langle \Sigma_1, T_1\rangle, \langle \Sigma_1, T_1'\rangle \in |\mathbf{TH}(\mathcal{I}_1)|$, $\alpha_{\Sigma_1}(T_1)^c \cap \alpha_{\Sigma_1}(T_1')^c = \alpha_{\Sigma_1}(T_1 \cap T_1')^c$.

PROOF. First, note that, for all $\Sigma_1 \in |\mathbf{Sign}_1|, \langle \Sigma_1, T_1\rangle \in |\mathbf{TH}(\mathcal{I}_1)|$, we have

$$\alpha_{\Sigma_1}(T_1)^c = \{\psi \in \mathrm{SEN}_2(F(\Sigma_1)) : \beta_{F(\Sigma_1)}(\psi) \subseteq \mathrm{SEN}_1(\eta_{\Sigma_1})(T_1)\}.$$

In fact,

$$\begin{gathered}\psi \in \alpha_{\Sigma_1}(T_1)^c \quad \text{iff, since } \langle G,\beta\rangle \text{ is an interpretation,}\\ \beta_{F(\Sigma_1)}(\psi) \subseteq \beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(T_1))^c \quad \text{iff, by Lemma 1.14,}\\ \beta_{F(\Sigma_1)}(\psi) \subseteq \mathrm{SEN}_1(\eta_{\Sigma_1})(T_1).\end{gathered}$$

Thus, we have

$$\begin{aligned}\alpha_{\Sigma_1}(T_1)^c \cap \alpha_{\Sigma_1}(T_1')^c \quad &= \\ &= \{\psi \in \mathrm{SEN}_2(F(\Sigma_1)) : \beta_{F(\Sigma_1)}(\psi) \subseteq \mathrm{SEN}_1(\eta_{\Sigma_1})(T_1)\}\\ &\quad \cap\{\psi \in \mathrm{SEN}_2(F(\Sigma_1)) : \beta_{F(\Sigma_1)}(\psi) \subseteq \mathrm{SEN}_1(\eta_{\Sigma_1})(T_1')\}\\ &= \{\psi \in \mathrm{SEN}_2(F(\Sigma_1)) : \beta_{F(\Sigma_1)}(\psi) \subseteq\\ &\quad \mathrm{SEN}_1(\eta_{\Sigma_1})(T_1) \cap \mathrm{SEN}_1(\eta_{\Sigma_1})(T_1')\}\\ &= \{\psi \in \mathrm{SEN}_2(F(\Sigma_1)) : \beta_{F(\Sigma_1)}(\psi) \subseteq \mathrm{SEN}_1(\eta_{\Sigma_1})(T_1 \cap T_1')\}\\ &\quad (\text{since } \eta_{\Sigma_1} \text{ is an isomorphism})\\ &= \alpha_{\Sigma_1}(T_1 \cap T_1')^c.\end{aligned}$$

■

THEOREM 3.20. *Let* $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_1|}\rangle, \mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_2|}\rangle$ *be two deductively equivalent* π*-institutions.* $\mathcal{I}_1$ *has disjunction if and only if* $\mathcal{I}_2$ *has disjunction.*

PROOF. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be deductively equivalent π-institutions via the interpretations $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ and the adjoint equivalence $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$. Suppose that $\mathcal{I}_1$ has disjunction and let $\bigvee : \mathcal{P}\mathrm{SEN}_1^2 \to \mathcal{P}\mathrm{SEN}_1$ be a disjunction for $\mathcal{I}_1$. Then, for all $\Sigma_2 \in |\mathbf{Sign}_2|, \Phi, \Gamma, \Delta \subseteq \mathrm{SEN}_2(\Sigma_2)$,

$$(\Phi \cup \Gamma)^c \cap (\Phi \cup \Delta)^c =$$

$$
\begin{aligned}
&= \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi\cup\Gamma))^c)\cap \\
&\quad \cap\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi\cup\Delta))^c) \\
&\quad \text{(by Lemma 1.14)} \\
&= \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi\cup\Gamma))^c\cap\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi\cup\Delta))^c) \\
&\quad \text{(since } \epsilon_{\Sigma_2} \text{ is an isomorphism)} \\
&= \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi)\cup\beta_{\Sigma_2}(\Gamma))^c\cap\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi)\cup\beta_{\Sigma_2}(\Delta))^c) \\
&= \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}((\beta_{\Sigma_2}(\Phi)\cup\beta_{\Sigma_2}(\Gamma))^c)^c\cap \\
&\quad \cap\alpha_{G(\Sigma_2)}((\beta_{\Sigma_2}(\Phi)\cup\beta_{\Sigma_2}(\Delta))^c)^c) \\
&\quad \text{(by Lemma 1.13)} \\
&= \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}((\beta_{\Sigma_2}(\Phi)\cup\beta_{\Sigma_2}(\Gamma))^c\cap(\beta_{\Sigma_2}(\Phi)\cup\beta_{\Sigma_2}(\Delta))^c)^c) \\
&\quad \text{(by Lemma 3.19)} \\
&= \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}((\beta_{\Sigma_2}(\Phi)\cup{\textstyle\bigvee_{G(\Sigma_2)}}(\beta_{\Sigma_2}(\Gamma),\beta_{\Sigma_2}(\Delta)))^c)^c) \\
&\quad \text{(since } \textstyle\bigvee \text{ is a disjunction for } \mathcal{I}_1\text{)} \\
&= \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi)\cup{\textstyle\bigvee_{G(\Sigma_2)}}(\beta_{\Sigma_2}(\Gamma),\beta_{\Sigma_2}(\Delta)))^c) \\
&\quad \text{(by Lemma 1.13)} \\
&= \mathrm{SEN}_2(\epsilon_{\Sigma_2})((\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi))\cup\alpha_{G(\Sigma_2)}({\textstyle\bigvee_{G(\Sigma_2)}}(\beta_{\Sigma_2}(\Gamma),\beta_{\Sigma_2}(\Delta))))^c) \\
&= \mathrm{SEN}_2(\epsilon_{\Sigma_2})((\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi))^c\cup\alpha_{G(\Sigma_2)}({\textstyle\bigvee_{G(\Sigma_2)}}(\beta_{\Sigma_2}(\Gamma),\beta_{\Sigma_2}(\Delta))))^c) \\
&= \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi))^c\cup\alpha_{G(\Sigma_2)}({\textstyle\bigvee_{G(\Sigma_2)}}(\beta_{\Sigma_2}(\Gamma),\beta_{\Sigma_2}(\Delta))))^c \\
&\quad \text{(by Corollary 1.7 and Lemma 1.5)} \\
&= (\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi))^c)\cup \\
&\quad \cup\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}({\textstyle\bigvee_{G(\Sigma_2)}}(\beta_{\Sigma_2}(\Gamma),\beta_{\Sigma_2}(\Delta)))))^c \\
&= (\Phi^c\cup\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}({\textstyle\bigvee_{G(\Sigma_2)}}(\beta_{\Sigma_2}(\Gamma),\beta_{\Sigma_2}(\Delta)))))^c \\
&\quad \text{(by Lemma 1.14)} \\
&= (\Phi\cup\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}({\textstyle\bigvee_{G(\Sigma_2)}}(\beta_{\Sigma_2}(\Gamma),\beta_{\Sigma_2}(\Delta)))))^c.
\end{aligned}
$$

Let $\bigvee' : \mathcal{P}\mathrm{SEN}_2^2 \to \mathcal{P}\mathrm{SEN}_2$ be defined by

$$\bigvee_{\Sigma_2}{}'(\Gamma,\Delta) = \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\bigvee_{G(\Sigma_2)}(\beta^2_{\Sigma_2}(\Gamma,\Delta)))),$$

for all $\Sigma_2 \in |\mathbf{Sign}_2|, \Gamma, \Delta \subseteq \mathrm{SEN}_2(\Sigma_2)$. $\bigvee' : \mathcal{P}\mathrm{SEN}_2^2 \to \mathcal{P}\mathrm{SEN}_2$ is a natural transformation, since it is the composite of the natural transformations $\beta^2 : \mathrm{SEN}_2^2 \to \mathcal{P}\mathrm{SEN}_1^2 G, \bigvee_G : \mathcal{P}\mathrm{SEN}_1^2 G \to \mathcal{P}\mathrm{SEN}_1 G, \alpha_G : \mathrm{SEN}_1 G \to \mathcal{P}\mathrm{SEN}_2 GF$ and $\mathrm{SEN}_2(\epsilon) : \mathrm{SEN}_2 GF \to \mathrm{SEN}_2$. Since, from what was just shown, we have

$$(\Phi \cup \bigvee_{\Sigma_2}{}'(\Gamma,\Delta))^c = (\Phi\cup\Gamma)^c \cap (\Phi\cup\Delta)^c,$$

$\bigvee'$ is a disjunction for $\mathcal{I}_2$. The converse follows by symmetry. ■

4. A Note on Conjunction

By analogy with the previous section, one may attempt to define conjunction for π-institutions as follows

DEFINITION 4.21. Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ be a π-institution. A natural transformation $\bigwedge : \mathcal{P}\mathrm{SEN}^2 \to \mathcal{P}\mathrm{SEN}$ will be called a **conjunction for** $\mathcal{I}$ if, for all $\Sigma \in |\mathbf{Sign}|, \Gamma, \Delta \subseteq \mathrm{SEN}(\Sigma)$,

$$(\Gamma \cup \Delta)^c = \bigwedge_\Sigma(\Gamma, \Delta)^c.$$

$\mathcal{I}$ will be said to **have conjunction** if there exists a conjunction for $\mathcal{I}$.

The property of conjunction will now be shown to be an intrinsic property of all π-institutions, owing to the fact that their sentence functor is postulated to map into **Set**.

LEMMA 4.22. *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$, *be a* π*-institution.* $\bigwedge : \mathcal{P}\mathrm{SEN}^2 \to \mathcal{P}\mathrm{SEN}$ *with*

$$\bigwedge_\Sigma(\Gamma, \Delta) = \Gamma \cup \Delta, \quad \textit{for all} \quad \Sigma \in |\mathbf{Sign}|, \Gamma, \Delta \subseteq \mathrm{SEN}(\Sigma),$$

is a natural transformation.

PROOF. Let $f : \Sigma \to \Sigma' \in \mathrm{Mor}(\mathbf{Sign})$. We need to show that the following diagram commutes. If $\Gamma, \Delta \subseteq \mathrm{SEN}(\Sigma)$, then

$$\begin{tikzcd}
\mathcal{P}\mathrm{SEN}^2(\Sigma) \arrow[r, "\bigwedge_\Sigma"] \arrow[d, "\mathcal{P}\mathrm{SEN}^2(f)"'] & \mathcal{P}\mathrm{SEN}(\Sigma) \arrow[d, "\mathcal{P}\mathrm{SEN}(f)"] \\
\mathcal{P}\mathrm{SEN}^2(\Sigma') \arrow[r, "\bigwedge_{\Sigma'}"'] & \mathcal{P}\mathrm{SEN}(\Sigma')
\end{tikzcd}$$

$$\begin{array}{rcl}
\mathcal{P}\mathrm{SEN}(f)(\bigwedge_\Sigma(\Gamma, \Delta)) & = & \mathcal{P}\mathrm{SEN}(f)(\Gamma \cup \Delta) \\
& = & \mathcal{P}\mathrm{SEN}(f)(\Gamma) \cup \mathcal{P}\mathrm{SEN}(f)(\Delta) \\
& = & \bigwedge_{\Sigma'}(\mathcal{P}\mathrm{SEN}(f)(\Gamma), \mathcal{P}\mathrm{SEN}(f)(\Delta)) \\
& = & \bigwedge_{\Sigma'}(\mathcal{P}\mathrm{SEN}^2(f)(\Gamma, \Delta)).
\end{array}$$

■

Lemma 4.22 directly yields

THEOREM 4.23. *Every* π*-institution has conjunction.*

5. Negation

Following the same line of thought that was followed in the previous sections, the property of negation for π-institutions is now introduced.

DEFINITION 5.24. Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ be a π-institution. A natural transformation $\neg : \mathcal{P}\mathrm{SEN} \to \mathcal{P}\mathrm{SEN}$ will be called a **negation for** $\mathcal{I}$ if, for all $\Sigma \in |\mathbf{Sign}|, \Phi, \Gamma \subseteq \mathrm{SEN}(\Sigma)$,

$$\Gamma \subseteq \Phi^c \quad \text{iff} \quad (\Phi \cup \neg_\Sigma \Gamma)^c = \mathrm{SEN}(\Sigma).$$

$\mathcal{I}$ will be said to **have negation** if there exists a negation for $\mathcal{I}$.

For the proof of the main theorem a lemma is needed first.

LEMMA 5.25. *Let* $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_1|}\rangle, \mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_2|}\rangle$ *be two deductively equivalent* π*-institutions via the interpretations* $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ *and the adjoint equivalence* $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$*. Then, for every* $\Sigma_1 \in |\mathbf{Sign}_1|$ *and for every* $\Sigma_2 \in |\mathbf{Sign}_2|$,

$$\alpha_{\Sigma_1}(\mathrm{SEN}_1(\Sigma_1))^c = \mathrm{SEN}_2(F(\Sigma_1)) \quad \textit{and} \quad \beta_{\Sigma_2}(\mathrm{SEN}_2(\Sigma_2))^c = \mathrm{SEN}_1(G(\Sigma_2)).$$

PROOF. Obviously, $\alpha_{\Sigma_1}(\mathrm{SEN}_1(\Sigma_1))^c \subseteq \mathrm{SEN}_2(F(\Sigma_1))$. Suppose that

$$\alpha_{\Sigma_1}(\mathrm{SEN}_1(\Sigma_1))^c \subset \mathrm{SEN}_2(F(\Sigma_1)).$$

Then, by Theorem 1.15 and the observation following it and Lemma 1.10, we have

$$\beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\mathrm{SEN}_1(\Sigma_1)))^c \subset \beta_{F(\Sigma_1)}(\mathrm{SEN}_2(F(\Sigma_1)))^c,$$

whence, since η_{Σ_1} is an isomorphism,

$$\mathrm{SEN}_1(\eta_{\Sigma_1}^{-1})(\beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(\mathrm{SEN}_1(\Sigma_1)))^c) \subset$$

$$\subset \mathrm{SEN}_1(\eta_{\Sigma_1}^{-1})(\beta_{F(\Sigma_1)}(\mathrm{SEN}_2(F(\Sigma_1)))^c),$$

i.e., by Lemma 1.14,

$$\mathrm{SEN}_1(\Sigma_1) \subset \mathrm{SEN}_1(\eta_{\Sigma_1}^{-1})(\beta_{F(\Sigma_1)}(\mathrm{SEN}_2(F(\Sigma_1)))^c),$$

which is absurd. ∎

THEOREM 5.26. *Let* $\mathcal{I}_1 = \langle \mathbf{Sign_1}, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign_1}|}\rangle, \mathcal{I}_2 = \langle \mathbf{Sign_2}, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign_2}|}\rangle$ *be two deductively equivalent* π*-institutions.* $\mathcal{I}_1$ *has negation if and only if* $\mathcal{I}_2$ *has negation.*

PROOF. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be deductively equivalent π-institutions via the interpretations $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ and the adjoint equivalence $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$. Suppose that $\mathcal{I}_1$ has negation and let $\neg : \mathcal{P}\mathrm{SEN}_1 \to \mathcal{P}\mathrm{SEN}_1$ be a negation for $\mathcal{I}_1$. Then, for all $\Sigma_2 \in |\mathbf{Sign}_2|, \Gamma \cup \Phi \subseteq \mathrm{SEN}_2(\Sigma_2)$,

$$\Gamma \subseteq \Phi^c \quad \text{iff}$$

$$\Gamma^c \subseteq \Phi^c \quad \text{iff}$$

$$\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Gamma))^c) \subseteq \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi))^c),$$

$$\text{by Lemma 1.14, iff}$$

$$\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Gamma))^c \subseteq \alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi))^c, \quad \text{since } \epsilon_{\Sigma_2} \text{ is an iso, iff}$$

$$\beta_{\Sigma_2}(\Gamma)^c \subseteq \beta_{\Sigma_2}(\Phi)^c, \quad \text{since } \langle F, \alpha\rangle \text{ is an interpretation, iff}$$

$$(\beta_{\Sigma_2}(\Phi) \cup \neg_{G(\Sigma_2)}\beta_{\Sigma_2}(\Gamma))^c = \mathrm{SEN}_1(G(\Sigma_2)),$$

$$\text{since } \neg \text{ is a negation for } \mathcal{I}_1\text{, iff}$$

$$\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi) \cup \neg_{G(\Sigma_2)}\beta_{\Sigma_2}(\Gamma))^c = \alpha_{G(\Sigma_2)}(\mathrm{SEN}_1(G(\Sigma_2)))^c,$$

$$\text{since } \langle F, \alpha\rangle \text{ is an int., iff}$$

$$(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi)) \cup \alpha_{G(\Sigma_2)}(\neg_{G(\Sigma_2)}\beta_{\Sigma_2}(\Gamma)))^c = \mathrm{SEN}_2(F(G(\Sigma_2))),$$

$$\text{by Lemma 5.25, iff}$$

$$(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi))^c \cup \alpha_{G(\Sigma_2)}(\neg_{G(\Sigma_2)}\beta_{\Sigma_2}(\Gamma)))^c = \mathrm{SEN}_2(F(G(\Sigma_2))), \quad \text{iff}$$

$$\mathrm{SEN}_2(\epsilon_{\Sigma_2})((\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi))^c \cup \alpha_{G(\Sigma_2)}(\neg_{G(\Sigma_2)}\beta_{\Sigma_2}(\Gamma)))^c) =$$

$$= \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\mathrm{SEN}_2(F(G(\Sigma_2)))),$$

$$\text{since } \epsilon_{\Sigma_2} \text{ is an iso, iff}$$

$$\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi))^c \cup \alpha_{G(\Sigma_2)}(\neg_{G(\Sigma_2)}\beta_{\Sigma_2}(\Gamma)))^c =$$

$$= \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\mathrm{SEN}_2(F(G(\Sigma_2)))),$$

$$\text{by Corollary 1.7 and Lemma 1.5, iff}$$

$$(\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(\Phi))^c) \cup \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\neg_{G(\Sigma_2)}\beta_{\Sigma_2}(\Gamma))))^c =$$

$$= \mathrm{SEN}_2(\Sigma_2),$$

$$\text{since } \epsilon_{\Sigma_2} \text{ is an iso, iff}$$

$$(\Phi^c \cup \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\neg_{G(\Sigma_2)}\beta_{\Sigma_2}(\Gamma))))^c = \mathrm{SEN}_2(\Sigma_2),$$
$$\text{by Lemma 1.14, iff}$$
$$(\Phi \cup \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\neg_{G(\Sigma_2)}\beta_{\Sigma_2}(\Gamma))))^c = \mathrm{SEN}_2(\Sigma_2).$$

Let $\neg' : \mathcal{P}\mathrm{SEN}_2 \to \mathcal{P}\mathrm{SEN}_2$ be defined by

$$\neg'_{\Sigma_2}\Gamma = \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\neg_{G(\Sigma_2)}\beta_{\Sigma_2}(\Gamma))),$$

for all $\Sigma_2 \in |\mathbf{Sign_2}|, \Gamma \subseteq \mathrm{SEN}_2(\Sigma_2)$. $\neg' : \mathcal{P}\mathrm{SEN}_2 \to \mathcal{P}\mathrm{SEN}_2$ is a natural transformation, since it is the composite of natural transformations. Thus, from what was just shown, we have

$$\Gamma \subseteq \Phi^c \quad \text{iff} \quad (\Phi \cup \neg'_{\Sigma_2}\Gamma)^c = \mathrm{SEN}_2(\Sigma_2),$$

i.e., $\neg'$ is a negation for $\mathcal{I}_2$. The converse follows by symmetry. ■

6. Craig Interpolation

Tarlecki [33] introduced and studied the Craig Interpolation Theorem for institutions.

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models \rangle$ be an institution and the following

$$\begin{tikzcd} \Sigma \arrow[r, "f'"] \arrow[d, "f''"'] & \Sigma' \arrow[d, "g'"] \\ \Sigma'' \arrow[r, "g''"'] & \Sigma''' \end{tikzcd}$$

a pushout diagram in **Sign**. According to [33], $\mathcal{I}$ is said to satisfy the *Craig Interpolation Theorem* if, for all $\phi' \in \mathrm{SEN}(\Sigma'), \phi'' \in \mathrm{SEN}(\Sigma'')$, with $\mathrm{SEN}(g')(\phi') \models \mathrm{SEN}(g'')(\phi'')$, there exists $\phi \in \mathrm{SEN}(\Sigma)$, such that $\phi' \models \mathrm{SEN}(f')(\phi)$ and $\mathrm{SEN}(f'')(\phi) \models \phi''$.

Modifying slightly Tarlecki's definition the following is obtained.

Definition 6.27. Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|} \rangle$ be a π-institution. $\mathcal{I}$ is said to have the **Craig Interpolation Property** (**CIP**) if, for all $\Sigma, \Sigma', \Sigma'' \in |\mathbf{Sign}|$ and pushout diagram

$$\begin{tikzcd} \Sigma \arrow[r, "f'"] \arrow[d, "f''"'] & \Sigma' \arrow[d, "g'"] \\ \Sigma'' \arrow[r, "g''"'] & \Sigma''' \end{tikzcd}$$

we have that, for all $\Phi' \subseteq \mathrm{SEN}(\Sigma'), \Phi'' \subseteq \mathrm{SEN}(\Sigma'')$, with $\mathrm{SEN}(g'')(\Phi'') \subseteq \mathrm{SEN}(g')(\Phi')^c$, there exists $\Phi \subseteq \mathrm{SEN}(\Sigma)$, such that $\mathrm{SEN}(f')(\Phi) \subseteq {\Phi'}^c$ and $\Phi'' \subseteq \mathrm{SEN}(f'')(\Phi)^c$.

Theorem 6.28. *Let* $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_1|}\rangle, \mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_2|}\rangle$ *be two deductively equivalent* π*-institutions.* $\mathcal{I}_1$ *has the* CIP *if and only if* $\mathcal{I}_2$ *has the* CIP.

Proof. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be deductively equivalent π-institutions via the interpretations $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ and the adjoint equivalence $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$. Suppose that $\mathcal{I}_1$ has the CIP and assume that

$$\begin{tikzcd} \Sigma_2 \arrow[r, "f_2'"] \arrow[d, "f_2''"'] & \Sigma_2' \arrow[d, "g_2'"] \\ \Sigma_2'' \arrow[r, "g_2''"'] & \Sigma_2''' \end{tikzcd}$$

is a pushout diagram in $\mathbf{Sign}_2$ and $\Phi_2' \subseteq \mathrm{SEN}_2(\Sigma_2'), \Phi_2'' \subseteq \mathrm{SEN}_2(\Sigma_2'')$, with

$$\mathrm{SEN}_2(g_2'')(\Phi_2'') \subseteq \mathrm{SEN}_2(g_2')(\Phi_2')^c.$$

Since left adjoints preserve colimits, the following is, then, a pushout diagram in $\mathbf{Sign}_1$.

$$\begin{tikzcd} G(\Sigma_2) \arrow[r, "G(f_2')"] \arrow[d, "G(f_2'')"'] & G(\Sigma_2') \arrow[d, "G(g_2')"] \\ G(\Sigma_2'') \arrow[r, "G(g_2'')"'] & G(\Sigma_2''') \end{tikzcd}$$

Moreover, since $\langle G, \beta\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ is an interpretation, we have

$$\beta_{\Sigma_2'''}(\mathrm{SEN}_2(g_2'')(\Phi_2'')) \subseteq \beta_{\Sigma_2'''}(\mathrm{SEN}_2(g_2')(\Phi_2'))^c.$$

Since β is a natural transformation,

$$\begin{tikzcd} \mathrm{SEN}_2(\Sigma_2') \arrow[r, "\beta_{\Sigma_2'}"] \arrow[d, "\mathrm{SEN}_2(g_2')"'] & \mathcal{P}\mathrm{SEN}_1(G(\Sigma_2')) \arrow[d, "\mathcal{P}\mathrm{SEN}_1(G(g_2'))"] \\ \mathrm{SEN}_2(\Sigma_2''') \arrow[r, "\beta_{\Sigma_2'''}"'] & \mathcal{P}\mathrm{SEN}_1(G(\Sigma_2''')) \end{tikzcd}$$

$$\begin{array}{ccc}\mathrm{SEN}_2(\Sigma_2'') & \xrightarrow{\beta_{\Sigma_2''}} & \mathcal{P}\mathrm{SEN}_1(G(\Sigma_2''))\\ {\scriptstyle \mathrm{SEN}_2(g_2'')}\downarrow & & \downarrow{\scriptstyle \mathcal{P}\mathrm{SEN}_1(G(g_2''))}\\ \mathrm{SEN}_2(\Sigma_2''') & \xrightarrow[\beta_{\Sigma_2'''}]{} & \mathcal{P}\mathrm{SEN}_1(G(\Sigma_2'''))\end{array}$$

we obtain

$$\mathrm{SEN}_1(G(g_2''))(\beta_{\Sigma_2''}(\Phi_2'')) \subseteq \mathrm{SEN}_1(G(g_2'))(\beta_{\Sigma_2'}(\Phi_2'))^c.$$

Since $\mathcal{I}_1$ has the CIP, there exists $\Phi_1 \subseteq \mathrm{SEN}_1(G(\Sigma_2))$, such that

$$\mathrm{SEN}_1(G(f_2'))(\Phi_1) \subseteq \beta_{\Sigma_2'}(\Phi_2')^c \quad \text{and} \quad \beta_{\Sigma_2''}(\Phi_2'') \subseteq \mathrm{SEN}_1(G(f_2''))(\Phi_1)^c.$$

Thus, since $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2$ is an interpretation,

$$\alpha_{G(\Sigma_2')}(\mathrm{SEN}_1(G(f_2'))(\Phi_1)) \subseteq \alpha_{G(\Sigma_2')}(\beta_{\Sigma_2'}(\Phi_2'))^c$$

and

$$\alpha_{G(\Sigma_2'')}(\beta_{\Sigma_2''}(\Phi_2'')) \subseteq \alpha_{G(\Sigma_2'')}(\mathrm{SEN}_1(G(f_2''))(\Phi_1))^c$$

and, since α is a natural transformation,

$$\begin{array}{ccc}\mathrm{SEN}_1(G(\Sigma_2)) & \xrightarrow{\alpha_{G(\Sigma_2)}} & \mathcal{P}\mathrm{SEN}_2(F(G(\Sigma_2)))\\ {\scriptstyle \mathrm{SEN}_1(G(f_2'))}\downarrow & & \downarrow{\scriptstyle \mathcal{P}\mathrm{SEN}_2(F(G(f_2')))}\\ \mathrm{SEN}_1(G(\Sigma_2')) & \xrightarrow[\alpha_{G(\Sigma_2')}]{} & \mathcal{P}\mathrm{SEN}_2(F(G(\Sigma_2')))\end{array}$$

$$\begin{array}{ccc}\mathrm{SEN}_1(G(\Sigma_2)) & \xrightarrow{\alpha_{G(\Sigma_2)}} & \mathcal{P}\mathrm{SEN}_2(F(G(\Sigma_2)))\\ {\scriptstyle \mathrm{SEN}_1(G(f_2''))}\downarrow & & \downarrow{\scriptstyle \mathcal{P}\mathrm{SEN}_2(F(G(f_2'')))}\\ \mathrm{SEN}_1(G(\Sigma_2'')) & \xrightarrow[\alpha_{G(\Sigma_2'')}]{} & \mathcal{P}\mathrm{SEN}_2(F(G(\Sigma_2'')))\end{array}$$

we obtain

$$\mathrm{SEN}_2(F(G(f_2')))(\alpha_{G(\Sigma_2)}(\Phi_1)) \subseteq \alpha_{G(\Sigma_2')}(\beta_{\Sigma_2'}(\Phi_2'))^c \quad \text{and}$$

$$\alpha_{G(\Sigma_2'')}(\beta_{\Sigma_2''}(\Phi_2'')) \subseteq \mathrm{SEN}_2(F(G(f_2'')))(\alpha_{G(\Sigma_2)}(\Phi_1))^c. \quad \text{Hence,}$$

$$\mathrm{SEN}_2(\epsilon_{\Sigma_2'})(\mathrm{SEN}_2(F(G(f_2')))(\alpha_{G(\Sigma_2)}(\Phi_1))) \subseteq \mathrm{SEN}_2(\epsilon_{\Sigma_2'})(\alpha_{G(\Sigma_2')}(\beta_{\Sigma_2'}(\Phi_2'))^c)$$

and

$$\mathrm{SEN}_2(\epsilon_{\Sigma_2''})(\alpha_{G(\Sigma_2'')}(\beta_{\Sigma_2''}(\Phi_2''))^c) \subseteq$$

$$\subseteq \mathrm{SEN}_2(\epsilon_{\Sigma''_2})(\mathrm{SEN}_2(F(G(f''_2)))(\alpha_{G(\Sigma_2)}(\Phi_1))^c),$$

i.e., by Lemma 1.14, $\mathrm{SEN}_2(\epsilon_{\Sigma'_2})(\mathrm{SEN}_2(F(G(f'_2)))(\alpha_{G(\Sigma_2)}(\Phi_1))) \subseteq \Phi'^c_2$

and

$$\Phi''_2 \subseteq \mathrm{SEN}_2(\epsilon_{\Sigma''_2})(\mathrm{SEN}_2(F(G(f''_2)))(\alpha_{G(\Sigma_2)}(\Phi_1))^c).$$

Thus,

$$\begin{tikzcd} F(G(\Sigma_2)) \arrow[r, "\epsilon_{\Sigma_2}"] \arrow[d, "F(G(f'_2))"'] & \Sigma_2 \arrow[d, "f'_2"] \\ F(G(\Sigma'_2)) \arrow[r, "\epsilon_{\Sigma'_2}"'] & \Sigma'_2 \end{tikzcd} \qquad \begin{tikzcd} F(G(\Sigma_2)) \arrow[r, "\epsilon_{\Sigma_2}"] \arrow[d, "F(G(f''_2))"'] & \Sigma_2 \arrow[d, "f''_2"] \\ F(G(\Sigma''_2)) \arrow[r, "\epsilon_{\Sigma''_2}"'] & \Sigma''_2 \end{tikzcd}$$

$$\mathrm{SEN}_2(f'_2\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\Phi_1)) \subseteq \Phi'^c_2 \quad \text{and} \quad \Phi''_2 \subseteq \mathrm{SEN}_2(f''_2\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\Phi_1))^c$$

and, therefore,

$$\mathrm{SEN}_2(f'_2)(\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\Phi_1))) \subseteq \Phi'^c_2$$

$$\text{and} \quad \Phi''_2 \subseteq \mathrm{SEN}_2(f''_2)(\mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(\Phi_1)))^c.$$

Thus, $\mathcal{I}_2$ has the CIP. The converse follows by symmetry. ■

7. Robinson Consistency

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ be a π-institution and $\Sigma \in |\mathbf{Sign}|$. Recall that a theory $\langle \Sigma, T\rangle \in |\mathbf{TH}(\mathcal{I})|$ is said to be **consistent** if $T \neq \mathrm{SEN}(\Sigma)$ and **complete** if, for every $\langle \Sigma, T'\rangle \in |\mathbf{TH}(\mathcal{I})|, T \subset T'$ implies $T' = \mathrm{SEN}(\Sigma)$.

Definition 7.29. Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ be a π-institution. $\mathcal{I}$ will be said to **have the Robinson Consistency Property** (**RCP**) if, for every consistent complete theory $\langle \Sigma, T\rangle$ and consistent theories $\langle \Sigma', T'\rangle, \langle \Sigma'', T''\rangle$, such that $f' : \langle \Sigma, T\rangle \to \langle \Sigma', T'\rangle, f'' : \langle \Sigma, T\rangle \to \langle \Sigma'', T''\rangle \in \mathrm{Mor}(\mathbf{TH}(\mathcal{I}))$, the theory $\langle \Sigma''', (\mathrm{SEN}(g')(T') \cup \mathrm{SEN}(g'')(T''))^c\rangle$ is consistent, where as before, the following diagram

$$\begin{tikzcd} \Sigma \arrow[r, "f'"] \arrow[d, "f''"'] & \Sigma' \arrow[d, "g'"] \\ \Sigma'' \arrow[r, "g''"'] & \Sigma''' \end{tikzcd}$$

is a pushout in **Sign**.

Before presenting our main result, a lemma is needed.

LEMMA 7.30. *Let* $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_1|}\rangle, \mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_2|}\rangle$ *be two deductively equivalent* π*-institutions via the interpretations* $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ *and the adjoint equivalence* $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$. *Then, for every* $\langle \Sigma_1, T_1\rangle \in |\mathbf{TH}(\mathcal{I}_1)|$, *if* $\langle \Sigma_1, T_1\rangle$ *is consistent, then so is* $\langle F(\Sigma_1), \alpha_{\Sigma_1}(T_1)^c\rangle$ *and if* $\langle \Sigma_1, T_1\rangle$ *is complete, then so is* $\langle F(\Sigma_1), \alpha_{\Sigma_1}(T_1)^c\rangle$.

PROOF. Suppose that $\langle \Sigma_1, T_1\rangle$ is consistent, i.e., that $T_1 \neq \mathrm{SEN}_1(\Sigma_1)$ and assume, to the contrary, that $\alpha_{\Sigma_1}(T_1)^c = \mathrm{SEN}_2(F(\Sigma_1))$. By Lemma 5.25, $\alpha_{\Sigma_1}(\mathrm{SEN}_1(\Sigma_1))^c = \mathrm{SEN}_2(F(\Sigma_1))$, whence $\alpha_{\Sigma_1}(T_1)^c = \alpha_{\Sigma_1}(\mathrm{SEN}_1(\Sigma_1))^c$. But this contradicts Theorem 1.15 and Lemma 1.10.

Next, suppose that $\langle \Sigma_1, T_1\rangle$ is complete, i.e., that, for every $\langle \Sigma_1, T_1'\rangle$, with $T_1 \subset T_1'$, we have $T_1' = \mathrm{SEN}_1(\Sigma_1)$. Suppose to the contrary, that $\langle F(\Sigma_1), \alpha_{\Sigma_1}(T_1)^c\rangle$ is not complete, i.e., that there exists $\langle F(\Sigma_1), T_2\rangle$, such that $\alpha_{\Sigma_1}(T_1)^c \subset T_2$, but $T_2 \neq \mathrm{SEN}_2(F(\Sigma_1))$. Then

$$\mathrm{SEN}_1(\eta_{\Sigma_1}^{-1})(\beta_{F(\Sigma_1)}(\alpha_{\Sigma_1}(T_1))^c) \subset \mathrm{SEN}_1(\eta_{\Sigma_1}^{-1})(\beta_{F(\Sigma_1)}(T_2)^c), \quad \text{i.e.,}$$

$$T_1 \subset \mathrm{SEN}_1(\eta_{\Sigma_1}^{-1})(\beta_{F(\Sigma_1)}(T_2)^c),$$

with $\mathrm{SEN}_1(\eta_{\Sigma_1}^{-1})(\beta_{F(\Sigma_1)}(T_2)^c) \neq \mathrm{SEN}_1(\Sigma_1)$, which contradicts our hypothesis. ∎

THEOREM 7.31. *Let* $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_1|}\rangle, \mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_2|}\rangle$ *be two deductively equivalent* π*-institutions.* $\mathcal{I}_1$ *has the* RCP *if and only if* $\mathcal{I}_2$ *has the* RCP.

PROOF. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be deductively equivalent π-institutions via the interpretations $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ and the adjoint equivalence $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$. Suppose that $\mathcal{I}_1$ has the RCP and assume that

$$\begin{tikzcd}
\Sigma_2 \arrow[r, "f_2'"] \arrow[d, "f_2''"'] & \Sigma_2' \arrow[d, "g_2'"] \\
\Sigma_2'' \arrow[r, "g_2''"'] & \Sigma_2'''
\end{tikzcd}$$

is a pushout diagram in $\mathbf{Sign}_2$ and that $\langle \Sigma_2, T_2\rangle$ is a consistent complete theory and $\langle \Sigma_2', T_2'\rangle, \langle \Sigma_2'', T_2''\rangle$ are consistent theories in $|\mathbf{TH}(\mathcal{I}_2)|$, such that

$f_2' : \langle \Sigma_2, T_2\rangle \to \langle \Sigma_2', T_2'\rangle, f_2'' : \langle \Sigma_2, T_2\rangle \to \langle \Sigma_2'', T_2''\rangle \in \mathrm{Mor}(\mathbf{TH}(\mathcal{I}_2))$. Since left adjoints preserve colimits, the following diagram

$$\begin{tikzcd} G(\Sigma_2) \arrow[r, "G(f_2')"] \arrow[d, "G(f_2'')"'] & G(\Sigma_2') \arrow[d, "G(g_2')"] \\ G(\Sigma_2'') \arrow[r, "G(g_2'')"'] & G(\Sigma_2''') \end{tikzcd}$$

is a pushout diagram in $\mathbf{Sign}_1$.

Consider the theories $\langle G(\Sigma_2), \beta_{\Sigma_2}(T_2)^c\rangle, \langle G(\Sigma_2'), \beta_{\Sigma_2'}(T_2')^c\rangle$ and $\langle G(\Sigma_2''), \beta_{\Sigma_2''}(T_2'')^c\rangle$ in $\mathbf{TH}(\mathcal{I}_1)$. By Lemma 7.30, $\langle G(\Sigma_2), \beta_{\Sigma_2}(T_2)^c\rangle$ is consistent and complete and

$$\langle G(\Sigma_2'), \beta_{\Sigma_2'}(T_2')^c\rangle, \langle G(\Sigma_2''), \beta_{\Sigma_2''}(T_2'')^c\rangle$$

are consistent. Moreover $G(f_2') : \langle G(\Sigma_2), \beta_{\Sigma_2}(T_2)^c\rangle \to \langle G(\Sigma_2'), \beta_{\Sigma_2'}(T_2')^c\rangle$ and $G(f_2'') : \langle G(\Sigma_2), \beta_{\Sigma_2}(T_2)^c\rangle \to \langle G(\Sigma_2''), \beta_{\Sigma_2''}(T_2'')^c\rangle$ are theory morphisms. Hence, since $\mathcal{I}_1$ has the RCP, the theory

$$\langle G(\Sigma_2'''), (\mathrm{SEN}_1(G(g_2'))(\beta_{\Sigma_2'}(T_2')) \cup \mathrm{SEN}_1(G(g_2''))(\beta_{\Sigma_2''}(T_2'')))^c\rangle$$

is a consistent theory in $\mathbf{TH}(\mathcal{I}_1)$. This theory is the same as

$$\begin{tikzcd} \mathrm{SEN}_2(\Sigma_2') \arrow[r, "\beta_{\Sigma_2'}"] \arrow[d, "\mathrm{SEN}_2(g_2')"'] & \mathcal{P}\mathrm{SEN}_1(G(\Sigma_2')) \arrow[d, "\mathcal{P}\mathrm{SEN}_1(G(g_2'))"] \\ \mathrm{SEN}_2(\Sigma_2''') \arrow[r, "\beta_{\Sigma_2'''}"'] & \mathcal{P}\mathrm{SEN}_1(G(\Sigma_2''')) \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{SEN}_2(\Sigma_2'') \arrow[r, "\beta_{\Sigma_2''}"] \arrow[d, "\mathrm{SEN}_2(g_2'')"'] & \mathcal{P}\mathrm{SEN}_1(G(\Sigma_2'')) \arrow[d, "\mathcal{P}\mathrm{SEN}_1(G(g_2''))"] \\ \mathrm{SEN}_2(\Sigma_2''') \arrow[r, "\beta_{\Sigma_2'''}"'] & \mathcal{P}\mathrm{SEN}_1(G(\Sigma_2''')) \end{tikzcd}$$

$$\langle G(\Sigma_2'''), (\beta_{\Sigma_2'''}(\mathrm{SEN}_2(g_2')(T_2')) \cup \beta_{\Sigma_2'''}(\mathrm{SEN}_2(g_2'')(T_2'')))^c\rangle$$

$$\text{i.e.,}\quad \langle G(\Sigma_2'''), \beta_{\Sigma_2'''}(\mathrm{SEN}_2(g_2')(T_2') \cup \mathrm{SEN}_2(g_2'')(T_2''))^c\rangle.$$

Consistency of this theory implies, by Lemma 7.30, consistency of

$$\langle F(G(\Sigma_2''')), \alpha_{G(\Sigma_2''')}(\beta_{\Sigma_2'''}(\mathrm{SEN}_2(g_2')(T_2') \cup \mathrm{SEN}_2(g_2'')(T_2'')))^c\rangle$$

and, therefore, since $\epsilon_{\Sigma_2'''}$ is an isomorphism, of

$$\langle \Sigma_2''', (\mathrm{SEN}_2(g_2')(T_2') \cup \mathrm{SEN}_2(g_2'')(T_2''))^c\rangle.$$

Thus, $\mathcal{I}_2$ has the RCP. The converse follows by symmetry. ∎

8. The Lindenbaum Property

DEFINITION 8.32. Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}\rangle$ be a π-institution. $\mathcal{I}$ will be said to **have the Lindenbaum Property** (**LP**) if, for all $\Sigma \in |\mathbf{Sign}|$, $\langle \Sigma, T\rangle \in |\mathbf{TH}(\mathcal{I})|$, if $\langle \Sigma, T\rangle$ is consistent, then there exists a consistent, complete theory $\langle \Sigma, T'\rangle$, such that $T \subseteq T'$.

THEOREM 8.33. *Let* $\mathcal{I}_1 = \langle \mathbf{Sign}_1, \mathrm{SEN}_1, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_1|}\rangle, \mathcal{I}_2 = \langle \mathbf{Sign}_2, \mathrm{SEN}_2, \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}_2|}\rangle$ *be two deductively equivalent* π*-institutions.* $\mathcal{I}_1$ *has the* LP *if and only if* $\mathcal{I}_2$ *has the* LP*.*

PROOF. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be deductively equivalent π-institutions via the interpretations $\langle F, \alpha\rangle : \mathcal{I}_1 \to \mathcal{I}_2, \langle G, \beta\rangle : \mathcal{I}_2 \to \mathcal{I}_1$ and the adjoint equivalence $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$. Suppose that $\mathcal{I}_1$ has the LP and let $\Sigma_2 \in |\mathbf{Sign}_2|, \langle \Sigma_2, T_2\rangle \in |\mathbf{TH}(\mathcal{I}_2)|$ a consistent theory. By Lemma 7.30, $\langle G(\Sigma_2), \beta_{\Sigma_2}(T_2)^c\rangle$ is a consistent theory in $\mathbf{TH}(\mathcal{I}_1)$. Thus, since $\mathcal{I}_1$ has the LP, there exists a consistent, complete theory $\langle G(\Sigma_2), T_1\rangle$, such that $\beta_{\Sigma_2}(T_2)^c \subseteq T_1$. But then, by Lemma 7.30, $\langle F(G(\Sigma_2)), \alpha_{G(\Sigma_2)}(T_1)^c\rangle$ is a consistent, complete theory of $\mathcal{I}_2$, such that $\alpha_{G(\Sigma_2)}(\beta_{\Sigma_2}(T_2))^c \subseteq \alpha_{G(\Sigma_2)}(T_1)^c$, whence $\langle \Sigma_2, \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(T_1))^c\rangle$ is a consistent, complete theory of $\mathbf{TH}(\mathcal{I}_2)$, such that $T_2 \subseteq \mathrm{SEN}_2(\epsilon_{\Sigma_2})(\alpha_{G(\Sigma_2)}(T_1))^c$. Hence, $\mathcal{I}_2$ has the LP. The converse follows by symmetry. ∎

Acknowledgements

It is beyond words to describe the effect of two persons' encouragement and support during the last few years: **Don Pigozzi** has been there when I needed him the most (and not only) for the last decade. **Charles Wells** has provided advice and new sources of knowledge and insights for the last four years.

References

[1] Bagchi, A., and Wells, C., *Graph-based Logic and Sketches I: The General Framework,* Preprint

[2] Bagchi, A., and Wells, C., *Graph-based Logic and Sketches II: Finite Product Categories and Equational Logic,* Preprint

[3] Barbour, G., and Raftery, J., *Unifying Algebraizability and Full Congruence Regularity (Abstract),* Workshop on Abstract Algebraic Logic, Centre de Recerca Matemàtica, Quaderns, Núm. 10/Gener 1998

[4] Barr, M., and Wells, C., *Toposes, Triples and Theories,* Springer-Verlag, New York 1985

[5] Barr, M., and Wells, C., *Category Theory for Computing Science, Third Edition,* Les Publications CRM, Montréal, 1999

[6] Bastiani, A., and Ehresmann, C., *Categories of Sketched Structures,* Cahiers de Topologie et Géométrie Différentielle, Vol. XIII (1972), No. 2, pp. 105-212

[7] Blok, W.J., and Pigozzi, D., *Protoalgebraic Logics,* Studia Logica, Vol. 45 (1986), pp. 337-369

[8] Blok, W.J., and Pigozzi, D., *Algebraizable Logics,* Memoirs of the American Mathematical Society, Vol. 77, No. 396 (1989)

[9] Blok, W.J., and Pigozzi, D., *Algebraic Semantics for Universal Horn Logic Without Equality,* in Universal Algebra and Quasigroup Theory, A. Romanowska and J.D.H. Smith, Eds., Heldermann Verlag, Berlin 1992

[10] Blok, W.J., and Pigozzi, D., *Abstract Algebraic Logic and the Deduction Theorem*, to appear in the Bulletin of Symbolic Logic

[11] Coppey, L., and Lair, C., *Leçons de Théorie des Esquisses, Partie I,* Diagrammes, Vol. 12 (1984), pp. I-X, 1-38

[12] Coppey, L., and Lair, C., *Leçons de Théorie des Esquisses, Partie II,* Diagrammes, Vol. 19 (1988), pp. 1-67

[13] Czelakowski, J., *Equivalential Logics I,II,* Studia Logica, Vol. 40 (1981), pp. 227-236, 355-372

[14] Czelakowski, J., *Logical Matrices and the Amalgamation Property,* Studia Logica, Vol. 41 (1982), No. 4, pp. 329-341

[15] Czelakowski, J., *Sentential Logics and Maehara Interpolation Property,* Studia Logica, Vol. 44 (1985), No. 3, pp. 265-283

[16] Czelakowski, J., *Local Deduction Theorems,* Studia Logica, Vol. 45 (1986), No. 4, pp. 377-391

[17] Czelakowski, J., *Consequence Operations Foundational Studies,* Reports of the Research Project *Theories, Models, Cognitive Schemata,* Polish Academy of Science 1992 (Preprint).

[18] Czelakowski, J., *Protoalgebraic Logics,* Studia Logica Library 10, Kluwer, Dordrecht, 2001

[19] Czelakowski, J., and Pigozzi, D., *Amalgamation and Interpolation in Abstract Algebraic Logic. Models, Algebras and Proofs,* in Lecture Notes in Pure and Applied Mathematics, Vol. 203, Dekker, New York 1999, pp. 187-265

[20] Fiadeiro, J., and Sernadas, A., *Structuring Theories on Consequence,* in Recent Trends in Data Type Specification, Donald Sannella and Andrzej Tarlecki, Eds., Lecture Notes in Computer Science, Vol. 332, Springer-Verlag, New York 1988, pp. 44-72

[21] Font, J.M., and Jansana, R., *A General Algebraic Semantics for Sentential Logics,* Lecture Notes in Logic, Vol. 7 (1996), Springer-Verlag, Berlin Heidelberg 1996

[22] Font, J.M., and Verdu, V., *Algebraic Logic for Classical Conjunction and Disjunction,* Studia Logica, Vol. 50 (1991), No. 3-4, pp. 391-419

[23] Goguen, J.A., and Burstall, R.M., *Introducing Institutions,* in Proceedings of the Logic of Programming Workshop, E. Clarke and D. Kozen, Eds., Lecture Notes in Computer Science, Vol. 164, Springer-Verlag, New York 1984, pp.221-256

[24] Goguen, J.A., and Burstall, R.M., *Institutions: Abstract Model Theory for Specification and Programming,* Journal of the Association for Computing Machinery, Vol. 39, No. 1 (1992), pp. 95-146

[25] Herrmann, B., *Equivalential Logics and Definability of Truth*, Dissertation, Freie Universitat Berlin, Berlin 1993

[26] Herrmann, B., *Equivalential and Algebraizable Logics,* Studia Logica, Vol. 57 (1996), pp. 419-436

[27] Herrmann, B., *Characterizing Equivalential and Algebraizable Logics by the Leibniz Operator,* Studia Logica, Vol. 58 (1997), pp. 305-323

[28] Hopcroft, J.E., and Ullman, J.D., *Introduction to Automata Theory, Languages and Computation, 2nd Edition,* Addison-Wesley, Reading, Massachussetts, 1979

[29] Lewis, H.R., and Papadimitriou, C., *Elements of the Theory of Computation,* Prentice Hall, Englewood Cliffs, New Jersey, 1981

[30] Mac Lane, S., *Category Theory for the Working Mathematician,* Springer-Verlag, New York 1971

[31] Martin, J.C., *Introduction to Languages and the Theory of Computation,* McGraw-Hill, New-York, 1991

[32] Pynko, A.P., *Definitional Equivalence and Algebraizability of Generalized Logical Systems,* Annals of Pure and Applied Logic, Vol. 98 (1999), pp. 1-68

[33] Tarlecki, A., *Bits and Pieces of the Theory of Institutions,* Category Theory and Computer Programming (Guildford, 1985), Lecture Notes in Computer Science, Vol. 240 (1986), pp. 334-363

[34] Voutsadakis, G., *Categorical Abstract Algebraic Logic,* Doctoral Dissertation, Iowa State University, Ames, Iowa 1998

[35] Voutsadakis, G., *Categorical Abstract Algebraic Logic: Equivalent Institutions,* (To appear in Studia Logica)

[36] Voutsadakis, G., *Categorical Abstract Algebraic Logic: Algebraizable Institutions,* (To appear in Applied Categorical Structures)

[37] Voutsadakis, G., *Categorical Abstract Algebraic Logic: Categorical Algebraization of Equational Logic,* (Submitted to the Logic Journal of the IGPL)

[38] Voutsadakis, G., *Categorical Abstract Algebraic Logic: Categorical Algebraization of First-Order Logic Without Terms,* (Submitted to the Archive for Mathematical Logic)

George Voutsadakis
School of Mathematics and Computer Science
Lake Superior State University
Sault Sainte Marie, MI 49783
USA
gvoutsad@gw.lssu.edu
http://pigozzi.lssu.edu/