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Categorical Abstract Algebraic Logic: Referential Algebraic Semantics

Abstract. Wójcicki has provided a characterization of selfextensional logics as those that can be endowed with a complete local referential semantics. His result was extended by Jansana and Palmigiano, who developed a duality between the category of reduced congruential atlases and that of reduced referential algebras over a fixed similarity type. This duality restricts to one between reduced atlas models and reduced referential algebra models of selfextensional logics. In this paper referential algebraic systems and congruential atlas systems are introduced, which abstract referential algebras and congruential atlases, respectively. This enables the formulation of an analog of Wójcicki's Theorem for logics formalized as π-institutions. Moreover, the results of Jansana and Palmigiano are generalized to obtain a duality between congruential atlas systems and referential algebraic systems over a fixed categorical algebraic signature. In future work, the duality obtained in this paper will be used to obtain one between atlas system models and referential algebraic system models of an arbitrary selfextensional π-institution. Using this latter duality, the characterization of fully selfextensional deductive systems among the selfextensional ones, that was obtained by Jansana and Palmigiano, can be extended to a similar characterization of fully selfextensional π-institutions among appropriately chosen classes of selfextensional ones.

Keywords: Abstract Algebraic Logic, Referential Algebraic Semantics, Wójcicki's Theorem, Generalized Matrix Model, Atlas, Algebraic Semantics, Duality, π-Institution, Selfextensional Logic, Fully Selfextensional Logic.

1. Introduction

In Abstract Algebraic Logic (AAL) [4, 8, 13, 14] the basic objects of study are *sentential logics* (or simply *logics*, also referred to as *deductive systems*), i.e., pairs of the form $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$, where $\mathcal{L}$ is a language type (a set of logical connectives or operation symbols of finite arities) and $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)) \times \mathrm{Fm}_{\mathcal{L}}(V)$ is a structural consequence relation on the set of $\mathcal{L}$-formulas $\mathrm{Fm}_{\mathcal{L}}(V)$ formed in the ordinary recursive way starting from the variables in a fixed denumerable set V and using the connectives in $\mathcal{L}$.

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On the other hand, in Categorical Abstract Algebraic Logic (CAAL) [21, 20], the basic objects of study are π-institutions [11] (see, also, [16, 17]), which are triples of the form $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, where **Sign** is an arbitrary category, $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is a set-valued functor and $C = \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ is a collection of closure operators $C_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \to \mathcal{P}(\mathrm{SEN}(\Sigma))$, such that, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$ and all $X \subseteq \mathrm{SEN}(\Sigma_1)$,

$$\mathrm{SEN}(f)(C_{\Sigma_1}(X)) \subseteq C_{\Sigma_2}(\mathrm{SEN}(f)(X)). \tag{1}$$

Due to the fact that the morphisms in the category **Sign** may be taken as abstractions of the operations of substituting formulas for variables in the AAL-context, Relation (1) is sometimes seen as abstracting the structurality of $\vdash_{\mathcal{S}}$ in the categorical context. To make this idea more precise, we provide, next, an example that illustrates how a sentential logic $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}}\rangle$ may be formalized as a π-institution $\mathcal{I}_{\mathcal{S}}$. This example, besides illustrating the notion of π-institution, which may be unfamiliar to some readers, also provides a first indication that sentential logics may be captured as rather trivial examples of π-institutions in a variety of possible ways. The additional expressive power availed by the structure of a π-institution provides room for the formalization of logical systems with multiple signatures and quantifiers and also some logical systems whose formulas are not string-based. Some examples of such higher-level logical systems may be found, e.g., in [9] and in the CAAL context in [21, 20].

Let $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}}\rangle$ be a sentential logic. Define $\mathcal{I}_{\mathcal{S}} = \langle \mathbf{Sign}_{\mathcal{L}}, \mathrm{SEN}_{\mathcal{L}}, C_{\mathcal{S}}\rangle$ (the signature category and the sentence functor depend only on $\mathcal{L}$) as follows:

- $\mathbf{Sign}_{\mathcal{L}}$ consists of a single element category, with element, say, V, and $\mathbf{Sign}_{\mathcal{L}}(V, V) = \mathrm{End}(\mathbf{Fm}_{\mathcal{L}}(V))$, where $\mathbf{Fm}_{\mathcal{L}}(V)$ denotes the $\mathcal{L}$-formula algebra and $\mathrm{End}(\mathbf{Fm}_{\mathcal{L}}(V))$ is the carrier of the monoid of its endomorphisms. Composition and identities are the usual ones in the monoid of endomorphisms.
- $\mathrm{SEN}_{\mathcal{L}}(V) = \mathrm{Fm}_{\mathcal{L}}(V)$ and, given $\sigma \in \mathbf{Sign}_{\mathcal{L}}(V, V)$, we set $\mathrm{SEN}_{\mathcal{L}}(\sigma)(\phi) = \sigma(\phi)$, for all $\phi \in \mathrm{Fm}_{\mathcal{L}}(V)$. This defines a functor $\mathrm{SEN}_{\mathcal{L}} : \mathbf{Sign}_{\mathcal{L}} \to \mathbf{Set}$.
- Finally, $C_{\mathcal{S}} : \mathcal{P}(\mathrm{SEN}_{\mathcal{L}}(V)) \to \mathcal{P}(\mathrm{SEN}_{\mathcal{L}}(V))$ is the closure operator associated with the consequence operator $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)) \to \mathrm{Fm}_{\mathcal{L}}(V)$, i.e., defined, for all $X \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$, by $C_{\mathcal{S}}(X) = \{\phi \in \mathrm{Fm}_{\mathcal{L}}(V) : X \vdash_{\mathcal{S}} \phi\}$.

It is not difficult to verify that $\mathcal{I}_{\mathcal{S}}$ is a π-institution. It is called the π-institution **associated with** the deductive system $\mathcal{S}$.

At the forefront of our investigations in this work will be selfextensional π-institutions. They abstract the notion of a selfextensional sentential logic, which, in turn generalizes that of a Fregean logic, expressing formally Frege's philosophical principle of compositionality, i.e., the principle that the meaning of a complex expression is determined by the meanings of its constituent expressions and the rules used to combine them.

A deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ is called *selfextensional* [23] if the relation of interderivability between formulas, defined by stipulating that, for all $\phi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$, ϕ and ψ are *interderivable* if $\phi \vdash_{\mathcal{S}} \psi$ and $\psi \vdash_{\mathcal{S}} \phi$ (abbreviated $\phi \dashv\vdash_{\mathcal{S}} \psi$), is a congruence relation on the formula algebra. Selfextensionality, in other words, is equivalent to the following condition holding, for all $\phi, \psi, \delta \in \mathrm{Fm}_{\mathcal{L}}(V)$ and every variable $v \in V$:

$$\phi \dashv\vdash_{\mathcal{S}} \psi \quad \text{implies} \quad \delta(v/\phi) \dashv\vdash_{\mathcal{S}} \delta(v/\psi),$$

where $\delta(v/\phi)$ indicates the result of uniform substitution of ϕ for v in δ and similarly for $\delta(v/\psi)$.

The main reason why selfextensional logics have received attention in abstract algebraic logic is that each of the main classes in the abstract algebraic hierarchy (also known as the Leibniz hierarchy) of logics contains selfextensional and non-selfextensional members. For example, as is pointed out in [18], the following are all selfextensional logics, but their classifications in the abstract algebraic hierarchy are different: classical propositional calculus and intuitionistic propositional calculus are algebraizable logics, the local consequence of the normal modal logic K is equivalential, the local consequence of the classical modal logic E is protoalgebraic and, finally, positive modal logic [7], Belnap's four-valued logic [3, 12], the conjunction-disjunction fragment of classical propositional logic [15] and Visser's logic [6, 19] are non-protoalgebraic. This presence of selfextensional logics across the boundaries of the Leibniz hierarchy creates an interesting environment where properties related to a class in the hierarchy may be equally applied to both selfextensional and non-selfextensional logics and, potentially, properties arising from selfextensionality might be exploited across the abstract algebraic hierarchy.

A π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, with N a category of natural transformations on SEN, is said to be *N-selfextensional* if the equivalence system $\Lambda(\mathcal{I}) = \{\Lambda_\Sigma(\mathcal{I})\}_{\Sigma \in |\mathbf{Sign}|}$, defined, for all $\Sigma \in |\mathbf{Sign}|$, by stipulating that, for all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi, \psi \rangle \in \Lambda_\Sigma(\mathcal{I}) \quad \text{iff} \quad C_\Sigma(\phi) = C_\Sigma(\psi),$$

is an N-congruence system on SEN.

The work of Jansana and Palmigiano [18] has its origins in Wójcicki's Theorem [23, 24], characterizing selfextensional logics, as well as in the various algebraic-topological dualities that have appeared in the logical literature, with prototypical examples of those between bounded lattices and Priestley spaces and Boolean algebras and Stone spaces.

Given an algebraic signature $\mathcal{L}$, an *$\mathcal{L}$-referential algebra* is a pair $\langle W, \mathcal{A}\rangle$, where W is a set of *reference points* and $\mathcal{A}$ is an $\mathcal{L}$-algebra on a collection of subsets of W, called the *algebra of propositions.* A referential algebra $\langle W, \mathcal{A}\rangle$ is said to be *reduced* if the relation $R_{\langle W,\mathcal{A}\rangle}$ on W, defined, for all $u, v \in W$, by

$$\langle u, v\rangle \in R_{\langle W,\mathcal{A}\rangle} \quad \text{iff} \quad (\forall X \in \mathcal{A})(u \in X \iff v \in X)$$

is the identity relation on W. If M is a class of $\mathcal{L}$-referential algebras, a consequence relation $\models_{\mathsf{M}}$ may be defined on $\mathrm{Fm}_{\mathcal{L}}(V)$ by setting, for all $\Phi \cup \{\psi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\begin{array}{lll} \Phi \models_{\mathsf{M}} \psi & \text{iff} & \text{for all } \langle W, \mathcal{A}\rangle \in \mathsf{M}, \text{ all homomorphisms } v : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathcal{A}, \\ & & \text{and all } w \in W, w \in v(\phi), \text{ for all } \phi \in \Phi, \text{ implies } w \in v(\psi). \end{array}$$

A deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}}\rangle$ is said to have a *complete local referential semantics* if there exists a class M of $\mathcal{L}$-referential algebras, such that $\vdash_{\mathcal{S}}$ coincides with $\models_{\mathsf{M}}$. Wójcicki showed that a logic $\mathcal{S}$ is selfextensional if and only if it admits a complete local referential semantics (Theorem 2.2 of [18]).

We revisit Wójcicki's Theorem and show in Theorem 3 that a similar result holds for any π-institution with a designated category of natural transformations N on its sentence functor SEN. More precisely, a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is N-selfextensional if and only if it admits a complete local N-referential semantics in a sense that will be made precise in Section 3. This result encompasses the original theorem as a special case.

Motivated by this characterization of selfextensional logics and the classical dualities between various classes of algebras and topological spaces, Jansana and Palmigiano [18] embark in establishing a general duality theorem between certain categories of atlases and of referential algebras. Given a language type $\mathcal{L}$, an *$\mathcal{L}$-atlas* [10] (also known as a *generalized matrix* [23, 8]) is a pair $\langle \mathbf{A}, \mathcal{B}\rangle$, where $\mathbf{A}$ is an $\mathcal{L}$-algebra and $\mathcal{B}$ is a family of subsets of the universe A of $\mathbf{A}$. Particular instances of atlases are the *abstract logics* of [5, 13], in which $\mathcal{B}$ forms a closure system. Given any logic $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}}\rangle$ and any $\mathcal{L}$-algebra $\mathbf{A}$, a subset $F \subseteq A$ of the universe of $\mathbf{A}$ is an *$\mathcal{S}$-filter on* $\mathbf{A}$ if, for all $\Phi \cup \{\psi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$ and every homomorphism $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$,

$$\Phi \vdash_{\mathcal{S}} \psi \quad \text{and} \quad h(\Phi) \subseteq F \quad \text{imply} \quad h(\psi) \in F.$$

For every $\mathcal{L}$-algebra $\mathbf{A}$, the collection $\mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ of all $\mathcal{S}$-filters on $\mathbf{A}$ forms a closure system on A and, hence $\langle \mathbf{A}, \mathrm{Fi}_{\mathcal{S}}\mathbf{A}\rangle$ is an abstract logic. The *Frege relation* $\Lambda_{\mathbf{A}}\mathcal{B}$ of an atlas $\langle \mathbf{A}, \mathcal{B}\rangle$ is defined, for all $a, b \in A$, by

$$\langle a, b\rangle \in \Lambda_{\mathbf{A}}\mathcal{B} \quad \text{iff} \quad (\forall B \in \mathcal{B})(a \in B \iff b \in B).$$

The Frege relation is not necessarily a congruence on $\mathbf{A}$. A given atlas $\langle \mathbf{A}, \mathcal{B}\rangle$ is said to be *congruential* if $\Lambda_{\mathbf{A}}\mathcal{B}$ is a congruence. Moreover, an atlas $\langle \mathbf{A}, \mathcal{B}\rangle$ is called *Frege-reduced* or, simply, *reduced* if $\Lambda_{\mathbf{A}}\mathcal{B}$ is the identity relation on A, the carrier of $\mathbf{A}$.

In their main duality theorem, Theorem 4.12 of [18], Jansana and Palmigiano prove that, given any algebraic signature $\mathcal{L}$, the categories of reduced congruential $\mathcal{L}$-atlases CA^* and of reduced $\mathcal{L}$-referential algebras RA^* are dually equivalent categories. In our main theorem, Theorem 40 of Section 10, this duality is extended to a more general duality between the category $\mathbf{CAS}^*_N$ of reduced congruential N-atlas systems and the category $\mathbf{RAS}^*_N$ of reduced referential N-algebraic systems, where N is a fixed category of natural transformations on a given set-valued functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$. Theorem 40 encompasses as a special case Theorem 4.12 of [18] and, therefore, encompasses, also, as Example 2.5 of [18] illustrates, many of the well-known algebraic-topological dualities that have appeared in the logic literature.

Given a deductive system $\mathcal{S}$ and an algebra $\mathbf{A}$, $\Lambda_{\mathbf{A}}\mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ is not necessarily a congruence on $\mathbf{A}$. A logic $\mathcal{S}$ is called *fully selfextensional* if, for every $\mathcal{L}$-algebra $\mathbf{A}$, $\Lambda_{\mathbf{A}}\mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ is a congruence on $\mathbf{A}$. It is obvious that every fully selfextensional logic is selfextensional, but the fact that fully selfextensional deductive systems form a *proper* subclass of the class of all selfextensional deductive systems was established by Babyonyshev [1]. Motivated by their general result on the duality between the category of reduced congruential $\mathcal{L}$-atlases CA^* and of reduced $\mathcal{L}$-referential algebras RA^*, Jansana and Palmigiano provide in Theorems 5.1 and 5.4 of [18] a characterization of the class of fully selfextensional logics inside the wider class of selfextensional logics. They obtain this result by first restricting the general duality result to a duality between the atlas semantics and the referential semantics of an arbitrary selfextensional logic. They, then, take advantage of a known correspondence between the reduced atlas semantics of a given selfextensional logic $\mathcal{S}$ and the class of algebras $\mathsf{Alg}\mathcal{S}$ associated with it by the theory of abstract algebraic logic [13] to show that a selfextensional logic $\mathcal{S}$ is fully selfextensional if and only if the category corresponding to $\mathsf{Alg}\mathcal{S}$ is dually equivalent to the category of reduced referential algebraic models of $\mathcal{S}$. In future work, we intend to continue the investigations begun in the present paper along similar lines. More precisely, the goal is to provide an analogous characterization of the class of fully N-selfextensional π-institutions inside the class of N-selfextensional π-institutions. Babyonyshev's results show that this inclusion is proper in general and a characterization theorem for logics formalized as π-institutions should encompass Theorem 5.4 of [18] as a special case.

2. A Few Preliminaries

Recall that the contravariant power-set functor $\overleftarrow{\mathcal{P}} : \mathbf{Set} \to \mathbf{Set}^{\mathrm{op}}$ sends a set X to its power-set $\mathcal{P}(X)$ and a function $f : X \to Y$ to the function $\overleftarrow{\mathcal{P}}(f) : \mathcal{P}(Y) \to \mathcal{P}(X)$, such that, for all $W \subseteq Y$, $\overleftarrow{\mathcal{P}}(f)(W) = f^{-1}(W) = \{x \in X : f(x) \in W\}$.

Recall, also, that, given a set-valued functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, the **clone of all natural transformations on** SEN is the locally small category with collection of objects $\{\mathrm{SEN}^\alpha : \alpha \text{ an ordinal}\}$ and collection of morphisms $\tau : \mathrm{SEN}^\alpha \to \mathrm{SEN}^\beta$ β-sequences of natural transformations $\tau : \mathrm{SEN}^\alpha \to \mathrm{SEN}$. Composition of $\langle \tau_i : i < \beta \rangle : \mathrm{SEN}^\alpha \to \mathrm{SEN}^\beta$ with $\langle \sigma_j : j < \gamma \rangle : \mathrm{SEN}^\beta \to \mathrm{SEN}^\gamma$

$$\mathrm{SEN}^\alpha \xrightarrow{\langle \tau_i : i < \beta \rangle} \mathrm{SEN}^\beta \xrightarrow{\langle \sigma_j : j < \gamma \rangle} \mathrm{SEN}^\gamma$$

is defined by

$$\langle \sigma_j : j < \gamma \rangle \circ \langle \tau_i : i < \beta \rangle = \langle \sigma_j(\langle \tau_i : i < \beta \rangle) : j < \gamma \rangle.$$

A subcategory N of this category with objects all objects of the form $\mathrm{SEN}^k : k < \omega$, and containing all projection morphisms $p^{k,i} : \mathrm{SEN}^k \to \mathrm{SEN}, i < k, k < \omega$, with $p^{k,i}_\Sigma : \mathrm{SEN}(\Sigma)^k \to \mathrm{SEN}$ given by

$$p^{k,i}_\Sigma(\vec{\phi}) = \phi_i, \text{ for all } \vec{\phi} \in \mathrm{SEN}(\Sigma)^k,$$

and such that, for every family $\{\tau_i : \mathrm{SEN}^k \to \mathrm{SEN} : i < l\}$ of natural transformations in N, $\langle \tau_i : i < l \rangle : \mathrm{SEN}^k \to \mathrm{SEN}^l$ is also in N, is referred to as a **category of natural transformations on** SEN.

Given a functor $F : \mathbf{Sign} \to \mathbf{Set}$, a functor $G : \mathbf{Sign} \to \mathbf{Set}$ is called a **simple subfunctor** of F, if, for all $\Sigma \in |\mathbf{Sign}|$, $G(\Sigma) \subseteq F(\Sigma)$ and, for all $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$ and all $\phi \in G(\Sigma_1)$, $G(f)(\phi) = F(f)(\phi)$, i.e., $G(f) = F(f)\restriction_{G(\Sigma_1)}$.

Given two functors $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ and $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, with categories of natural transformations N, N' on SEN and SEN', N and N' will be called **similar** if there exists a bijective functor $F : N \to N'$, that preserves the projection natural transformations and, hence, also the arities of the natural transformations. We use the notation $\sigma' := F(\sigma)$ for the image in N' under F of a natural transformation in N. When two categories of natural transformations are assumed similar, such a bijective correspondence between their categories of natural transformations will be assumed given and fixed. This notion captures in the context of functors the idea of similarity between two ordinary universal algebras over the same algebraic signature.

Given two functors $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ and $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, with similar categories of natural transformations N, N' on SEN and SEN$'$, respectively, an (N, N')-**epimorphic translation** $\langle F, \alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$ consists of a functor $F : \mathbf{Sign} \to \mathbf{Sign}'$ and a natural transformation $\alpha : \mathrm{SEN} \to \mathrm{SEN}' \circ F$, such that, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^n$,

$$\begin{tikzcd}
\mathrm{SEN}(\Sigma)^n \arrow[r, "\sigma_\Sigma"] \arrow[d, "\alpha_\Sigma^n"'] & \mathrm{SEN}(\Sigma) \arrow[d, "\alpha_\Sigma"] \\
\mathrm{SEN}'(F(\Sigma))^n \arrow[r, "\sigma'_{F(\Sigma)}"'] & \mathrm{SEN}'(F(\Sigma))
\end{tikzcd}$$

$$\alpha_\Sigma(\sigma_\Sigma(\vec{\phi})) = \sigma'_{F(\Sigma)}(\alpha_\Sigma^n(\vec{\phi})).$$

An (N, N')-epimorphic translation will also be referred to as an N-**algebraic morphism**.

3. Referential Semantics and Wójcicki's Theorem

Let **Sign** be a category, $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ a set-valued functor and N a category of natural transformations on SEN. An N-**referential algebraic system** $\mathcal{F}$ is a triple $\mathcal{F} = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F'\rangle\rangle$, consisting of:

- A contravariant functor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}^{\mathrm{op}}$;
- A simple subfunctor $\mathrm{SEN}'_s : \mathbf{Sign}' \to \mathbf{Set}$ of the (covariant) functor $\overleftarrow{\mathcal{P}}\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$;
- A similar to N category N'_s of natural transformations on SEN'_s, with $F' : N \to N'_s$ the functor witnessing the similarity.

Given an N-referential algebraic system, the elements of the set $\mathrm{SEN}'(\Sigma)$, $\Sigma \in |\mathbf{Sign}'|$, will be referred to as Σ-**points**, Σ-**reference points**, Σ-**indices** or Σ-**states**. The N-referential algebraic system $\mathcal{F}$ is said to be **based on** the functor SEN'. In this context, N-algebraic morphisms $\langle F,\alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'_s$ are referred to as **interpretations**. Note that, for all $\Sigma,\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma,\Sigma')$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\begin{tikzcd}
\mathrm{SEN}(\Sigma) \arrow[r, "\alpha_\Sigma"] \arrow[d, "\mathrm{SEN}(f)"'] & \mathrm{SEN}'_s(F(\Sigma)) \arrow[d, "\mathrm{SEN}'_s(F(f))"] \\
\mathrm{SEN}(\Sigma') \arrow[r, "\alpha_{\Sigma'}"'] & \mathrm{SEN}'_s(F(\Sigma'))
\end{tikzcd}$$

$$\mathrm{SEN}'_s(F(f))(\alpha_\Sigma(\phi)) = \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)),$$

and, also, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^n$,

$$\begin{tikzcd}
\mathrm{SEN}(\Sigma)^n \arrow[r, "\sigma_\Sigma"] \arrow[d, "\alpha^n_\Sigma"'] & \mathrm{SEN}(\Sigma) \arrow[d, "\alpha_\Sigma"] \\
\mathrm{SEN}'_s(F(\Sigma))^n \arrow[r, "\sigma'_{F(\Sigma)}"'] & \mathrm{SEN}'_s(F(\Sigma))
\end{tikzcd}$$

$$\alpha_\Sigma(\sigma_\Sigma(\vec{\phi})) = \sigma'_{F(\Sigma)}(\alpha^n_\Sigma(\vec{\phi})).$$

Given an interpretation $\langle F,\alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'_s$, $\Sigma \in |\mathbf{Sign}|$, $\phi \in \mathrm{SEN}(\Sigma)$ and $w \in \mathrm{SEN}'(F(\Sigma))$, ϕ **is true at** w **under the interpretation** $\langle F,\alpha\rangle$ if $w \in \alpha_\Sigma(\phi)$. Otherwise, we say that ϕ **is false at** w **under** $\langle F,\alpha\rangle$.

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a set-valued functor and N a category of natural transformations on SEN. An **augmented N-referential algebraic system** is a pair $\mathfrak{F}' = \langle \mathcal{F}', \langle F,\alpha\rangle\rangle$, where

- $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F'\rangle\rangle$ is an N-referential algebraic system;
- $\langle F,\alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'_s$ is an interpretation.

A Σ-sentence $\phi \in \mathrm{SEN}(\Sigma)$ is **true at** an $F(\Sigma)$-point $w \in \mathrm{SEN}'(F(\Sigma))$ **under** $\mathfrak{F}'$ if and only if ϕ is true at w under the interpretation $\langle F,\alpha\rangle$. Otherwise, ϕ is **false at** w **under** $\mathfrak{F}'$.

Given a functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, with N a category of natural transformations on SEN, and an augmented N-referential algebraic system $\mathfrak{F}'$, as above, define on SEN the families of closure systems $C^{\mathfrak{F}'}, C^{\mathfrak{F}',g}$ as follows: For all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\psi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\psi \in C^{\mathfrak{F}'}_\Sigma(\Phi) \quad \text{iff} \quad \begin{array}{l}\bigcap_{\phi\in\Phi} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \subseteq \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi)),\\ \text{for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma,\Sigma').\end{array}$$

and

$$\begin{array}{lll}\psi \in C^{\mathfrak{F}',g}_\Sigma(\Phi) & \text{iff} & \bigcap_{\phi\in\Phi}\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) = \mathrm{SEN}'(F(\Sigma')) \\ & & \text{implies } \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi)) = \mathrm{SEN}'(F(\Sigma')) \\ & & \text{for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma,\Sigma').\end{array}$$

It will be shown, next, that the triples $\mathcal{I}^{\mathfrak{F}'} = \langle\mathbf{Sign}, \mathrm{SEN}, C^{\mathfrak{F}'}\rangle$ and $\mathcal{I}^{\mathfrak{F}',g} = \langle\mathbf{Sign}, \mathrm{SEN}, C^{\mathfrak{F}',g}\rangle$ are π-institutions.

LEMMA 1. *Suppose that* SEN : **Sign** → **Set** *is a functor and N a category of natural transformations on* SEN. *Let $\mathfrak{F}' = \langle\mathcal{F}', \langle F, \alpha\rangle\rangle$ be an augmented N-referential algebraic system, with $\mathcal{F}' = \langle\mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F'\rangle\rangle$. Then the triple $\mathcal{I}^{\mathfrak{F}'} = \langle\mathbf{Sign}, \mathrm{SEN}, C^{\mathfrak{F}'}\rangle$ is a π-institution. The same holds for the triple $\mathcal{I}^{\mathfrak{F}',g} = \langle\mathbf{Sign}, \mathrm{SEN}, C^{\mathfrak{F}',g}\rangle$.*

PROOF. We must show that $C^{\mathfrak{F}'}$ is a closure system on SEN. Reflexivity and monotonicity are straightforward. For transitivity, suppose that $\Sigma \in |\mathbf{Sign}|$, $\Phi\cup\{\psi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\psi \in C^{\mathfrak{F}'}_\Sigma(C^{\mathfrak{F}'}_\Sigma(\Phi))$. Then, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma,\Sigma')$, $\bigcap_{\chi\in C^{\mathfrak{F}'}_\Sigma(\Phi)}\alpha_{\Sigma'}(\mathrm{SEN}(f)(\chi)) \subseteq \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi))$. But, for every $\chi \in C^{\mathfrak{F}'}_\Sigma(\Phi)$, we have that

$$\bigcap_{\phi\in\Phi}\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \subseteq \alpha_{\Sigma'}(\mathrm{SEN}(f)(\chi)),$$

whence it follows that

$$\begin{array}{lll}\bigcap_{\phi\in\Phi}\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) & \subseteq & \bigcap_{\chi\in C^{\mathfrak{F}'}_\Sigma(\Phi)}\alpha_{\Sigma'}(\mathrm{SEN}(f)(\chi)) \\ & \subseteq & \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi)).\end{array}$$

Therefore, $\psi \in C^{\mathfrak{F}'}_\Sigma(\Phi)$ and $C^{\mathfrak{F}'}$ is transitive. Now, it only remains to show that $C^{\mathfrak{F}'}$ is structural. To this end, suppose that $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1,\Sigma_2)$ and $\Phi\cup\{\psi\} \subseteq \mathrm{SEN}(\Sigma_1)$, such that $\psi \in C^{\mathfrak{F}'}_{\Sigma_1}(\Phi)$. Then, we have, for all $\Sigma' \in |\mathbf{Sign}|, g \in \mathbf{Sign}(\Sigma_1,\Sigma')$, $\bigcap_{\phi\in\Phi}\alpha_{\Sigma'}(\mathrm{SEN}(g)(\phi)) \subseteq \alpha_{\Sigma'}(\mathrm{SEN}(g)(\psi))$.

$$\begin{tikzcd} \Sigma_1 \arrow[rr, "f"] \arrow[dr, "g"'] & & \Sigma_2 \arrow[dl, "k"] \\ & \Sigma' & \end{tikzcd}$$

Therefore, for all $k \in \mathbf{Sign}(\Sigma_2,\Sigma')$, we get that

$$\bigcap_{\phi\in\Phi}\alpha_{\Sigma'}(\mathrm{SEN}(kf)(\phi)) \subseteq \alpha_{\Sigma'}(\mathrm{SEN}(kf)(\psi)),$$

i.e., that $\bigcap_{\phi\in\Phi}\alpha_{\Sigma'}(\mathrm{SEN}(k)(\mathrm{SEN}(f)(\phi)))\subseteq\alpha_{\Sigma'}(\mathrm{SEN}(k)(\mathrm{SEN}(f)(\psi)))$. This proves that $\mathrm{SEN}(f)(\psi)\in C^{\mathfrak{F}'}_{\Sigma_2}(\mathrm{SEN}(f)(\Phi))$. Therefore $C^{\mathfrak{F}'}$ is structural and $\mathcal{I}^{\mathfrak{F}'}$ is indeed a π-institution.

The proof for $\mathcal{I}^{\mathfrak{F}',g}$ is similar and will be omitted. ■

The closure system $C^{\mathfrak{F}'}$ is called the **local closure system on** SEN **associated with** $\mathfrak{F}'$ and $C^{\mathfrak{F}',g}$ the **global closure system on** SEN **associated with** $\mathfrak{F}'$. Moreover, if $\mathtt{F}$ is a family of augmented N-referential algebraic systems, define

$$C^{\mathtt{F}}=\bigcap\{C^{\mathfrak{F}'}:\mathfrak{F}'\in\mathtt{F}\}\quad\text{and}\quad C^{\mathtt{F},g}=\bigcap\{C^{\mathfrak{F}',g}:\mathfrak{F}'\in\mathtt{F}\}.$$

It will be shown, next, that N-selfextensional π-institutions on SEN are exactly those that are determined by the local closure systems on SEN associated with classes of augmented N-referential algebraic systems. Theorem 3 is essentially a result first obtained for selfextensional deductive systems by Wójcicki [23, 24] and revisited in [18] (see Theorem 2.2). Proposition 2 establishes the fact that all π-institutions locally induced by a class of augmented referential algebraic systems are self-extensional.

Proposition 2. *Let* SEN : **Sign** → **Set** *be a set-valued functor and* N *a category of natural transformations on* SEN. *For every class* $\mathtt{F}=\{\langle\mathcal{F}^i,\langle F^i,\alpha^i\rangle\rangle:i\in I\}$, *with* $\mathcal{F}^i=\langle\mathrm{SEN}^i,\mathrm{SEN}^i_s,\langle N^i_s,F^i\rangle\rangle$, *of augmented N-referential algebraic systems, the local π-institution* $\mathcal{I}^{\mathtt{F}}=\langle\mathbf{Sign},\mathrm{SEN},C^{\mathtt{F}}\rangle$ *associated with* $\mathtt{F}$ *is a self-extensional π-institution.*

Proof. Suppose $\Sigma\in|\mathbf{Sign}|,\phi,\psi\in\mathrm{SEN}(\Sigma)$, such that $C^{\mathtt{F}}_\Sigma(\phi)=C^{\mathtt{F}}_\Sigma(\psi)$. It must be shown that $\langle\phi,\psi\rangle\in\widetilde{\Omega}^N_\Sigma(\mathcal{I}^{\mathtt{F}})$. This will be done by employing the characterization (Theorem 4 of [22]) of the Tarski N-congruence system of $\mathcal{I}^{\mathtt{F}}$. We have, by the definition of $C^{\mathtt{F}}$, that, for all $i\in I$, all $\Sigma'\in|\mathbf{Sign}|$ and all $f\in\mathbf{Sign}(\Sigma,\Sigma')$, $\alpha^i_{\Sigma'}(\mathrm{SEN}(f)(\phi))=\alpha^i_{\Sigma'}(\mathrm{SEN}(f)(\psi))$. Thus, for all $\Sigma''\in|\mathbf{Sign}|$, all $g\in\mathbf{Sign}(\Sigma',\Sigma'')$,

$$\Sigma\xrightarrow{f}\Sigma'\xrightarrow{g}\Sigma''$$

all $\sigma:\mathrm{SEN}^n\to\mathrm{SEN}$ in N and all $\vec{\chi}\in\mathrm{SEN}(\Sigma'')^{n-1}$, we get that

$$\begin{aligned}&\mathrm{SEN}^i_s(F^i(g))(\sigma^i_{F^i(\Sigma')}(\alpha^i_{\Sigma'}(\mathrm{SEN}(f)(\phi)),\alpha^i_{\Sigma'}(\vec{\chi})))=\\&\qquad\mathrm{SEN}^i_s(F^i(g))(\sigma^i_{F^i(\Sigma')}(\alpha^i_{\Sigma'}(\mathrm{SEN}(f)(\psi)),\alpha^i_{\Sigma'}(\vec{\chi}))).\end{aligned}$$

Hence, we obtain

$$\begin{tikzcd}
\mathrm{SEN}(\Sigma')^n \arrow[r, "(\alpha^i_{\Sigma'})^n"] \arrow[d, "\sigma_{\Sigma'}"'] & (\mathrm{SEN}^i_s(F^i(\Sigma'))^n \arrow[d, "\sigma^i_{F^i(\Sigma')}"] \\
\mathrm{SEN}(\Sigma') \arrow[r, "\alpha^i_{\Sigma'}"'] & \mathrm{SEN}^i_s(F^i(\Sigma'))
\end{tikzcd}$$

$$\begin{aligned}\mathrm{SEN}^i_s(F^i(g))(\alpha^i_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}))) = \\ \mathrm{SEN}^i_s(F^i(g))(\alpha^i_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}))).\end{aligned}$$

Finally, this gives

$$\begin{tikzcd}
\mathrm{SEN}(\Sigma') \arrow[r, "\alpha^i_{\Sigma'}"] \arrow[d, "\mathrm{SEN}(g)"'] & \mathrm{SEN}^i_s(F^i(\Sigma')) \arrow[d, "\mathrm{SEN}^i_s(F^i(g))"] \\
\mathrm{SEN}(\Sigma'') \arrow[r, "\alpha^i_{\Sigma''}"'] & \mathrm{SEN}^i_s(F^i(\Sigma''))
\end{tikzcd}$$

$$\alpha^i_{\Sigma''}(\mathrm{SEN}(g)(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi}))) = \alpha^i_{\Sigma''}(\mathrm{SEN}(g)(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi}))).$$

Therefore, by the definition of C^{F}, we get that

$$C^{\mathsf{F}}_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\phi),\vec{\chi})) = C^{\mathsf{F}}_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi),\vec{\chi})).$$

Since this holds for every $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma,\Sigma')$, we get, by Theorem 4 of [22], that $\langle\phi,\psi\rangle \in \widetilde{\Omega}^N_\Sigma(\mathcal{I}^{\mathsf{F}})$, i.e., that $\mathcal{I}^{\mathsf{F}}$ is N-self-extensional. ■

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on SEN. An augmented N-referential algebraic system $\mathfrak{F}' = \langle \mathcal{F}', \langle F,\alpha\rangle\rangle$ is a **local** N**-model** of $\mathcal{I}$ if $C \leq C^{\mathfrak{F}'}$, i.e., if, for all $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\psi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\psi \in C_\Sigma(\Phi) \quad \text{implies} \quad \begin{array}{l}\bigcap_{\phi\in\Phi} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \subseteq \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi)),\\ \text{for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma,\Sigma').\end{array}$$

Similarly, $\mathfrak{F}'$ is a **global** N**-model** of $\mathcal{I}$ if $C \leq C^{\mathfrak{F}',g}$, i.e., if, for all $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\psi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\psi \in C_\Sigma(\Phi) \quad \text{implies that, if} \quad \begin{array}{l}\bigcap_{\phi\in\Phi} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) = \mathrm{SEN}'(F(\Sigma')),\\ \text{then } \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi)) = \mathrm{SEN}'(F(\Sigma')),\\ \text{for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma,\Sigma').\end{array}$$

A class $\mathtt{F}$ of augmented N-referential algebraic systems is a **complete local N-referential semantics** for $\mathcal{I}$ if $C = C^{\mathtt{F}}$ and, in this case, it is said that $\mathcal{I}$ **admits a complete local N-referential semantics**.

Wójcicki's Theorem, characterizing self-extensional logics in terms of the existence of a complete local referential semantics, has the following extension in the π-institution framework.

THEOREM 3 (Wójcicki's Theorem). *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be a π-institution. Then $\mathcal{I}$ is N-self-extensional iff it admits a complete local N-referential semantics.*

PROOF. The implication from right to left follows from Proposition 2. For the converse implication, assume that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is an N-self-extensional π-institution. Construct the triple $\mathcal{F}^{\mathcal{I}} = \langle \mathrm{SEN}^{\mathcal{I}}, \mathrm{SEN}^{\mathcal{I}}_s, \langle N^{\mathcal{I}}_s, F^{\mathcal{I}}\rangle\rangle$ as follows:

- $\mathrm{SEN}^{\mathcal{I}} : \mathbf{Sign} \to \mathbf{Set}^{\mathrm{op}}$ is the (contravariant) functor that maps $\Sigma \in |\mathbf{Sign}|$ to the collection of all Σ-theories of $\mathcal{I}$, i.e., $\mathrm{SEN}^{\mathcal{I}}(\Sigma) = \{T : T \in \mathrm{Th}_\Sigma(\mathcal{I})\}$, and that maps $f \in \mathbf{Sign}(\Sigma, \Sigma')$, to the morphism $\mathrm{SEN}^{\mathcal{I}}(f) : \mathrm{Th}_{\Sigma'}(\mathcal{I}) \to \mathrm{Th}_\Sigma(\mathcal{I})$, given by $T' \mapsto \mathrm{SEN}(f)^{-1}(T')$.
- $\mathrm{SEN}^{\mathcal{I}}_s : \mathbf{Sign} \to \mathbf{Set}$ is the subfunctor of $\overleftarrow{\mathcal{P}}\mathrm{SEN}^{\mathcal{I}}$ defined by
$$\mathrm{SEN}^{\mathcal{I}}_s(\Sigma) = \{\{T \in \mathrm{Th}_\Sigma(\mathcal{I}) : \phi \in T\} : \phi \in \mathrm{SEN}(\Sigma)\}.$$
- The subfunctor $\mathrm{SEN}^{\mathcal{I}}_s : \mathbf{Sign} \to \mathbf{Set}$ of $\overleftarrow{\mathcal{P}}\mathrm{SEN}^{\mathcal{I}}$ is endowed with a category of natural transformations $N^{\mathcal{I}}_s$, defined, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, by $\sigma^{\mathcal{I}} : (\mathrm{SEN}^{\mathcal{I}}_s)^n \to \mathrm{SEN}^{\mathcal{I}}_s$, given by
$$\sigma^{\mathcal{I}}_\Sigma(\eta^{\mathcal{I}}_\Sigma(\phi_0), \ldots, \eta^{\mathcal{I}}_\Sigma(\phi_{n-1})) = \eta^{\mathcal{I}}_\Sigma(\sigma_\Sigma(\phi_0, \ldots, \phi_{n-1})),$$
for all $\Sigma \in |\mathbf{Sign}|, \phi_0, \ldots, \phi_{n-1} \in \mathrm{SEN}(\Sigma)$, where, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $\eta^{\mathcal{I}}_\Sigma(\phi) = \{T \in \mathrm{Th}_\Sigma(\mathcal{I}) : \phi \in T\} \in \mathrm{SEN}^{\mathcal{I}}_s(\Sigma)$.
- Finally, $F^{\mathcal{I}} : N \to N^{\mathcal{I}}_s$ maps σ to $\sigma^{\mathcal{I}}$, as defined previously.

The triple $\mathcal{F}^{\mathcal{I}} = \langle \mathrm{SEN}^{\mathcal{I}}, \mathrm{SEN}^{\mathcal{I}}_s, \langle N^{\mathcal{I}}_s, F^{\mathcal{I}}\rangle\rangle$, thus defined, is an N-referential algebraic system. We only check a few of the properties that need to be satisfied. First, $\mathrm{SEN}^{\mathcal{I}} : \mathbf{Sign} \to \mathbf{Set}^{\mathrm{op}}$ is well-defined: It is clear that $\mathrm{SEN}^{\mathcal{I}}(i_\Sigma) = i_{\mathrm{SEN}^{\mathcal{I}}(\Sigma)}$, for all $\Sigma \in |\mathbf{Sign}|$. For all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $T' \in \mathrm{Th}_{\Sigma'}(\mathcal{I})$, if $\phi \in C_\Sigma(\mathrm{SEN}(f)^{-1}(T'))$, then

$$\begin{array}{rcl}\mathrm{SEN}(f)(\phi) & \in & \mathrm{SEN}(f)(C_\Sigma(\mathrm{SEN}(f)^{-1}(T'))) \\ & \subseteq & C_{\Sigma'}(\mathrm{SEN}(f)(\mathrm{SEN}(f)^{-1}(T'))) \\ & \subseteq & C_{\Sigma'}(T') \\ & = & T',\end{array}$$

whence $\phi \in \mathrm{SEN}(f)^{-1}(T')$, showing that $\mathrm{SEN}(f)^{-1}(T') \in \mathrm{Th}_\Sigma(\mathcal{I})$. Therefore, $\mathrm{SEN}^{\mathcal{I}}$ is well-defined on morphisms. Moreover, for all $\Sigma, \Sigma', \Sigma'' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma'), g \in \mathbf{Sign}(\Sigma', \Sigma'')$ and all $T \in \mathrm{SEN}^{\mathcal{I}}(\Sigma'')$, we have that

$$\begin{array}{rcl}\mathrm{SEN}^{\mathcal{I}}(gf)(T'') & = & \mathrm{SEN}(gf)^{-1}(T'') \\ & = & (\mathrm{SEN}(g)\mathrm{SEN}(f))^{-1}(T'') \\ & = & \mathrm{SEN}(f)^{-1}(\mathrm{SEN}(g)^{-1}(T'')) \\ & = & \mathrm{SEN}^{\mathcal{I}}(f)(\mathrm{SEN}^{\mathcal{I}}(g)(T'')).\end{array}$$

Thus, $\mathrm{SEN}^{\mathcal{I}} : \mathbf{Sign} \to \mathbf{Set}^{\mathrm{op}}$ is a functor.

To see that $\mathrm{SEN}^{\mathcal{I}}_s$ is a subfunctor of $\overleftarrow{\mathcal{P}}\mathrm{SEN}^{\mathcal{I}}$, it suffices to show that, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $\mathrm{SEN}^{\mathcal{I}}_s(f)(\{T \in \mathrm{Th}_\Sigma(\mathcal{I}) : \phi \in T\}) \in \mathrm{SEN}^{\mathcal{I}}_s(\Sigma')$. To see this, we prove that, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\mathrm{SEN}^{\mathcal{I}}_s(f)(\eta^{\mathcal{I}}_\Sigma(\phi)) = \eta^{\mathcal{I}}_{\Sigma'}(\mathrm{SEN}(f)(\phi)). \quad (2)$$

In fact, we have

$$\begin{array}{rcl}\mathrm{SEN}^{\mathcal{I}}_s(f)(\eta^{\mathcal{I}}_\Sigma(\phi)) & = & \overleftarrow{\mathcal{P}}\mathrm{SEN}^{\mathcal{I}}(f)(\eta^{\mathcal{I}}_\Sigma(\phi)) \\ & = & \{T' \in \mathrm{Th}_{\Sigma'}(\mathcal{I}) : \mathrm{SEN}^{\mathcal{I}}(f)(T') \in \eta^{\mathcal{I}}_\Sigma(\phi)\} \\ & = & \{T' \in \mathrm{Th}_{\Sigma'}(\mathcal{I}) : \phi \in \mathrm{SEN}^{\mathcal{I}}(f)(T')\} \\ & = & \{T' \in \mathrm{Th}_{\Sigma'}(\mathcal{I}) : \phi \in \mathrm{SEN}(f)^{-1}(T')\} \\ & = & \{T' \in \mathrm{Th}_{\Sigma'}(\mathcal{I}) : \mathrm{SEN}(f)(\phi) \in T'\} \\ & = & \eta^{\mathcal{I}}_{\Sigma'}(\mathrm{SEN}(f)(\phi)).\end{array}$$

Finally, $\sigma^{\mathcal{I}} : (\mathrm{SEN}^{\mathcal{I}}_s)^n \to \mathrm{SEN}^{\mathcal{I}}_s$ is well-defined, i.e., independent of the choice of representatives, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N. Indeed, if $\Sigma \in |\mathbf{Sign}|$, $\phi_0, \psi_0, \ldots, \phi_{n-1}, \psi_{n-1} \in \mathrm{SEN}(\Sigma)$ are such that $\eta^{\mathcal{I}}_\Sigma(\phi_i) = \eta^{\mathcal{I}}_\Sigma(\psi_i)$, for all $i < n$, then $C_\Sigma(\phi_i) = C_\Sigma(\psi_i)$, whence, by self-extensionality, $\langle \phi_i, \psi_i \rangle \in \widetilde{\Omega}^N_\Sigma(\mathcal{I})$, implying that $\langle \sigma_\Sigma(\vec{\phi}), \sigma_\Sigma(\vec{\psi}) \rangle \in \widetilde{\Omega}^N_\Sigma(\mathcal{I})$ and, therefore, $C_\Sigma(\sigma_\Sigma(\vec{\phi})) = C_\Sigma(\sigma_\Sigma(\vec{\psi}))$. Hence we get that $\eta^{\mathcal{I}}_\Sigma(\sigma_\Sigma(\vec{\phi})) = \eta^{\mathcal{I}}_\Sigma(\sigma_\Sigma(\vec{\psi}))$, i.e., that $\sigma^{\mathcal{I}}_\Sigma(\eta^{\mathcal{I}}_\Sigma(\vec{\phi})) = \sigma^{\mathcal{I}}_\Sigma(\eta^{\mathcal{I}}_\Sigma(\vec{\psi}))$.

Construct, next, the triple $\mathfrak{F}^{\mathcal{I}} = \langle \mathcal{F}^{\mathcal{I}}, \langle \mathrm{I}_{\mathbf{Sign}}, \eta^{\mathcal{I}} \rangle \rangle$ as follows:

- $\mathcal{F}^{\mathcal{I}}$ is the N-referential algebraic system constructed above.
- $\mathrm{I}_{\mathbf{Sign}} : \mathbf{Sign} \to \mathbf{Sign}$ is the identity functor on $\mathbf{Sign}$.
- $\eta^{\mathcal{I}} : \mathrm{SEN} \to \mathrm{SEN}^{\mathcal{I}}_s$ is defined as above, i.e., by letting for all $\Sigma \in |\mathbf{Sign}|$, $\eta^{\mathcal{I}}_\Sigma : \mathrm{SEN}(\Sigma) \to \mathrm{SEN}^{\mathcal{I}}_s(\Sigma)$ be given by

$$\eta^{\mathcal{I}}_\Sigma(\phi) = \{T \in \mathrm{Th}_\Sigma(\mathcal{I}) : \phi \in T\}, \quad \text{for all } \phi \in \mathrm{SEN}(\Sigma).$$

To see that $\mathfrak{F}$ is a valid augmented N-referential algebraic system, we remind the reader that, by Equation (2), $\eta^{\mathcal{I}} : \mathrm{SEN} \to \mathrm{SEN}_s^{\mathcal{I}}$ is a natural transformation, i.e., the following rectangle commutes.

$$\begin{tikzcd}
\mathrm{SEN}(\Sigma) \arrow[r, "\eta^{\mathcal{I}}_{\Sigma}"] \arrow[d, "\mathrm{SEN}(f)"'] & \mathrm{SEN}_s^{\mathcal{I}}(\Sigma) \arrow[d, "\mathrm{SEN}_s^{\mathcal{I}}(f)"] \\
\mathrm{SEN}(\Sigma') \arrow[r, "\eta^{\mathcal{I}}_{\Sigma'}"'] & \mathrm{SEN}_s^{\mathcal{I}}(\Sigma')
\end{tikzcd}$$

Finally, to conclude the proof, it suffices to show that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\psi\} \subseteq \mathrm{SEN}(\Sigma)$, $\psi \in C_{\Sigma}^{\mathfrak{F}^{\mathcal{I}}}(\Phi)$ if and only if $\psi \in C_{\Sigma}(\Phi)$.

Suppose, first, that $\psi \in C_{\Sigma}^{\mathfrak{F}^{\mathcal{I}}}(\Phi)$. This is equivalent to

$$\bigcap_{\phi \in \Phi} \eta^{\mathcal{I}}_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \subseteq \eta^{\mathcal{I}}_{\Sigma'}(\mathrm{SEN}(f)(\psi)), \tag{3}$$

for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$. Let $T = C_{\Sigma}(\Phi)$. Then $T \in \bigcap_{\phi \in \Phi} \eta^{\mathcal{I}}_{\Sigma}(\phi)$. Therefore, by Inclusion (3), $T \in \eta^{\mathcal{I}}_{\Sigma}(\psi)$, which shows that $\psi \in T$ and, hence, that $\psi \in C_{\Sigma}(\Phi)$, as required.

Suppose, conversely, that $\psi \in C_{\Sigma}(\Phi)$. By structurality, this gives that, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\mathrm{SEN}(f)(\psi) \in C_{\Sigma'}(\mathrm{SEN}(f)(\Phi)). \tag{4}$$

Suppose that $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $T \in \mathrm{Th}_{\Sigma'}(\mathcal{I})$ is such that $T \in \bigcap_{\phi \in \Phi} \eta^{\mathcal{I}}_{\Sigma'}(\mathrm{SEN}(f)(\phi))$. Hence, for all $\phi \in \Phi$, $T \in \eta^{\mathcal{I}}_{\Sigma'}(\mathrm{SEN}(f)(\phi))$, which yields that, for all $\phi \in \Phi$, $\mathrm{SEN}(f)(\phi) \in T$. Therefore $\mathrm{SEN}(f)(\Phi) \subseteq T$ and, hence, by (4), we get that $\mathrm{SEN}(f)(\psi) \in T$. Thus $T \in \eta^{\mathcal{I}}_{\Sigma'}(\mathrm{SEN}(f)(\psi))$. This concludes the proof that $\bigcap_{\phi \in \Phi} \eta^{\mathcal{I}}_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \subseteq \eta^{\mathcal{I}}_{\Sigma'}(\mathrm{SEN}(f)(\psi))$, proving that $\psi \in C_{\Sigma}^{\mathfrak{F}^{\mathcal{I}}}(\Phi)$. ■

4. Reduced N-Referential Algebraic Systems

In the remainder of the paper, only local N-models and local N-referential semantics will be considered in the discussion and the adjective "local" will be omitted. Also, instead of N-model of $\mathcal{I}$, the term **augmented $\mathcal{I}$-N-referential algebraic system** may sometimes be used.

Let $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F' \rangle\rangle$ be an N-referential algebraic system. Define the equivalence family $R^{\mathcal{F}'} = \{R^{\mathcal{F}'}_{\Sigma}\}_{\Sigma \in |\mathbf{Sign}'|}$ on SEN', by letting, for all $\Sigma \in |\mathbf{Sign}'|$, $R^{\mathcal{F}'}_\Sigma$ be given, for all $\phi, \psi \in \mathrm{SEN}'(\Sigma)$, by

$$\langle\phi,\psi\rangle \in R^{\mathcal{F}'}_\Sigma \quad \text{iff} \quad (\forall X \in \mathrm{SEN}'_s(\Sigma))(\phi \in X \Leftrightarrow \psi \in X).$$

It is shown, now, that this equivalence family is in fact an equivalence system in the sense of categorical abstract algebraic logic.

PROPOSITION 4. *Let* **Sign** *be a category and* SEN : **Sign** $\rightarrow$ **Set** *a functor, with N a category of natural transformations on* SEN. *Suppose that $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F'\rangle\rangle$ is an N-referential algebraic system. Then $R^{\mathcal{F}'}$ is an equivalence system on* SEN$'$.

PROOF. Suppose that $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}'|, f \in \mathbf{Sign}'(\Sigma_2, \Sigma_1)$ and $\langle\phi,\psi\rangle \in R^{\mathcal{F}'}_{\Sigma_2}$. Then, for all $Y \in \mathrm{SEN}'_s(\Sigma_2)$ we have that $\phi \in Y$ if and only if $\psi \in Y$. Now, let $X \in \mathrm{SEN}'_s(\Sigma_1)$ and suppose that $\mathrm{SEN}'(f)(\phi) \in X$. Then $\phi \in \mathrm{SEN}'(f)^{-1}(X) = \mathrm{SEN}'_s(f)(X)$. Therefore, by the hypothesis, $\psi \in \mathrm{SEN}'_s(f)(X) = \mathrm{SEN}'(f)^{-1}(X)$, which yields that $\mathrm{SEN}'(f)(\psi) \in X$. By symmetry, we obtain that, for all $X \in \mathrm{SEN}'_s(\Sigma_1)$, $\mathrm{SEN}'(f)(\phi) \in X$ if and only if $\mathrm{SEN}'(f)(\psi) \in X$, i.e., that $\langle \mathrm{SEN}'(f)(\phi), \mathrm{SEN}'(f)(\psi)\rangle \in R^{\mathcal{F}'}_{\Sigma_1}$. Hence $R^{\mathcal{F}'}$ is indeed an equivalence system on SEN$'$. ■

Given an N-referential algebraic system $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F'\rangle\rangle$, $\mathcal{F}'$ is said to be **reduced** if $R^{\mathcal{F}'}$ is the identity equivalence system on SEN$'$.

Terminological Convention: The definitions above will also be applied to an augmented N-referential algebraic system always in reference to its N-referential algebraic system component. For instance, if $\mathfrak{F}' = \langle \mathcal{F}', \langle F, \alpha\rangle\rangle$ is an augmented N-referential algebraic system, we will denote by $R^{\mathfrak{F}'}$ the equivalence system $R^{\mathcal{F}'}$ and we will call $\mathfrak{F}'$ **reduced** if $\mathcal{F}'$ is a reduced N-referential algebraic system.

Any N-referential algebraic system $\mathcal{F}'$, as above, can be reduced. In other words, it can be associated with a reduced N-referential algebraic system "counterpart". Moreover, if the N-referential algebraic system is augmented, i.e., it is accompanied by a given interpretation, then the reduced counterpart may also be endowed with an accompanying interpretation in such a way that it induces the same local and global closure systems as the original augmented N-referential algebraic system. The reduction is obtained by identifying sentences via the equivalence system $R^{\mathcal{F}'}$. Next, we construct this reduced N-referential algebraic system associated with $\mathcal{F}'$ in some detail.

Let $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F'\rangle\rangle$ be an N-referential algebraic system. Define $\mathcal{F}'/R^{\mathcal{F}'} = \langle \mathrm{SEN}'^{R^{\mathcal{F}'}}, \mathrm{SEN}'^{R^{\mathcal{F}'}}_s, \langle N'^{R^{\mathcal{F}'}}_s, F'^{R^{\mathcal{F}'}}\rangle\rangle$, where

- $\mathrm{SEN}'^{R^{\mathcal{F}'}}(\Sigma) = \mathrm{SEN}'(\Sigma)/R^{\mathcal{F}'}_\Sigma$, for all $\Sigma \in |\mathbf{Sign'}|$, and
$$\mathrm{SEN}'^{R^{\mathcal{F}'}}(f)(\phi/R^{\mathcal{F}'}_{\Sigma_2}) = \mathrm{SEN}'(f)(\phi)/R^{\mathcal{F}'}_{\Sigma_1},$$
for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign'}|$, $f \in \mathbf{Sign'}(\Sigma_1, \Sigma_2)$ and all $\phi \in \mathrm{SEN}'(\Sigma_2)$;
- $\mathrm{SEN}'^{R^{\mathcal{F}'}}_s(\Sigma) = \mathrm{SEN}'_s(\Sigma)/R^{\mathcal{F}'}_\Sigma = \{\{\phi/R^{\mathcal{F}'}_\Sigma : \phi \in X\} : X \in \mathrm{SEN}'_s(\Sigma)\}$, for all $\Sigma \in |\mathbf{Sign'}|$;
- For all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, let $\sigma'^{R^{\mathcal{F}'}} : (\mathrm{SEN}'^{R^{\mathcal{F}'}}_s)^n \to \mathrm{SEN}'^{R^{\mathcal{F}'}}_s$ be given, for all $\Sigma \in |\mathbf{Sign'}|$ and all $X_0, \ldots, X_{n-1} \in \mathrm{SEN}'_s(\Sigma)$, by
$$\sigma'^{R^{\mathcal{F}'}}_\Sigma(X_0/R^{\mathcal{F}'}_\Sigma, \ldots, X_{n-1}/R^{\mathcal{F}'}_\Sigma) = \sigma'_\Sigma(X_0, \ldots, X_{n-1})/R^{\mathcal{F}'}_\Sigma;$$
- Finally $F'^{R^{\mathcal{F}'}}$ is defined by $\sigma \mapsto \sigma'^{R^{\mathcal{F}'}}$, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N.

In the next proposition it is shown that the triple just defined is in fact an N-referential algebraic system.

Proposition 5. *Let* **Sign** *be a category and* SEN : **Sign** $\to$ **Set** *a functor, with N a category of natural transformations on* SEN. *Let $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F' \rangle\rangle$ be an N-referential algebraic system. Then the triple*
$$\mathcal{F}'/R^{\mathcal{F}'} = \langle \mathrm{SEN}'^{R^{\mathcal{F}'}}, \mathrm{SEN}'^{R^{\mathcal{F}'}}_s, \langle N'^{R^{\mathcal{F}'}}_s, F'^{R^{\mathcal{F}'}}\rangle\rangle$$
is also an N-referential algebraic system.

Proof. First, since, by Proposition 4, $R^{\mathcal{F}'}$ is an equivalence system on SEN', the mapping $\mathrm{SEN}'^{R^{\mathcal{F}'}}$ is well-defined and it is not difficult to see that it does form a contravariant functor $\mathrm{SEN}'^{R^{\mathcal{F}'}} : \mathbf{Sign'} \to \mathbf{Set}^{\mathrm{op}}$ (see [22]).

To see that $\mathrm{SEN}'^{R^{\mathcal{F}'}}_s : \mathbf{Sign'} \to \mathbf{Set}$ is properly defined, suppose that $\Sigma_1, \Sigma_2 \in |\mathbf{Sign'}|$, $f \in \mathbf{Sign'}(\Sigma_1, \Sigma_2)$ and $X/R^{\mathcal{F}'}_{\Sigma_1} \in \mathrm{SEN}'^{R^{\mathcal{F}'}}_s(\Sigma_1)$. We need to show that $\mathrm{SEN}'^{R^{\mathcal{F}'}}_s(f)(X/R^{\mathcal{F}'}_{\Sigma_1}) \in \mathrm{SEN}'^{R^{\mathcal{F}'}}_s(\Sigma_2)$. To this end, it suffices to show that
$$\mathrm{SEN}'_s(f)(X)/R^{\mathcal{F}'}_{\Sigma_2} = \mathrm{SEN}'^{R^{\mathcal{F}'}}_s(f)(X/R^{\mathcal{F}'}_{\Sigma_1}).$$
To this end, let $\phi \in \mathrm{SEN}'(\Sigma_2)$. We have
$$\begin{array}{lll}\phi/R^{\mathcal{F}'}_{\Sigma_2} \in \mathrm{SEN}'_s(f)(X)/R^{\mathcal{F}'}_{\Sigma_2} & \text{iff} & \phi \in \mathrm{SEN}'_s(f)(X)\\ & \text{iff} & \mathrm{SEN}'(f)(\phi) \in X\\ & \text{iff} & \mathrm{SEN}'(f)(\phi)/R^{\mathcal{F}'}_{\Sigma_1} \in X/R^{\mathcal{F}'}_{\Sigma_1}\\ & \text{iff} & \mathrm{SEN}'^{R^{\mathcal{F}'}}(f)(\phi/R^{\mathcal{F}'}_{\Sigma_2}) \in X/R^{\mathcal{F}'}_{\Sigma_1}\\ & \text{iff} & \phi/R^{\mathcal{F}'}_{\Sigma_2} \in \mathrm{SEN}'^{R^{\mathcal{F}'}}_s(f)(X/R^{\mathcal{F}'}_{\Sigma_1}).\end{array}$$

Finally, for every $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, the definition of $\sigma'^{R^{\mathcal{F}'}}$ is sound because, if $X, Y \in \mathrm{SEN}'_s(\Sigma)$, such that $X/R^{\mathcal{F}'}_\Sigma = Y/R^{\mathcal{F}'}_\Sigma$, we get, by the definition of $R^{\mathcal{F}'}$, that $X = Y$. ■

Suppose, next that $\mathfrak{F}' = \langle \mathcal{F}', \langle F, \alpha \rangle \rangle$ is an augmented N-referential algebraic system, with $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F' \rangle \rangle$. Then we define the pair $\mathfrak{F}'^{R^{\mathfrak{F}'}} = \langle \mathcal{F}'^{R^{\mathcal{F}'}}, \langle F^{R^{\mathfrak{F}'}}, \alpha^{R^{\mathfrak{F}'}} \rangle \rangle$ as follows:

- $\mathcal{F}'^{R^{\mathcal{F}'}} = \langle \mathrm{SEN}'^{R^{\mathcal{F}'}}, \mathrm{SEN}'^{R^{\mathcal{F}'}}_s, \langle N'^{R^{\mathcal{F}'}}_s, F'^{R^{\mathcal{F}'}} \rangle \rangle$, as defined previously;
- $F^{R^{\mathfrak{F}'}} = F : \mathbf{Sign} \to \mathbf{Sign}'$;
- $\alpha^{R^{\mathfrak{F}'}}_\Sigma : \mathrm{SEN}(\Sigma) \to \mathrm{SEN}'^{R^{\mathcal{F}'}}_s(F(\Sigma))$ is given by $\alpha^{R^{\mathfrak{F}'}}_\Sigma(\phi) = \alpha_\Sigma(\phi)/R^{\mathcal{F}'}_{F(\Sigma)}$, for all $\Sigma \in |\mathbf{Sign}|$.

It is shown, now, that the pair $\mathfrak{F}'^{R^{\mathfrak{F}'}}$ is also an augmented N-referential algebraic system.

Proposition 6. *Let* **Sign** *be a category and* SEN : **Sign** $\to$ **Set** *a functor, with N a category of natural transformations on* SEN. *Let $\mathfrak{F}' = \langle \mathcal{F}', \langle F, \alpha \rangle \rangle$ be an augmented N-referential algebraic system, with $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N', F' \rangle \rangle$. Then $\mathfrak{F}'^{R^{\mathfrak{F}'}} = \langle \mathcal{F}'^{R^{\mathcal{F}'}}, \langle F^{R^{\mathfrak{F}'}}, \alpha^{R^{\mathfrak{F}'}} \rangle \rangle$ is also an augmented N-referential algebraic system.*

Proof. In view of Proposition 5, it suffices to show that $\alpha^{R^{\mathfrak{F}'}} : \mathrm{SEN} \to \mathrm{SEN}'^{R^{\mathcal{F}'}}_s \circ F$ is a natural transformation. To this end suppose that $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$ and $\phi \in \mathrm{SEN}(\Sigma_1)$. We have

$$\begin{tikzcd}
\mathrm{SEN}(\Sigma_1) \arrow[r, "\alpha^{R^{\mathfrak{F}'}}_{\Sigma_1}"] \arrow[d, "\mathrm{SEN}(f)"'] & \mathrm{SEN}'_s(F(\Sigma_1))/R^{\mathcal{F}'}_{F(\Sigma_1)} \arrow[d, "\mathrm{SEN}'^{R^{\mathcal{F}'}}_s(F(f))"] \\
\mathrm{SEN}(\Sigma_2) \arrow[r, "\alpha^{R^{\mathfrak{F}'}}_{\Sigma_2}"'] & \mathrm{SEN}'_s(F(\Sigma_2))/R^{\mathcal{F}'}_{F(\Sigma_2)}
\end{tikzcd}$$

$$\begin{array}{rcl}
\alpha^{R^{\mathfrak{F}'}}_{\Sigma_2}(\mathrm{SEN}(f)(\phi)) & = & \alpha_{\Sigma_2}(\mathrm{SEN}(f)(\phi))/R^{\mathcal{F}'}_{F(\Sigma_2)} \\
& = & \mathrm{SEN}'_s(F(f))(\alpha_{\Sigma_1}(\phi))/R^{\mathcal{F}'}_{F(\Sigma_2)} \\
& = & \mathrm{SEN}'^{R^{\mathcal{F}'}}_s(F(f))(\alpha_{\Sigma_1}(\phi)/R^{\mathcal{F}'}_{F(\Sigma_1)}) \\
& = & \mathrm{SEN}'^{R^{\mathcal{F}'}}_s(F(f))(\alpha^{R^{\mathfrak{F}'}}_{\Sigma_1}(\phi)).
\end{array}$$

This concludes the proof that the pair $\mathfrak{F}'^{R^{\mathfrak{F}'}} = \langle \mathcal{F}'^{R^{\mathcal{F}'}}, \langle F^{R^{\mathfrak{F}'}}, \alpha^{R^{\mathfrak{F}'}} \rangle \rangle$ is an augmented N-referential algebraic system. ■

It is finally shown that the reduced augmented N-referential algebraic system $\mathfrak{F}'^{R^{\mathfrak{F}'}}$ induces the same closure system $C^{\mathfrak{F}'^{R^{\mathfrak{F}'}}}$ on SEN as does the augmented N-referential algebraic system $\mathfrak{F}'$.

PROPOSITION 7. *Let* **Sign** *be a category and* SEN : **Sign** $\to$ **Set** *a functor, with* N *a category of natural transformations on* SEN. *Let* $\mathfrak{F}' = \langle \mathcal{F}', \langle F, \alpha \rangle \rangle$ *be an augmented* N*-referential algebraic system, with* $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F' \rangle \rangle$. *Then* $C^{\mathfrak{F}'} = C^{\mathfrak{F}'^{R^{\mathfrak{F}'}}}$.

PROOF. Suppose that $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\psi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\psi \in C^{\mathfrak{F}'}_\Sigma(\Phi)$. Then, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, we have that

$$\bigcap_{\phi \in \Phi} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \subseteq \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi)). \tag{5}$$

To show that $\psi \in C^{\mathfrak{F}'^{R^{\mathfrak{F}'}}}_\Sigma(\Phi)$, suppose that $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\chi \in \mathrm{SEN}(\Sigma')$ is such that $\chi / R^{\mathfrak{F}'}_{F(\Sigma')} \in \bigcap_{\phi \in \Phi} \alpha^{R^{\mathfrak{F}'}}_{\Sigma'}(\mathrm{SEN}(f)(\phi))$. Then, for all $\phi \in \Phi$, we have that $\chi / R^{\mathcal{F}'}_{F(\Sigma')} \in \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) / R^{\mathcal{F}'}_{F(\Sigma')}$, which shows that $\chi \in \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi))$, for all $\phi \in \Phi$. Therefore, by Inclusion (5), we get that $\chi \in \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi))$, showing that $\bigcap_{\phi \in \Phi} \alpha^{R^{\mathcal{F}'}}_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \subseteq \alpha^{R^{\mathcal{F}'}}_{\Sigma'}(\mathrm{SEN}(f)(\psi))$, i.e., that $\psi \in C^{\mathfrak{F}'^{R^{\mathfrak{F}'}}}_\Sigma(\Phi)$.

Suppose, conversely, that $\psi \in C^{\mathfrak{F}'^{R^{\mathfrak{F}'}}}_\Sigma(\Phi)$. Then, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, we have that

$$\bigcap_{\phi \in \Phi} \alpha^{R^{\mathcal{F}'}}_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \subseteq \alpha^{R^{\mathcal{F}'}}_{\Sigma'}(\mathrm{SEN}(f)(\psi)). \tag{6}$$

To see that $\psi \in C^{\mathfrak{F}'}_\Sigma(\Phi)$, let $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\chi \in \mathrm{SEN}(\Sigma')$ be such that $\chi \in \bigcap_{\phi \in \Phi} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi))$. Then $\chi \in \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi))$, for all $\phi \in \Phi$. This yields that

$$\begin{array}{rcl}
\chi / R^{\mathcal{F}'}_{F(\Sigma')} & \in & (\bigcap_{\phi \in \Phi} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi))) / R^{\mathcal{F}'}_{F(\Sigma')} \\
& \subseteq & \bigcap_{\phi \in \Phi} (\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) / R^{\mathcal{F}'}_{F(\Sigma')}) \\
& = & \bigcap_{\phi \in \Phi} \alpha^{R^{\mathcal{F}'}}_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \\
& \subseteq & \alpha^{R^{\mathcal{F}'}}_{\Sigma'}(\mathrm{SEN}(f)(\psi)) \quad \text{(by (6))} \\
& = & \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi)) / R^{\mathcal{F}'}_{\Sigma'},
\end{array}$$

Thus $\chi \in \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi))$. This verifies that $\bigcap_{\phi \in \Phi} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \subseteq \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi))$, i.e., that $\psi \in C^{\mathfrak{F}'}_\Sigma(\Phi)$. ■

5. N-Referential Algebraic System Morphisms

Consider, now, two N-referential algebraic systems $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F' \rangle \rangle$ and $\mathcal{F}'' = \langle \mathrm{SEN}'', \mathrm{SEN}''_s, \langle N''_s, F'' \rangle \rangle$. A pair $\langle F, \alpha \rangle$ is a **morphism of N-referential algebraic systems** from $\mathcal{F}'$ to $\mathcal{F}''$, denoted $\langle F, \alpha \rangle : \mathcal{F}' \to \mathcal{F}''$, if

- $\langle F, \alpha \rangle : \mathrm{SEN}' \to \mathrm{SEN}''$ is a singleton translation, with $F : \mathbf{Sign}' \to \mathbf{Sign}''$ an isomorphism;
- $\langle F^{-1}, \alpha^{-1} \rangle : \mathrm{SEN}''_s \to \mathrm{SEN}'_s$ is an (N'', N')-epimorphic translation.

The morphism $\langle F, \alpha \rangle : \mathcal{F}' \to \mathcal{F}''$ is said to be **strict** if, for all $\Sigma \in |\mathbf{Sign}''|$, $\alpha_\Sigma^{-1} : \mathrm{SEN}''_s(F(\Sigma)) \to \mathrm{SEN}'_s(\Sigma)$ is surjective, i.e., $\langle F^{-1}, \alpha^{-1} \rangle : \mathrm{SEN}''_s \to \mathrm{SEN}'_s$ is a surjective (N''_s, N'_s)-epimorphic translation.

With these morphisms the collection of all N-referential algebraic systems forms a category, which will be denoted by $\mathbf{RAS}_N$. The same collection of objects with strict morphisms between them forms a subcategory of $\mathbf{RAS}_N$, which will be denoted by $\mathbf{sRAS}_N$. Furthermore, reduced N-referential algebraic systems form a full subcategory of $\mathbf{RAS}_N$, which will be denoted by $\mathbf{RAS}^*_N$.

In the remainder of this section we show, first, that the construction of the N-referential algebraic system $\mathcal{F}'^{R^{\mathcal{F}'}}$ from $\mathcal{F}'$ extends to a functor from the category $\mathbf{RAS}_N$ to the category $\mathbf{RAS}^*_N$ and, second, provide some interesting properties of morphisms of N-referential algebraic systems. To simplify notation, whenever an N-referential algebraic system $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F' \rangle \rangle$ is at play, its reduction $\mathcal{F}'^{R^{\mathcal{F}'}} = \langle \mathrm{SEN}'^{R^{\mathfrak{F}'}}, \mathrm{SEN}'^{R^{\mathcal{F}'}}_s, \langle N'^{R^{\mathcal{F}'}}_s, F'^{R^{\mathcal{F}'}} \rangle \rangle$ will be denoted by $\mathcal{F}'^* = \langle \mathrm{SEN}'^*, \mathrm{SEN}'^*_s, \langle N'^*_s, F'^* \rangle \rangle$.

Let $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F' \rangle \rangle$, $\mathcal{F}'' = \langle \mathrm{SEN}'', \mathrm{SEN}''_s, \langle N''_s, F'' \rangle \rangle$ be two N-referential algebraic systems and $\langle F, \alpha \rangle : \mathcal{F}' \to \mathcal{F}''$ be a morphism of N-referential algebraic systems. Define the pair $\langle F^*, \alpha^* \rangle : \mathcal{F}'^* \to \mathcal{F}''^*$ as follows:

- $F^* : \mathbf{Sign}' \to \mathbf{Sign}''$ is defined by $F^* = F$;
- $\alpha^* : \mathrm{SEN}'^* \to \mathrm{SEN}''^* \circ F$ is defined, for all $\Sigma \in |\mathbf{Sign}'|$, by letting $\alpha^*_\Sigma : \mathrm{SEN}'^*(\Sigma) \to \mathrm{SEN}''^*(F(\Sigma))$ be given by

$$\alpha^*_\Sigma(\phi^*) = \alpha_\Sigma(\phi)^*, \quad \text{for all } \phi \in \mathrm{SEN}'(\Sigma).$$

We show that $\langle F^*, \alpha^* \rangle$ is a morphism of N-referential algebraic systems, and that, moreover, the assignments $\mathcal{F} \mapsto \mathcal{F}^*$ and $\langle F, \alpha \rangle \mapsto \langle F^*, \alpha^* \rangle$ define a functor $^* : \mathbf{RAS}_N \to \mathbf{RAS}^*_N$.

Proposition 8. *Let* **Sign** *be a category and* SEN : **Sign** $\to$ **Set** *a functor, with* N *a category of natural transformations on* SEN. *Suppose that* $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F'\rangle\rangle$, $\mathcal{F}'' = \langle \mathrm{SEN}'', \mathrm{SEN}''_s, \langle N''_s, F''\rangle\rangle$ *are two* N*-referential algebraic systems and* $\langle F, \alpha\rangle : \mathcal{F}' \to \mathcal{F}''$ *a morphism of* N*-referential algebraic systems. Then* $\langle F^*, \alpha^*\rangle : \mathcal{F}'^* \to \mathcal{F}''^*$ *is also a morphism of* N*-referential algebraic systems.*

Proof. It will be shown, first, that $\alpha^*_\Sigma : \mathrm{SEN}'^*(\Sigma) \to \mathrm{SEN}''^*(F(\Sigma))$ is well-defined, for all $\Sigma \in |\mathbf{Sign}'|$, and that $\alpha^* : \mathrm{SEN}'^* \to \mathrm{SEN}''^* \circ F$ is a natural transformation.

Suppose that $\Sigma \in |\mathbf{Sign}'|$ and $\phi, \psi \in \mathrm{SEN}'(\Sigma)$, such that $\phi^* = \psi^*$, i.e., that $\phi/R^{\mathcal{F}'}_\Sigma = \psi/R^{\mathcal{F}'}_\Sigma$. Let $X \in \mathrm{SEN}''_s(F(\Sigma))$, such that $\alpha_\Sigma(\phi) \in X$. Hence $\phi \in \alpha^{-1}_\Sigma(X) \in \mathrm{SEN}'_s(\Sigma)$, by the definition of a morphism of N-referential algebraic systems. Now, since $\langle \phi, \psi\rangle \in R^{\mathcal{F}'}_\Sigma$, we get that $\psi \in \alpha^{-1}_\Sigma(X)$. Thus, we obtain that $\alpha_\Sigma(\psi) \in X$. By symmetry, $\alpha_\Sigma(\phi) \in X$ iff $\alpha_\Sigma(\psi) \in X$, for all $X \in \mathrm{SEN}''_s(F(\Sigma))$, showing that $\alpha_\Sigma(\phi)^* = \alpha_\Sigma(\psi)^*$. Therefore, $\alpha^*_\Sigma(\phi^*) = \alpha_\Sigma(\phi)^* = \alpha_\Sigma(\psi)^* = \alpha^*_\Sigma(\psi^*)$ and α^*_Σ is well-defined.

To see that $\alpha^* : \mathrm{SEN}'^* \to \mathrm{SEN}''^* \circ F$ is a natural transformation, consider $\Sigma, \Sigma' \in |\mathbf{Sign}'|$, $f \in \mathbf{Sign}'(\Sigma, \Sigma')$ and $\phi \in \mathrm{SEN}'(\Sigma)$. We have

$$\begin{tikzcd}
\mathrm{SEN}'^*(\Sigma) \arrow[r, "\alpha^*_\Sigma"] \arrow[d, "\mathrm{SEN}'^*(f)"'] & \mathrm{SEN}''^*(F(\Sigma)) \arrow[d, "\mathrm{SEN}''^*(F(f))"] \\
\mathrm{SEN}'^*(\Sigma') \arrow[r, "\alpha^*_{\Sigma'}"'] & \mathrm{SEN}''^*(F(\Sigma'))
\end{tikzcd}$$

$$\begin{array}{ll}
\alpha^*_{\Sigma'}(\mathrm{SEN}'^*(f)(\phi^*)) & \\
\quad = \alpha^*_{\Sigma'}(\mathrm{SEN}'(f)(\phi)^*) & \text{(by the definition of } \mathrm{SEN}'^*) \\
\quad = \alpha_{\Sigma'}(\mathrm{SEN}'(f)(\phi))^* & \text{(by the definition of } \alpha^*) \\
\quad = \mathrm{SEN}''(F(f))(\alpha_\Sigma(\phi))^* & \text{(since } \alpha \text{ is a nat. transf.)} \\
\quad = \mathrm{SEN}''^*(F(f))(\alpha_\Sigma(\phi)^*) & \text{(by the defin. of } \mathrm{SEN}''^*) \\
\quad = \mathrm{SEN}''^*(F(f))(\alpha^*_\Sigma(\phi^*)) & \text{(by the definition of } \alpha^*).
\end{array}$$

Finally, it remains to show that $\langle F^{-1}, \alpha^{*^{-1}}\rangle : \mathrm{SEN}''^*_s \to \mathrm{SEN}'^*_s$ is an (N''^*_s, N'^*_s)-epimorphic translation. To do this, we start by proving that, for all $\Sigma \in |\mathbf{Sign}'|$ and all $X \in \mathrm{SEN}''_s(F(\Sigma))$, we have

$$\alpha^{*^{-1}}_\Sigma(X^*) = \alpha^{-1}_\Sigma(X)^*. \tag{7}$$

In fact, for all $\phi \in \mathrm{SEN}'(\Sigma)$, we have

$$\begin{array}{rll}\phi^* \in \alpha_\Sigma^{-1}(X)^* & \text{iff} & \phi \in \alpha_\Sigma^{-1}(X)\\ & \text{iff} & \alpha_\Sigma(\phi) \in X\\ & \text{iff} & \alpha_\Sigma(\phi)^* \in X^*\\ & \text{iff} & \alpha^*_\Sigma(\phi^*) \in X^*\\ & \text{iff} & \phi^* \in {\alpha^*_\Sigma}^{-1}(X^*).\end{array}$$

Thus, ${\alpha^*_\Sigma}^{-1}$ is a well-defined map from $\mathrm{SEN}''^*_s(F(\Sigma))$ into $\mathrm{SEN}'^*_s(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}'|$. To see that it is an (N''^*_s, N'^*_s)-epimorphic translation, let $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N and consider $\Sigma \in |\mathbf{Sign}'|$ and $X_0, \ldots, X_{n-1} \in \mathrm{SEN}''_s(F(\Sigma))$. Then we have

$$\begin{array}{rcl}{\alpha^*_\Sigma}^{-1}(\sigma''^*_{F(\Sigma)}(X^*_0, \ldots, X^*_{n-1})) & = & {\alpha^*_\Sigma}^{-1}(\sigma''_{F(\Sigma)}(X_0, \ldots, X_{n-1})^*)\\ & = & \alpha_\Sigma^{-1}(\sigma''_{F(\Sigma)}(X_0, \ldots, X_{n-1}))^*\\ & = & \sigma'_\Sigma(\alpha_\Sigma^{-1}(X_0), \ldots, \alpha_\Sigma^{-1}(X_{n-1}))^*\\ & = & \sigma'^*_\Sigma(\alpha_\Sigma^{-1}(X_0)^*, \ldots, \alpha_\Sigma^{-1}(X_{n-1})^*)\\ & = & \sigma'^*_\Sigma({\alpha^*_\Sigma}^{-1}(X^*_0), \ldots, {\alpha^*_\Sigma}^{-1}(X^*_{n-1})).\end{array}$$

This concludes the proof that $\langle F^*, \alpha^*\rangle : \mathcal{F}'^* \to \mathcal{F}''^*$ is a well-defined morphism of N-referential algebraic systems. ∎

PROPOSITION 9. *Let* **Sign** *be a category and* SEN : **Sign** → **Set** *a functor, with* N *a category of natural transformations on* SEN. *The assignments* $\mathcal{F} \mapsto \mathcal{F}^*$ *and* $\langle F, \alpha\rangle \mapsto \langle F^*, \alpha^*\rangle$ *define a functor* $(\ )^* : \mathbf{RAS}_N \to \mathbf{RAS}^*_N$.

PROOF. The property of the identities is easily demonstrated. For composition, we have

$$\mathcal{F}' \xrightarrow{\langle F, \alpha\rangle} \mathcal{F}'' \xrightarrow{\langle G, \beta\rangle} \mathcal{F}'''$$

$$\begin{array}{rcl}\beta^*_{F(\Sigma)}(\alpha^*_\Sigma(\phi^*)) & = & \beta^*_{F(\Sigma)}(\alpha_\Sigma(\phi)^*)\\ & = & \beta_{F(\Sigma)}(\alpha_\Sigma(\phi))^*\\ & = & (\beta_{F(\Sigma)} \circ \alpha_\Sigma)^*(\phi^*).\end{array}$$

Therefore $\langle G, \beta\rangle^* \circ \langle F, \alpha\rangle^* = (\langle G, \beta\rangle \circ \langle F, \alpha\rangle)^*$. ∎

We prove, next, some propositions addressing several properties that morphisms of N-referential algebraic systems possess. We start with a proposition asserting that a strict morphism, whose domain is a reduced N-referential algebraic system has injective natural transformation components. In fact, since by definition, its functor component is an isomorphism, the conclusion is equivalent to the morphism being injective.

Proposition 10. *Let* **Sign** *be a category and* SEN : **Sign** $\to$ **Set** *a functor, with N a category of natural transformations on* SEN. *Let* $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F' \rangle\rangle$, $\mathcal{F}'' = \langle \mathrm{SEN}'', \mathrm{SEN}''_s, \langle N''_s, F''\rangle\rangle$ *be two N-referential algebraic systems, such that $\mathcal{F}'$ is reduced. Then, every strict morphism* $\langle F, \alpha\rangle : \mathcal{F}' \to \mathcal{F}''$ *is injective.*

Proof. Suppose that $\Sigma \in |\mathbf{Sign}'|, \phi, \psi \in \mathrm{SEN}'(\Sigma)$, such that $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$. Let $X \in \mathrm{SEN}'_s(\Sigma)$, such that $\phi \in X$. Then, since $\langle F, \alpha\rangle$ is strict, there exists $Y \in \mathrm{SEN}''_s(F(\Sigma))$, such that $\phi \in X = \alpha_\Sigma^{-1}(Y)$. Thus, $\alpha_\Sigma(\psi) = \alpha_\Sigma(\phi) \in Y$. Hence $\psi \in \alpha_\Sigma^{-1}(Y) = X$. By symmetry, we obtain that, for all $X \in \mathrm{SEN}'_s(\Sigma)$, $\phi \in X$ if and only if $\psi \in X$, whence, since $\mathcal{F}'$ is reduced, $\phi = \psi$. Therefore $\langle F, \alpha\rangle$ is injective. ∎

In the following proposition we deal with the setting in which a morphism $\langle F, \alpha\rangle : \mathcal{F}' \to \mathcal{F}''$ relates two N-referential algebraic systems in such a way that it forms a commutative diagram with interpretations $\langle F', \alpha'\rangle$ and $\langle F'', \alpha''\rangle$ into $\mathcal{F}'$ and $\mathcal{F}''$, respectively.

$$\begin{tikzcd}
 & \mathrm{SEN} \arrow[dl, "{\langle F', \alpha'\rangle}"'] \arrow[dr, "{\langle F'', \alpha''\rangle}"] & \\
\mathcal{F}' \arrow[rr, "{\langle F, \alpha\rangle}"'] & & \mathcal{F}''
\end{tikzcd}$$

Let $\mathfrak{F}'$ and $\mathfrak{F}''$ be the two augmented N-referential algebraic systems obtained from $\mathcal{F}'$ and $\mathcal{F}''$, respectively, by adjoining the corresponding interpretations. In this setting, it is shown that, if $\langle F, \alpha\rangle$ is strict, then $C^{\mathfrak{F}''} \leq C^{\mathfrak{F}'}$ and, if $\langle F, \alpha\rangle$ is surjective, then $C^{\mathfrak{F}'} \leq C^{\mathfrak{F}''}$. Therefore, if $\langle F, \alpha\rangle$ is both strict and surjective, then one may infer that $\mathfrak{F}'$ and $\mathfrak{F}''$ generate identical closure systems on SEN.

We start by formulating two lemmas to the effect that the given triangle may be completed if any one of its two legs are provided.

Lemma 11. *Let* **Sign** *be a category and* SEN : **Sign** $\to$ **Set** *a functor, with N a category of natural transformations on* SEN. *Let* $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F'\rangle\rangle$, $\mathcal{F}'' = \langle \mathrm{SEN}'', \mathrm{SEN}''_s, \langle N''_s, F''\rangle\rangle$ *be two N-referential algebraic systems and* $\langle F, \alpha\rangle : \mathcal{F}' \to \mathcal{F}''$ *a morphism of N-referential algebraic systems. Let, also,* $\mathfrak{F}'' = \langle \mathcal{F}'', \langle F'', \alpha''\rangle\rangle$ *be an augmented N-referential algebraic system.*

$$\begin{tikzcd} & \mathrm{SEN} \arrow[dr, "\langle F'', \alpha'' \rangle"] & \\ \mathcal{F}' \arrow[rr, "\langle F, \alpha \rangle"'] & & \mathcal{F}'' \end{tikzcd}$$

Define $F' = F^{-1} \circ F''$ *and, for all* $\Sigma \in |\mathbf{Sign}|$, $\alpha'_\Sigma : \mathrm{SEN}(\Sigma) \to \mathrm{SEN}'_s(F'(\Sigma))$ *by*

$$\alpha'_\Sigma(\phi) = \alpha^{-1}_{F'(\Sigma)}(\alpha''_\Sigma(\phi)), \quad \textit{for all } \phi \in \mathrm{SEN}(\Sigma).$$

Then $\mathfrak{F}' = \langle \mathcal{F}', \langle F', \alpha' \rangle\rangle$ *is also an augmented* N*-referential algebraic system.*

Proof. We must show that $\alpha' : \mathrm{SEN} \to \mathrm{SEN}'_s \circ F'$ is a natural transformation. To this end, suppose $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\phi \in \mathrm{SEN}(\Sigma)$. We have

$$\begin{tikzcd} \mathrm{SEN}(\Sigma) \arrow[r, "\alpha'_\Sigma"] \arrow[d, "\mathrm{SEN}(f)"'] & \mathrm{SEN}'_s(F'(\Sigma)) \arrow[d, "\mathrm{SEN}'_s(F(f))"] \\ \mathrm{SEN}(\Sigma') \arrow[r, "\alpha'_{\Sigma'}"'] & \mathrm{SEN}'_s(F'(\Sigma')) \end{tikzcd}$$

$$\begin{array}{rcl} \alpha'_{\Sigma'}(\mathrm{SEN}(f)(\phi)) & = & \alpha^{-1}_{F'(\Sigma')}(\alpha''_{\Sigma'}(\mathrm{SEN}(f)(\phi))) \\ & = & \alpha^{-1}_{F'(\Sigma')}(\mathrm{SEN}''_s(F''(f))(\alpha''_\Sigma(\phi)) \\ & = & \mathrm{SEN}'_s(F'(f))(\alpha^{-1}_{F'(\Sigma)}(\alpha''_\Sigma(\phi))) \\ & = & \mathrm{SEN}'_s(F'(f))(\alpha'_\Sigma(\phi)). \end{array}$$ ■

Lemma 12. *Let* $\mathbf{Sign}$ *be a category and* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *a functor, with* N *a category of natural transformations on* SEN*. Let* $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F' \rangle\rangle$, $\mathcal{F}'' = \langle \mathrm{SEN}'', \mathrm{SEN}''_s, \langle N''_s, F'' \rangle\rangle$ *be two* N*-referential algebraic systems and* $\langle F, \alpha \rangle : \mathcal{F}' \to \mathcal{F}''$ *a strict surjective morphism of* N*-referential algebraic systems. Let, also,* $\mathfrak{F}' = \langle \mathcal{F}', \langle F', \alpha' \rangle\rangle$ *be an augmented* N*-referential algebraic system.*

$$\begin{tikzcd} & \mathrm{SEN} \arrow[dl, "\langle F', \alpha' \rangle"'] & \\ \mathcal{F}' \arrow[rr, "\langle F, \alpha \rangle"'] & & \mathcal{F}'' \end{tikzcd}$$

Define $F'' = F \circ F'$ *and, for all* $\Sigma \in |\mathbf{Sign}|$, $\alpha''_\Sigma : \mathrm{SEN}(\Sigma) \to \mathrm{SEN}''_s(F''(\Sigma))$ *by*

$$\alpha''_\Sigma(\phi) = \alpha_{F'(\Sigma)}(\alpha'_\Sigma(\phi)), \quad \textit{for all } \phi \in \mathrm{SEN}(\Sigma).$$

Then $\mathfrak{F}'' = \langle \mathcal{F}'', \langle F'', \alpha'' \rangle\rangle$ is also an augmented N-referential algebraic system.

Proof. The crucial point in the proof is that $\alpha'' : \mathrm{SEN} \to \mathrm{SEN}''_s \circ F''$ is well-defined, i.e., that $\alpha''_\Sigma(\phi) \in \mathrm{SEN}''_s(F''(\Sigma))$, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$. To obtain this conclusion, we need the strictness and the surjectivity of $\langle F, \alpha\rangle$. In fact, since $\langle F, \alpha\rangle$ is strict, there exists $X'' \in \mathrm{SEN}''_s(F''(\Sigma))$, such that $\alpha^{-1}_{F'(\Sigma)}(X'') = \alpha'_\Sigma(\phi)$. Therefore,

$$\begin{array}{rcl} \alpha''_\Sigma(\phi) & = & \alpha_{F'(\Sigma)}(\alpha'_\Sigma(\phi)) \\ & = & \alpha_{F'(\Sigma)}(\alpha^{-1}_{F'(\Sigma)}(X'')) \\ & = & X'' \\ & \in & \mathrm{SEN}''_s(F''(\Sigma)), \end{array}$$

where in the last equality the surjectivity of $\langle F, \alpha\rangle$ was used. That α'' is a natural transformation is easy to see. Hence, $\langle F'', \alpha''\rangle$ is indeed an interpretation into $\mathcal{F}''$ and $\mathfrak{F}''$ is an augmented N-referential algebraic system. ■

Next, the main result that was promised before Lemmas 11 and 12 is presented.

Proposition 13. *Let* **Sign** *be a category and* SEN : **Sign** → **Set** *a functor, with N a category of natural transformations on* SEN. *Let $\mathfrak{F}' = \langle \mathcal{F}', \langle F', \alpha'\rangle\rangle$, $\mathfrak{F}'' = \langle \mathcal{F}'', \langle F'', \alpha''\rangle\rangle$ be two augmented N-referential algebraic systems, with $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F'\rangle\rangle$, $\mathcal{F}'' = \langle \mathrm{SEN}'', \mathrm{SEN}''_s, \langle N''_s, F''\rangle\rangle$, and $\langle F, \alpha\rangle : \mathcal{F}' \to \mathcal{F}''$ be a morphism of N-referential algebraic systems, that makes the following triangle commute.*

$$\begin{tikzcd} & \mathrm{SEN} \arrow[dl, "\langle F', \alpha'\rangle"'] \arrow[dr, "\langle F'', \alpha''\rangle"] & \\ \mathcal{F}' \arrow[rr, "\langle F, \alpha\rangle"'] & & \mathcal{F}'' \end{tikzcd}$$

1. *If $\langle F, \alpha\rangle$ is surjective, then $C^{\mathfrak{F}'} \le C^{\mathfrak{F}''}$.*
2. *If $\langle F, \alpha\rangle$ is strict and surjective, then $C^{\mathfrak{F}'} = C^{\mathfrak{F}''}$.*

Proof. Suppose, first, that $\langle F, \alpha\rangle$ is surjective. Let $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\psi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\psi \in C^{\mathfrak{F}'}_\Sigma(\Phi)$. Thus, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\bigcap_{\phi \in \Phi} \alpha'_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \subseteq \alpha'_{\Sigma'}(\mathrm{SEN}(f)(\psi)). \quad (8)$$

Now, let $\chi'' \in \mathrm{SEN}''(F''(\Sigma))$ be such that $\chi'' \in \bigcap_{\phi\in\Phi} \alpha''_{\Sigma'}(\mathrm{SEN}(f)(\phi))$, i.e., such that $\chi'' \in \alpha''_{\Sigma'}(\mathrm{SEN}(f)(\phi))$, for all $\phi \in \Phi$. Since $\langle F, \alpha\rangle$ is surjective, there exists a $\chi' \in \mathrm{SEN}'(F'(\Sigma'))$, such that $\alpha_{F'(\Sigma')}(\chi') = \chi''$. Therefore, $\alpha_{F'(\Sigma')}(\chi') \in \alpha''_{\Sigma'}(\mathrm{SEN}(f)(\phi))$, for all $\phi \in \Phi$. This gives that $\chi' \in \alpha^{-1}_{F'(\Sigma')}(\alpha''_{\Sigma'}(\mathrm{SEN}(f)(\phi)))$, for all $\phi \in \Phi$. Therefore, by Inclusion (8), $\chi' \in \alpha^{-1}_{F'(\Sigma')}(\alpha''_{\Sigma'}(\mathrm{SEN}(f)(\psi)))$, and, hence, $\chi'' = \alpha_{F'(\Sigma')}(\chi') \in \alpha''_{\Sigma'}(\mathrm{SEN}(f)(\psi))$ proving that $\psi \in C^{\mathfrak{F}''}_{\Sigma}(\Phi)$.

Suppose, next, that $\langle F, \alpha\rangle$ is both strict and surjective. Consider $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\psi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\psi \in C^{\mathfrak{F}''}_{\Sigma}(\Phi)$. Thus, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, we have

$$\bigcap_{\phi\in\Phi} \alpha''_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \subseteq \alpha''_{\Sigma'}(\mathrm{SEN}(f)(\psi)). \tag{9}$$

Let $\chi' \in \mathrm{SEN}'(F'(\Sigma))$ be such that $\chi' \in \bigcap_{\phi\in\Phi} \alpha'_{\Sigma'}(\mathrm{SEN}(f)(\phi))$, i.e., $\chi' \in \alpha'_{\Sigma'}(\mathrm{SEN}(f)(\phi))$, for all $\phi \in \Phi$. Then $\alpha_{F(\Sigma')}(\chi') \in \alpha_{F(\Sigma')}(\alpha'_{\Sigma'}(\mathrm{SEN}(f)(\phi)))$, for all $\phi \in \Phi$. This gives that $\alpha_{F(\Sigma')}(\chi') \in \alpha''_{\Sigma'}(\mathrm{SEN}(f)(\phi))$, for all $\phi \in \Phi$, whence, by Inclusion (9), we get that $\alpha_{F(\Sigma')}(\chi') \in \alpha''_{\Sigma'}(\mathrm{SEN}(f)(\psi))$, i.e., that $\alpha_{F(\Sigma')}(\chi') \in \alpha_{F(\Sigma')}(\alpha'_{\Sigma'}(\mathrm{SEN}(f)(\psi)))$. Hence, using strictness and surjectivity, we get that $\chi' \in \alpha'_{\Sigma'}(\mathrm{SEN}(f)(\psi))$, showing that $\psi \in C^{\mathfrak{F}'}_{\Sigma}(\Phi)$. ■

Proposition 13 yields the following corollary in reference to augmented $\mathcal{I}$-N-referential algebraic systems.

Corollary 14. *Suppose $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, with N a category of natural transformations on* SEN. *Let $\mathfrak{F}' = \langle \mathcal{F}', \langle F', \alpha'\rangle\rangle$, $\mathfrak{F}'' = \langle \mathcal{F}'', \langle F'', \alpha''\rangle\rangle$ be two augmented N-referential algebraic systems, with $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F'\rangle\rangle$, $\mathcal{F}'' = \langle \mathrm{SEN}'', \mathrm{SEN}''_s, \langle N''_s, F''\rangle\rangle$, and $\langle F, \alpha\rangle : \mathcal{F}' \to \mathcal{F}''$ be a morphism of N-referential algebraic systems, that makes the following triangle commute.*

$$\begin{tikzcd}
& \mathrm{SEN} \arrow[dl, "\langle F', \alpha'\rangle"'] \arrow[dr, "\langle F'', \alpha''\rangle"] & \\
\mathcal{F}' \arrow[rr, "\langle F, \alpha\rangle"'] & & \mathcal{F}''
\end{tikzcd}$$

1. *If $\langle F, \alpha\rangle$ is surjective and $\mathfrak{F}'$ is an augmented $\mathcal{I}$-N-referential algebraic system, then so is $\mathfrak{F}''$.*
2. *If $\langle F, \alpha\rangle$ is strict and surjective, then $\mathfrak{F}'$ is an augmented $\mathcal{I}$-N-referential algebraic system iff $\mathfrak{F}''$ is also.*

Moreover, since, given an N-referential algebraic system $\mathcal{F}'$, the natural projection $\langle \mathrm{I}, \pi\rangle : \mathcal{F}' \to \mathcal{F}'^{R^{\mathcal{F}'}}$ is a strict and surjective morphism of N-referential algebraic systems, we obtain right away the following

Proposition 15. *Suppose $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, with N a category of natural transformations on* SEN. *Let $\mathfrak{F} = \langle \mathcal{F}, \langle F, \alpha\rangle\rangle$ be an augmented N-referential algebraic system, with $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F'\rangle\rangle$. Then $C^{\mathfrak{F}'} = C^{\mathfrak{F}'^{R^{\mathfrak{F}'}}}$.*

6. Atlas Semantics

Recall that given a functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, with a category N of natural transformations on SEN, an N-**algebraic system** $\mathbf{A}' = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$ is a triple consisting of a functor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, a category N' of natural transformations on SEN' and a surjective functor $F' : N \to N'$, that preserves projections and, as a result, also preserves the arities of all natural transformations.

An N-**atlas system** is a pair $\mathcal{A}' = \langle \mathbf{A}', \mathcal{X}'\rangle$, where $\mathbf{A}'$ is an N-algebraic system, as above, and $\mathcal{X}' = \{\mathcal{X}'_\Sigma\}_{\Sigma \in |\mathbf{Sign}'|}$ is a collection of families $\mathcal{X}_\Sigma$ of subsets of $\mathrm{SEN}'(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}'|$, such that, for all $\Sigma, \Sigma' \in |\mathbf{Sign}'|$, all $f \in \mathbf{Sign}'(\Sigma, \Sigma')$ and all $X \in \mathcal{X}_{\Sigma'}$, $\mathrm{SEN}(f)^{-1}(X) \in \mathcal{X}_\Sigma$.

An **augmented** N-**atlas system** is a pair $\mathfrak{A}' = \langle \mathcal{A}', \langle F, \alpha\rangle\rangle$, where $\mathcal{A}' = \langle \mathbf{A}', \mathcal{X}'\rangle$ is an N-atlas system and $\langle F, \alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$ is an (N, N')-epimorphic translation.

Similarly with the case of augmented N-referential algebraic systems, augmented N-atlas systems generate closure systems on the sentence functor SEN. Define $C^{\mathfrak{A}'} = \{C^{\mathfrak{A}'}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, by letting, for all $\Sigma \in |\mathbf{Sign}|$, $C^{\mathfrak{A}'}_\Sigma : \mathcal{P}\mathrm{SEN}(\Sigma) \to \mathcal{P}\mathrm{SEN}(\Sigma)$ be given, for all $\Phi \cup \{\psi\} \subseteq \mathrm{SEN}(\Sigma)$, by

$$\begin{array}{lll}\psi \in C^{\mathfrak{A}'}_\Sigma(\Phi) & \text{iff} & \alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq X \text{ implies} \\ & & \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi)) \in X, \text{ for all} \\ & & \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma'), X \in \mathcal{X}'_{\Sigma'}.\end{array}$$

The next proposition, an analog of Lemma 1 for augmented N-atlas systems, asserts that the structure $\mathcal{I}^{\mathfrak{A}'} = \langle \mathbf{Sign}, \mathrm{SEN}, C^{\mathfrak{A}'}\rangle$ is a π-institution.

Lemma 16. *Let* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *be a functor, with N a category of natural transformations on* SEN. *If $\mathfrak{A}' = \langle \mathcal{A}', \langle F, \alpha\rangle\rangle$ is an augmented N-atlas system, with $\mathcal{A}' = \langle \mathbf{A}', \mathcal{X}'\rangle$ and $\mathbf{A}' = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, then the triple $\mathcal{I}^{\mathfrak{A}'} = \langle \mathbf{Sign}, \mathrm{SEN}, C^{\mathfrak{A}'}\rangle$ is a π-institution.*

Proof. Reflexivity and monotonicity are straightforward. To show transitivity, suppose $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\psi\} \subseteq \mathrm{SEN}(\Sigma)$ are such that $\psi \in C^{\mathfrak{A}'}_\Sigma(C^{\mathfrak{A}'}_\Sigma(\Phi))$. This means that, for all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $X \in \mathcal{X}'_{\Sigma'}$, we have $\alpha_{\Sigma'}(\mathrm{SEN}(f)(C^{\mathfrak{A}'}_\Sigma(\Phi))) \subseteq X$ implies $\alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi)) \in X$. But, by the definition of $C^{\mathfrak{A}'}$, for all $\phi \in C^{\mathfrak{A}'}_\Sigma(\Phi)$, we also have that $\alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq X$ implies $\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \in X$. Therefore, we obtain

$$\begin{array}{lll}\alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq X & \text{implies} & \alpha_{\Sigma'}(\mathrm{SEN}(f)(C^{\mathfrak{A}'}_\Sigma(\Phi))) \subseteq X \\ & \text{implies} & \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi)) \in X,\end{array}$$

i.e., that $\psi \in C^{\mathfrak{A}'}_\Sigma(\Phi)$, proving the transitivity of $C^{\mathfrak{A}'}$.

Finally, to see that $C^{\mathfrak{A}'}$ is structural, suppose that $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$ and $\Phi \cup \{\psi\} \subseteq \mathrm{SEN}(\Sigma_1)$, such that $\psi \in C^{\mathfrak{A}'}_{\Sigma_1}(\Phi)$. Thus, we have that, for all $\Sigma' \in |\mathbf{Sign}|$, all $g \in \mathbf{Sign}(\Sigma_1, \Sigma')$ and all $X \in \mathcal{X}'_{\Sigma'}$,

$$\begin{tikzcd} \Sigma_1 \arrow[rr, "f"] \arrow[dr, "g"'] & & \Sigma_2 \arrow[dl, "k"] \\ & \Sigma' & \end{tikzcd}$$

$\alpha_{\Sigma'}(\mathrm{SEN}(g)(\Phi)) \subseteq X$ implies $\alpha_{\Sigma'}(\mathrm{SEN}(g)(\psi)) \in X$. Since this holds for all $g \in \mathbf{Sign}(\Sigma_1, \Sigma')$, we have, for all $k \in \mathbf{Sign}(\Sigma_2, \Sigma')$, that $\alpha_{\Sigma'}(\mathrm{SEN}(kf)(\Phi)) \subseteq X$ implies $\alpha_{\Sigma'}(\mathrm{SEN}(kf)(\psi)) \in X$, whence $\alpha_{\Sigma'}(\mathrm{SEN}(k)(\mathrm{SEN}(f)(\Phi))) \subseteq X$ implies $\alpha_{\Sigma'}(\mathrm{SEN}(k)(\mathrm{SEN}(f)(\psi))) \in X$, for all $k \in \mathbf{Sign}(\Sigma_2, \Sigma')$, showing that $\mathrm{SEN}(f)(\psi) \in C^{\mathfrak{A}'}_{\Sigma_2}(\mathrm{SEN}(f)(\Phi))$, i.e., that $C^{\mathfrak{A}'}$ is also structural.

This concludes the proof that the triple $\mathcal{I}^{\mathfrak{A}'} = \langle \mathbf{Sign}, \mathrm{SEN}, C^{\mathfrak{A}'}\rangle$ is a π-institution. ■

If $\mathtt{A}$ is a family of augmented N-atlas systems, then we set, as before,

$$C^{\mathtt{A}} = \bigcap_{\mathfrak{A} \in \mathtt{A}} C^{\mathfrak{A}}.$$

This is also a closure system on SEN and, hence, the triple $\mathcal{I}^{\mathtt{A}} = \langle \mathbf{Sign}, \mathrm{SEN}, C^{\mathtt{A}}\rangle$ is a π-institution as well.

Let, now, $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on SEN. Consider an augmented N-atlas system $\mathfrak{A}' = \langle \mathcal{A}', \langle F, \alpha\rangle\rangle$, with $\mathcal{A}' = \langle \mathbf{A}', \mathcal{X}'\rangle$, $\mathbf{A}' = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$. $\mathfrak{A}'$ is said to be an N-**model** of $\mathcal{I}$ or an **augmented $\mathcal{I}$-N-atlas system** if $C \leq C^{\mathfrak{A}'}$. Moreover, $\mathcal{I}$ will be said to be **complete with respect to a class** $\mathtt{A}$ of augmented

N-atlas system models and $\mathtt{A}$ a **complete** N**-atlas system semantics for** $\mathcal{I}$ if $C = C^{\mathtt{A}}$.

REMARK. Of course, every π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is complete with respect to the class $\mathtt{I} = \{\mathfrak{I}\}$, where $\mathfrak{I} = \langle \mathcal{I}, \langle \mathrm{I}_{\mathbf{Sign}}, \iota\rangle\rangle$, $\mathcal{I} = \langle \mathbf{I}, \mathcal{C}\rangle$ and $\mathbf{I} = \langle \mathrm{SEN}, \langle N, \mathrm{I}_N\rangle\rangle$, with the obvious definitions pertaining to the identity functors and natural transformations involved, $\mathcal{C}$ denoting the closure set system corresponding to the closure operator system C and an overloading of notation for the symbol $\mathcal{I}$, which, hopefully, is clear from the context.

In the previous section, it was shown that every N-referential algebraic system may be reduced by applying the process of dividing out by the relation identifying elements belonging to the same sets of the underlying algebraic system of sets. A similar process may be applied to N-atlas systems.

Suppose that $\mathcal{A}' = \langle \mathbf{A}', \mathcal{X}'\rangle$ is an N-atlas system, with $\mathbf{A}' = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$. The **Frege relation system** $\Lambda(\mathcal{A}') = \{\Lambda_\Sigma(\mathcal{A}')\}_{\Sigma\in|\mathbf{Sign}'|}$ (sometimes denoted $\Lambda^{\mathbf{A}'}(\mathcal{X}') = \{\Lambda^{\mathbf{A}'}_\Sigma(\mathcal{X}')\}_{\Sigma\in|\mathbf{Sign}'|}$) is the collection defined, for all $\Sigma \in |\mathbf{Sign}'|$, by setting, for all $\phi, \psi \in \mathrm{SEN}'(\Sigma)$,

$$\langle \phi, \psi\rangle \in \Lambda_\Sigma(\mathcal{A}') \quad \text{iff} \quad \phi \in X \text{ iff } \psi \in X, \text{ for all } X \in \mathcal{X}_\Sigma.$$

It is very easy to see that the Frege relation system $\Lambda(\mathcal{A}')$ is indeed a relation system in the usual sense of categorical abstract algebraic logic, i.e., that for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}'|$ and all $f \in \mathbf{Sign}'(\Sigma_1, \Sigma_2)$, if $\langle \phi, \psi\rangle \in \Lambda_{\Sigma_1}(\mathcal{A}')$, then $\langle \mathrm{SEN}'(f)(\phi), \mathrm{SEN}'(f)(\psi)\rangle \in \Lambda_{\Sigma_2}(\mathcal{A}')$. In fact, one has, under the hypothesis that $\langle \phi, \psi\rangle \in \Lambda_{\Sigma_1}(\mathcal{A}')$, for all $Y \in \mathcal{X}'_{\Sigma_2}$,

$$\begin{aligned} \mathrm{SEN}'(f)(\phi) \in Y \quad &\text{iff} \quad \phi \in \mathrm{SEN}'(f)^{-1}(Y) \\ &\text{iff} \quad \psi \in \mathrm{SEN}'(f)^{-1}(Y) \\ &\text{iff} \quad \mathrm{SEN}'(f)(\psi) \in Y. \end{aligned}$$

If $\Lambda(\mathcal{A}')$ happens to be an N'-congruence system on SEN', then $\mathcal{A}'$ is said to be a **congruential** N**-atlas system** and if $\Lambda(\mathcal{A}')$ happens to be the identity N'-congruence system on SEN', then $\mathcal{A}'$ is called **Frege reduced**. Since, by default, $\Lambda_\Sigma(\mathcal{A}')$ is compatible with every set in $\mathcal{X}'_\Sigma$, for all $\Sigma \in |\mathbf{Sign}'|$, if it happens to be an N'-congruence system on SEN', then the reduction $\mathcal{A}'^* = \mathcal{A}'/\Lambda(\mathcal{A}')$ is well-defined (see [22] for more details) and it is easily seen that $\mathcal{A}'^*$ is a Frege reduced N-atlas system. The reduction extends to an augmented N-atlas system $\mathfrak{A}' = \langle \mathcal{A}', \langle F, \alpha\rangle\rangle$ by composing the interpretation $\langle F, \alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$ with the natural projection $\langle \mathrm{I}, \pi\rangle : \mathrm{SEN}' \to \mathrm{SEN}'/\Lambda(\mathcal{A}')$ to obtain the pair $\mathfrak{A}'^* = \langle \mathcal{A}'^*, \langle F, \pi_F\alpha\rangle\rangle$. Sometimes we write $\langle F^*, \alpha^*\rangle$ to denote the interpretation $\langle F, \pi_F\alpha\rangle : \mathrm{SEN}' \to \mathrm{SEN}'/\Lambda(\mathcal{A}')$.

Remark. If the π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is N-selfextensional, then the N-atlas system $\mathcal{I} = \langle \mathbf{I}, \mathcal{C}\rangle$ is a congruential N-atlas system. Therefore, every N-selfextensional π-institution $\mathcal{I}$ is complete with respect to the class of all Frege reductions of its congruential N-atlas system models. It is now a tradition in abstract algebraic logic to call the reduction of the augmented N-atlas system $\mathfrak{I} = \langle \mathcal{I}, \langle \mathrm{I}_{\mathbf{Sign}}, \iota\rangle\rangle$, with $\mathcal{I} = \langle \mathbf{I}, C\rangle$, the **Lindenbaum-Tarski N-atlas system model** of $\mathcal{I}$.

7. Morphisms of N-Atlas Systems

We turn, now, to the study of morphisms of N-atlas systems. Suppose that $\mathcal{A}' = \langle \mathbf{A}', \mathcal{X}'\rangle$, $\mathcal{A}'' = \langle \mathbf{A}'', \mathcal{X}''\rangle$, with $\mathbf{A}' = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, $\mathbf{A}'' = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$, are two N-atlas systems. A **morphism of N-atlas systems** $\langle F, \alpha\rangle : \mathcal{A}' \to \mathcal{A}''$ is an N-algebraic system morphism $\langle F, \alpha\rangle : \mathrm{SEN}' \to \mathrm{SEN}''$, such that $\alpha^{-1}(\mathcal{X}'') \leq \mathcal{X}'$, i.e., for all $\Sigma \in |\mathbf{Sign}'|$ and all $X \in \mathcal{X}''_{F(\Sigma)}$ $\alpha_\Sigma^{-1}(X) \in \mathcal{X}'_\Sigma$.

A morphism of N-atlas systems $\langle F, \alpha\rangle : \mathcal{A}' \to \mathcal{A}''$ is said to be **strict** if $\alpha^{-1}(\mathcal{X}'') = \mathcal{X}'$, i.e., if, for all $\Sigma \in |\mathbf{Sign}'|$ and all $X \in \mathcal{X}'_\Sigma$, there exists a $Y \in \mathcal{X}''_{F(\Sigma)}$, such that $X = \alpha_\Sigma^{-1}(Y)$.

The collection of all N-atlas systems with morphisms of N-atlas systems between them forms a category, denoted by $\mathbf{ATS}_N$. The same holds for the collection of all N-atlas systems with strict morphisms of N-atlas systems between them. This category will be denoted by $\mathbf{sATS}_N$. On the other hand, the category with objects all congruential N-atlas systems will be denoted by $\mathbf{CAS}_N$, its subcategory with objects all congruential N-atlas systems with strict morphisms of N-atlas systems between them is denoted by $\mathbf{sCAS}_N$ and the full subcategory of $\mathbf{CAS}_N$ consisting of Frege reduced congruential N-atlas systems by $\mathbf{CAS}^*_N$.

Next, it is shown that every morphism between two congruential N-atlas systems gives rise in a functorial way to a morphism between their Frege reduced counterparts.

Let $\mathcal{A}' = \langle \mathbf{A}', \mathcal{X}'\rangle$, $\mathcal{A}'' = \langle \mathbf{A}'', \mathcal{X}''\rangle$ be two congruential N-atlas systems, with $\mathbf{A}' = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, $\mathbf{A}'' = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$, and $\langle F, \alpha\rangle : \mathcal{A}' \to \mathcal{A}''$ a morphism of N-atlas systems. Define the pair $\langle F^*, \alpha^*\rangle$ by setting $F^* = F$ and, for all $\Sigma \in |\mathbf{Sign}'|$, $\alpha^*_\Sigma : \mathrm{SEN}'^*(\Sigma) \to \mathrm{SEN}''^*(F(\Sigma))$ given, for all $\phi \in \mathrm{SEN}'(\Sigma)$, by

$$\alpha^*_\Sigma(\phi/\Lambda_\Sigma(\mathcal{A}')) = \alpha_\Sigma(\phi)/\Lambda_{F(\Sigma)}(\mathcal{A}'').$$

Sometimes, we write $\phi^* = \phi/\Lambda_\Sigma(\mathcal{A}')$, $\alpha_\Sigma(\phi)^* = \alpha_\Sigma(\phi)/\Lambda_{F(\Sigma)}(\mathcal{A}'')$, etc.

Thus, the definition above may be rewritten as $\alpha^*_\Sigma(\phi^*) = \alpha_\Sigma(\phi)^*$. The notation refers to the appropriate congruence class modulo the appropriate Frege equivalence relation.

PROPOSITION 17. *Let* SEN : **Sign** $\rightarrow$ **Set** *be a functor, with N a category of natural transformations on* SEN*. Let, also, $\mathcal{A}' = \langle \mathbf{A}', \mathcal{X}' \rangle$, $\mathcal{A}'' = \langle \mathbf{A}'', \mathcal{X}'' \rangle$ be two congruential N-atlas systems, with $\mathbf{A}' = \langle \mathrm{SEN}', \langle N', F' \rangle\rangle$, $\mathbf{A}'' = \langle \mathrm{SEN}'', \langle N'', F'' \rangle\rangle$, and $\langle F, \alpha \rangle : \mathcal{A}' \rightarrow \mathcal{A}''$ a morphism of N-atlas systems. Then $\langle F^*, \alpha^* \rangle : \mathcal{A}'^* \rightarrow \mathcal{A}''^*$ is a morphism of N-atlas systems.*

PROOF. We show, first, that $\langle F^*, \alpha^* \rangle$ is well-defined. Consider $\Sigma \in |\mathbf{Sign}'|$ and $\phi, \psi \in \mathrm{SEN}'(\Sigma)$, such that $\langle \phi, \psi \rangle \in \Lambda_\Sigma(\mathcal{A}')$. Then, for all $X \in \mathcal{X}'_\Sigma$, $\phi \in X$ iff $\psi \in X$. Suppose, now, that $Y \in \mathcal{X}''_{F(\Sigma)}$, such that $\alpha_\Sigma(\phi) \in Y$. Then $\phi \in \alpha_\Sigma^{-1}(Y) \in \mathcal{X}'_\Sigma$. Therefore, $\psi \in \alpha_\Sigma^{-1}(Y)$, i.e., $\alpha_\Sigma(\psi) \in Y$. By symmetry, we get that, for all $Y \in \mathcal{X}''_{F(\Sigma)}$, $\alpha_\Sigma(\phi) \in Y$ iff $\alpha_\Sigma(\psi) \in Y$. Thus, $\alpha_\Sigma(\phi)^* = \alpha_\Sigma(\psi)^*$, showing that $\alpha^*_\Sigma(\phi^*) = \alpha^*_\Sigma(\psi^*)$, i.e., α^* is well-defined.

To see that $\alpha^* : \mathrm{SEN}'^* \rightarrow \mathrm{SEN}''^* \circ F$ is a natural transformation, consider $\Sigma, \Sigma' \in |\mathbf{Sign}'|$, $f \in \mathbf{Sign}'(\Sigma, \Sigma')$ and $\phi \in \mathrm{SEN}'(\Sigma)$. Then we have

$$\begin{tikzcd}
\mathrm{SEN}'^*(\Sigma) \arrow[r, "\alpha^*_\Sigma"] \arrow[d, "\mathrm{SEN}'^*(f)"'] & \mathrm{SEN}''^*(F(\Sigma)) \arrow[d, "\mathrm{SEN}''^*(F(f))"] \\
\mathrm{SEN}'^*(\Sigma') \arrow[r, "\alpha^*_{\Sigma'}"'] & \mathrm{SEN}''^*(F(\Sigma'))
\end{tikzcd}$$

$$\begin{aligned}
\alpha^*_{\Sigma'}(\mathrm{SEN}'^*(f)(\phi^*)) &= \alpha^*_{\Sigma'}(\mathrm{SEN}'(f)(\phi)^*) \\
&= \alpha_{\Sigma'}(\mathrm{SEN}'(f)(\phi))^* \\
&= \mathrm{SEN}''(F(f))(\alpha_\Sigma(\phi))^* \\
&= \mathrm{SEN}''^*(F(f))(\alpha_\Sigma(\phi)^*) \\
&= \mathrm{SEN}''^*(F(f))(\alpha^*_\Sigma(\phi^*)).
\end{aligned}$$

Finally, it remains to show that $\langle F^*, \alpha^* \rangle : \mathcal{A}'^* \rightarrow \mathcal{A}''^*$ is a morphism of N-atlas systems, i.e., that, for all $\Sigma \in |\mathbf{Sign}'|$ and all $X^* \in \mathcal{X}''^*_{F(\Sigma)}$, ${\alpha^*_\Sigma}^{-1}(X^*) \in \mathcal{X}'^*_\Sigma$. First, note that, for all $\Sigma \in |\mathbf{Sign}'|$ and all $\phi \in \mathrm{SEN}'\Sigma$ and $X \in \mathcal{X}'_\Sigma$, we have

$$\phi^* \in X^* \quad \text{iff} \quad \phi \in X.$$

Thus, for all $\Sigma \in |\mathbf{Sign}'|$, all $\phi \in \mathrm{SEN}'(\Sigma)$ and all $X \in \mathcal{X}''_{F(\Sigma)}$,

$$\begin{array}{rll}\phi^* \in \alpha_\Sigma^{-1}(X)^* & \text{iff} & \phi \in \alpha_\Sigma^{-1}(X)\\ & \text{iff} & \alpha_\Sigma(\phi) \in X\\ & \text{iff} & \alpha_\Sigma(\phi)^* \in X^*\\ & \text{iff} & \alpha^*_\Sigma(\phi^*) \in X^*\\ & \text{iff} & \phi^* \in {\alpha^*_\Sigma}^{-1}(X^*).\end{array}$$

This shows that, for all $\Sigma \in |\mathbf{Sign}'|$ and all $X \in \mathcal{X}'_\Sigma$, ${\alpha^*_\Sigma}^{-1}(X^*) = \alpha_\Sigma^{-1}(X)^*$. Therefore, we conclude that, for all $\Sigma \in |\mathbf{Sign}'|$,

$$\begin{array}{rll}X \in \mathcal{X}''^*_{F(\Sigma)} & \text{iff} & \pi''^{-1}_{F(\Sigma)}(X) \in \mathcal{X}''_{F(\Sigma)}\\ & \text{implies} & \alpha_\Sigma^{-1}(\pi''^{-1}_{F(\Sigma)}(X)) \in \mathcal{X}'_\Sigma\\ & \text{iff} & \alpha_\Sigma^{-1}(\pi''^{-1}_{F(\Sigma)}(X))^* \in \mathcal{X}'^*_\Sigma\\ & \text{iff} & {\alpha^*_\Sigma}^{-1}(\pi''^{-1}_{F(\Sigma)}(X)^*) \in \mathcal{X}'^*_\Sigma\\ & \text{iff} & {\alpha^*_\Sigma}^{-1}(X) \in \mathcal{X}'^*_\Sigma.\end{array}$$

■

Now it is not difficult to see that the following proposition holds, that provides an extension of the construction of the Frege reduction from the objects of the category of congruential N-institutions to a functor into the category of reduced congruential N-institutions.

Proposition 18. *Let* SEN : **Sign** → **Set** *be a functor, with* N *a category of natural transformations on* SEN. *The assignment* $\mathcal{A} \mapsto \mathcal{A}^*$ *and* $\langle F, \alpha\rangle \mapsto \langle F^*, \alpha^*\rangle$ *defines a functor from the category* $\mathbf{CAS}_N$ *of congruential* N*-atlas systems with* N*-atlas system morphisms between them into the category* $\mathbf{CAS}^*_N$ *of Frege reduced congruential* N*-atlas systems.*

Finally, the section concludes with some properties of morphisms of N-atlas systems paralleling properties that were previously supplied for morphisms between N-referential algebraic systems. These properties will be helpful in the development of the main duality that will be presented in the following sections.

Proposition 19. *Let* SEN : **Sign** → **Set** *be a functor, with* N *a category of natural transformations on* SEN. *Let* $\mathcal{A}' = \langle \mathbf{A}', \mathcal{X}'\rangle$ *and* $\mathcal{A}'' = \langle \mathbf{A}'', \mathcal{X}''\rangle$ *be* N*-atlas systems, with* $\mathbf{A}' = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle, \mathbf{A}'' = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$, *such that* $\mathcal{A}'$ *is Frege reduced, and* $\langle F, \alpha\rangle : \mathcal{A}' \to \mathcal{A}''$ *a strict morphism of* N*-atlas systems. Then* $\langle F, \alpha\rangle$ *has injective natural transformation components.*

Proof. To see this, suppose that $\Sigma \in |\mathbf{Sign}'|$ and $\phi, \psi \in \mathrm{SEN}'(\Sigma)$, such that $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$. Let $X \in \mathcal{X}'_\Sigma$, such that $\phi \in X$. Since $\langle F, \alpha\rangle$ is strict, there exists $Y \in \mathcal{X}''_{F(\Sigma)}$, such that $X = \alpha_\Sigma^{-1}(Y)$. Hence $\phi \in \alpha_\Sigma^{-1}(Y)$, i.e., $\alpha_\Sigma(\phi) \in Y$. Thus, $\alpha_\Sigma(\psi) = \alpha_\Sigma(\phi) \in Y$, showing that $\psi \in \alpha_\Sigma^{-1}(Y) = X$. By symmetry, we conclude that, for all $X \in \mathcal{X}'_\Sigma$, $\phi \in X$ iff $\psi \in X$ and, since $\mathcal{A}'$ is Frege reduced, $\phi = \psi$. Therefore, $\langle F, \alpha\rangle$ has injective natural transformation components. ■

In the sequel, we examine the configuration

$$\begin{tikzcd} & \mathrm{SEN} \arrow[dl, "\langle F', \alpha'\rangle"'] \arrow[dr, "\langle F'', \alpha''\rangle"] & \\ \mathcal{A}' \arrow[rr, "\langle F, \alpha\rangle"'] & & \mathcal{A}'' \end{tikzcd}$$

where $\mathcal{A}', \mathcal{A}''$ are N-atlas systems, forming augmented N-atlas systems $\mathfrak{A}'$, $\mathfrak{A}''$ with the interpretations $\langle F', \alpha'\rangle, \langle F'', \alpha''\rangle$, respectively, and $\langle F, \alpha\rangle : \mathcal{A}' \to \mathcal{A}''$ is a morphism of N-atlas systems. As was the case with morphisms of augmented N-referential algebraic systems, it is shown that this triangle may be appropriately completed under suitable conditions if only one of its two legs are given.

Recall, first, that a functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, with a N a category of natural transformations on SEN, is said to **admit lifting of quotients** if every diagram of morphisms of N-algebraic systems like the one on the left below, with $\langle F, \alpha\rangle$ surjective, may be completed to a commutative diagram like the one on the right below.

$$\begin{tikzcd} & \mathrm{SEN} \arrow[dr, "\langle F'', \alpha''\rangle"] & \\ \mathrm{SEN}' \arrow[rr, two heads, "\langle F, \alpha\rangle"'] & & \mathrm{SEN}'' \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{SEN} \arrow[dl, "\langle F', \alpha'\rangle"'] \arrow[dr, "\langle F'', \alpha''\rangle"] & \\ \mathrm{SEN}' \arrow[rr, two heads, "\langle F, \alpha\rangle"'] & & \mathrm{SEN}'' \end{tikzcd} \tag{10}$$

In that case, it is also said that SEN **admits lifting of N-quotients**.

Directly from the relevant definitions, we obtain the following lemma to the effect that the completion of a triangle, as above, when one of its two legs, having the structure of an augmented N-atlas system, is given, results in both its legs having that same structure.

Lemma 20. *Let* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *be a functor, with N a category of natural transformations on* SEN. *Let* $\mathcal{A}' = \langle \mathbf{A}', \mathcal{X}'\rangle$, $\mathcal{A}'' = \langle \mathbf{A}'', \mathcal{X}''\rangle$ *be two*

N-atlas systems, with $\mathbf{A}' = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, $\mathbf{A}'' = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$, and $\langle F, \alpha\rangle : \mathcal{A}' \to \mathcal{A}''$ a morphism of N-atlas systems.

1. *If $\langle F, \alpha\rangle$ is surjective, $\mathfrak{A}'' = \langle \mathcal{A}'', \langle F'', \alpha''\rangle\rangle$ is an augmented N-atlas system and* SEN *admits lifting of N-quotients, then $\mathfrak{A}' = \langle \mathcal{A}', \langle F', \alpha'\rangle\rangle$, with $\langle F', \alpha'\rangle$ completing the diagram on the left of* (10)*, is an augmented N-atlas system.*
2. *If $\mathfrak{A}' = \langle \mathcal{A}', \langle F', \alpha'\rangle\rangle$ is an augmented N-atlas system, then $\mathfrak{A}'' = \langle \mathcal{A}'', \langle F'', \alpha''\rangle\rangle$, with $\langle F'', \alpha''\rangle := \langle F, \alpha\rangle \circ \langle F', \alpha'\rangle$ is an augmented N-atlas system.*

Proposition 21. *Let* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *be a functor, with N a category of natural transformations on* SEN. *Let $\mathcal{A}' = \langle \mathbf{A}', \mathcal{X}'\rangle$, $\mathcal{A}'' = \langle \mathbf{A}'', \mathcal{X}''\rangle$ be two N-atlas systems, with $\mathbf{A}' = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, $\mathbf{A}'' = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$, and $\langle F, \alpha\rangle : \mathcal{A}' \to \mathcal{A}''$ a morphism of N-atlas systems.*

1. *If* SEN *admits lifting of N-quotients, $\langle F, \alpha\rangle$ is surjective, $\mathfrak{A}'' = \langle \mathcal{A}'', \langle F'', \alpha''\rangle\rangle$ is an augmented N-atlas system and $\langle F', \alpha'\rangle$ results as in Diagram* (10)*, then $C^{\mathfrak{A}'} \leq C^{\mathfrak{A}''}$, where $\mathfrak{A}' = \langle \mathcal{A}', \langle F', \alpha'\rangle\rangle$.*
2. *If $\mathfrak{A}' = \langle \mathcal{A}', \langle F', \alpha'\rangle\rangle$ is an augmented N-atlas system, then $C^{\mathfrak{A}'} \leq C^{\mathfrak{A}''}$, where $\mathfrak{A}'' = \langle \mathcal{A}'', \langle F'', \alpha''\rangle\rangle$, with $\langle F'', \alpha''\rangle := \langle F, \alpha\rangle \circ \langle F', \alpha'\rangle$. Moreover, if $\langle F, \alpha\rangle$ is strict, then $C^{\mathfrak{A}'} = C^{\mathfrak{A}''}$.*

Proof. 1. Suppose that $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\psi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\psi \in C^{\mathfrak{A}'}_\Sigma(\Phi)$, i.e., for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $X \in \mathcal{X}'_{F'(\Sigma')}$,

$$\alpha'_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq X \quad \text{implies} \quad \alpha'_{\Sigma'}(\mathrm{SEN}(f)(\psi)) \in X.$$

Let, now $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $Y \in \mathcal{X}''_{F''(\Sigma')}$, such that $\alpha''_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq Y$. Then $\alpha_{F'(\Sigma')}(\alpha'_{\Sigma'}(\mathrm{SEN}(f)(\Phi))) \subseteq Y$. Therefore, $\alpha'_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq \alpha^{-1}_{F'(\Sigma')}(Y)$. This gives that $\alpha'_{\Sigma'}(\mathrm{SEN}(f)(\psi)) \in \alpha^{-1}_{F'(\Sigma')}(Y)$, i.e., that $\alpha_{F'(\Sigma')}(\alpha'_{\Sigma'}(\mathrm{SEN}(f)(\psi))) \in Y$. This is equivalent to $\alpha''_{\Sigma'}(\mathrm{SEN}(f)(\psi)) \in Y$, showing that $\psi \in C^{\mathfrak{A}''}_\Sigma(\Phi)$. Thus, we get that $C^{\mathfrak{A}'} \leq C^{\mathfrak{A}''}$.

2. That $C^{\mathfrak{A}'} \leq C^{\mathfrak{A}''}$ follows exactly in the same way as in Part 1. Suppose, conversely, that $\langle F, \alpha\rangle$ is strict, $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\psi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\psi \in C^{\mathfrak{A}''}_\Sigma(\Phi)$. Then we have, for all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $Y \in \mathcal{X}''_{F''(\Sigma')}$

$$\alpha''_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq Y \quad \text{implies} \quad \alpha''_{\Sigma'}(\mathrm{SEN}(f)(\psi)) \in Y.$$

Let, now $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $X \in \mathcal{X}'_{F'(\Sigma')}$, such that $\alpha'_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq X$. By strictness, there exists $Y \in \mathcal{X}''_{F(F'(\Sigma'))}$, such that $X = \alpha^{-1}_{F'(\Sigma')}(Y)$. Therefore, we get $\alpha'_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq \alpha^{-1}_{F'(\Sigma')}(Y)$, i.e., $\alpha_{F'(\Sigma')}(\alpha'_{\Sigma'}(\mathrm{SEN}(f)(\Phi))) \subseteq Y$, which implies, using the implication displayed above, that $\alpha_{F'(\Sigma')}(\alpha'_{\Sigma'}(\mathrm{SEN}(f)(\psi))) \in Y$. Thus, we get that $\alpha'_{\Sigma'}(\mathrm{SEN}(f)(\psi)) \in \alpha^{-1}_{F'(\Sigma')}(Y) = X$. This proves that $\psi \in C^{\mathfrak{A}'}_{\Sigma}(\Phi)$ and, hence, that $C^{\mathfrak{A}'} = C^{\mathfrak{A}''}$. ■

Very easily from Proposition 21 one obtains the following

Corollary 22. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN. *Let $\mathcal{A}' = \langle \mathbf{A}', \mathcal{X}'\rangle$, $\mathcal{A}'' = \langle \mathbf{A}'', \mathcal{X}''\rangle$ be two N-atlas systems, with $\mathbf{A}' = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, $\mathbf{A}'' = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$, and $\langle F, \alpha\rangle : \mathcal{A}' \to \mathcal{A}''$ a morphism of N-atlas systems.*

1. *If* SEN *admits lifting of N-quotients, $\langle F, \alpha\rangle$ is surjective, $\mathfrak{A}'' = \langle \mathcal{A}'', \langle F'', \alpha''\rangle\rangle$ is an augmented N-atlas system and $\langle F', \alpha'\rangle$ results as in Diagram* (10)*, then, if $\mathfrak{A}' = \langle \mathcal{A}', \langle F', \alpha'\rangle\rangle$ is an augmented $\mathcal{I}$-N-atlas system, then so is $\mathfrak{A}''$.*
2. *If $\mathfrak{A}' = \langle \mathcal{A}', \langle F', \alpha'\rangle\rangle$ is an augmented $\mathcal{I}$-N-atlas system, then $\mathfrak{A}'' = \langle \mathcal{A}'', \langle F, \alpha\rangle \circ \langle F', \alpha'\rangle\rangle$ is also. Moreover, if $\langle F, \alpha\rangle$ is strict, the implication becomes an equivalence.*

In view of the fact that, for every augmented N-atlas system $\mathfrak{A}' = \langle \mathcal{A}', \langle F, \alpha\rangle\rangle$, with $\mathcal{A}'$ a congruential N-atlas system, we have that $\mathfrak{A}'^* = \langle \mathcal{A}'^*, \langle F^*, \alpha^*\rangle\rangle$ is also an augmented N-atlas system, with $\langle \mathrm{I}, \pi\rangle : \mathcal{A}' \to \mathcal{A}'^*$ a strict morphism of N-atlas systems, that makes the following triangle commute

$$\begin{tikzcd}
 & \mathrm{SEN} \arrow[dl, "{\langle F,\alpha\rangle}"'] \arrow[dr, "{\langle F^*,\alpha^*\rangle}"] & \\
\mathcal{A}' \arrow[rr, "{\langle \mathrm{I},\pi\rangle}"'] & & \mathcal{A}'^*
\end{tikzcd}$$

we also obtain:

Proposition 23. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN. *For every augmented N-atlas system $\mathfrak{A}'$, we have that $C^{\mathfrak{A}'} = C^{\mathfrak{A}'^*}$.*

Finally, the following proposition addresses several cases where some conclusion may be drawn on the preservation of the property of being congruential across morphisms of N-atlas systems.

Proposition 24. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN. *Let $\mathcal{A}' = \langle \mathbf{A}', \langle F', \alpha'\rangle\rangle$, $\mathcal{A}'' = \langle \mathbf{A}'', \langle F'', \alpha''\rangle\rangle$ be two N-atlas systems, with $\mathbf{A}' = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, $\mathbf{A}'' = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$, and $\langle F, \alpha\rangle : \mathcal{A}' \to \mathcal{A}''$ a morphism of N-atlas systems.*

1. *$\Lambda(\mathcal{A}') \leq \alpha^{-1}(\Lambda(\mathcal{A}''))$.*
2. *If $\langle F, \alpha\rangle$ is strict, then $\alpha^{-1}(\Lambda(\mathcal{A}'')) = \Lambda(\mathcal{A}')$.*
3. *If $\langle F, \alpha\rangle$ is strict and surjective, then $\mathcal{A}'$ is congruential iff $\mathcal{A}''$ is congruential.*

Proof. 1. Suppose, first, that $\Sigma \in |\mathbf{Sign}'|, \phi, \psi \in \mathrm{SEN}'(\Sigma)$, such that $\langle \phi, \psi\rangle \in \Lambda_\Sigma(\mathcal{A}')$. Then for every $X \in \mathcal{X}'_\Sigma$, $\phi \in X$ iff $\psi \in X$. Now, let $Y \in \mathcal{X}''_{F(\Sigma)}$, such that $\alpha_\Sigma(\phi) \in Y$. Then $\phi \in \alpha_\Sigma^{-1}(Y)$, whence $\psi \in \alpha_\Sigma^{-1}(Y)$, showing that $\alpha_\Sigma(\psi) \in Y$. By symmetry, for all $Y \in \mathcal{X}''_{F(\Sigma)}$, $\alpha_\Sigma(\phi) \in Y$ iff $\alpha_\Sigma(\psi) \in Y$, showing that $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in \Lambda_{F(\Sigma)}(\mathcal{A}'')$.

2. Suppose, now, that $\langle F, \alpha\rangle$ is strict and let $\Sigma \in |\mathbf{Sign}'|, \phi, \psi \in \mathrm{SEN}'(\Sigma)$, such that $\langle \phi, \psi\rangle \in \alpha_\Sigma^{-1}(\Lambda_{F(\Sigma)}(\mathcal{A}''))$. Then $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in \Lambda_{F(\Sigma)}(\mathcal{A}'')$. This means that, for every $Y \in \mathcal{X}''_{F(\Sigma)}$, $\alpha_\Sigma(\phi) \in Y$ iff $\alpha_\Sigma(\psi) \in Y$. Now, let $X \in \mathcal{X}'_\Sigma$, such that $\phi \in X$. Since $\langle F, \alpha\rangle$ is strict, there exists a $Y \in \mathcal{X}''_{F(\Sigma)}$, such that $X = \alpha_\Sigma^{-1}(Y)$. Thus, $\phi \in \alpha_\Sigma^{-1}(Y)$, i.e., $\alpha_\Sigma(\phi) \in Y$, which yields $\alpha_\Sigma(\psi) \in Y$. This shows that $\psi \in \alpha_\Sigma^{-1}(Y) = X$. By symmetry, for all $X \in \mathcal{X}'_\Sigma$, $\phi \in X$ iff $\psi \in X$, i.e., $\langle \phi, \psi\rangle \in \Lambda_\Sigma(\mathcal{A}')$. Therefore, $\alpha^{-1}(\Lambda(\mathcal{A}'')) \leq \Lambda(\mathcal{A}')$.

3. Suppose that $\mathcal{A}'$ is congruential. Let $\Sigma'' \in |\mathbf{Sign}''|$ and suppose $\phi'', \psi'' \in \mathrm{SEN}''(\Sigma'')$, such that $\langle \phi'', \psi''\rangle \in \Lambda_{\Sigma''}(\mathcal{A}'')$. Then, since $\langle F, \alpha\rangle$ is surjective, there exist $\Sigma' \in |\mathbf{Sign}'|$ and $\phi', \psi' \in \mathrm{SEN}'(\Sigma')$, such that $\Sigma'' = F(\Sigma')$ and $\phi'' = \alpha_{\Sigma'}(\phi')$ and $\psi'' = \alpha_{\Sigma'}(\psi')$. Thus, $\langle \alpha_{\Sigma'}(\phi'), \alpha_{\Sigma'}(\psi')\rangle \in \Lambda_{F(\Sigma')}(\mathcal{A}'')$. This yields, by Part 2 and the strictness of $\langle F, \alpha\rangle$, that $\langle \phi', \psi'\rangle \in \Lambda_{\Sigma'}(\mathcal{A}')$. But $\mathcal{I}'$ is congruential, whence $\langle \phi', \psi'\rangle \in \widetilde{\Omega}^{N'}_{\Sigma'}(\mathcal{A}')$, which yields, by Theorem 21 of [22], that $\langle \phi'', \psi''\rangle = \langle \alpha_{\Sigma'}(\phi'), \alpha_{\Sigma'}(\psi')\rangle \in \widetilde{\Omega}^{N''}_{\Sigma''}(\mathcal{A}'')$, whence $\mathcal{A}''$ is also congruential.

   Suppose, conversely, that $\mathcal{A}''$ is congruential, $\Sigma \in |\mathbf{Sign}'|$ and $\phi', \psi' \in \mathrm{SEN}'(\Sigma)$, such that $\langle \phi', \psi'\rangle \in \Lambda_\Sigma(\mathcal{A}')$. Therefore, by Part 1, $\langle \alpha_\Sigma(\phi'), \alpha_\Sigma(\psi')\rangle \in \Lambda_{F(\Sigma)}(\mathcal{A}'')$. But $\mathcal{A}''$ is congruential, which yields that $\langle \alpha_\Sigma(\phi'), \alpha_\Sigma(\psi')\rangle \in \widetilde{\Omega}^{N''}_{F(\Sigma)}(\mathcal{A}'')$. Thus, again by Theorem 21 of [22], we get that $\langle \phi', \psi'\rangle \in \widetilde{\Omega}^{N'}_\Sigma(\mathcal{A}')$. Therefore $\mathcal{A}'$ is indeed congruential. ■

8. From Atlas to Referential Algebraic Systems

In this section a functor $(\ )^\circ : \mathbf{CAS}_N \to \mathbf{RAS}_N$ from the category of congruential N-atlas systems to the category of N-referential algebraic systems will be constructed. We will be dealing with a fixed functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, with N a category of natural transformations on SEN, and both atlas systems and referential algebraic systems will be in reference to this functor.

Suppose $\mathcal{A}' = \langle \mathbf{A}', \mathcal{X}' \rangle$ is a congruential N-atlas system, with $\mathbf{A}' = \langle \mathrm{SEN}', \langle N', F' \rangle \rangle$, where $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$. Define the N-referential algebraic system $\mathcal{A}'^\circ := \langle \mathrm{SEN}'^\circ, \mathrm{SEN}'^\circ_s, \langle N'^\circ_s, F'^\circ \rangle \rangle$ as follows:

- $\mathrm{SEN}'^\circ : \mathbf{Sign}' \to \mathbf{Set}^{\mathrm{op}}$ is defined on objects by
$$\mathrm{SEN}'^\circ(\Sigma) = \mathcal{X}'_\Sigma, \quad \text{for all } \Sigma \in |\mathbf{Sign}'|,$$
and on morphisms, for all $f \in \mathbf{Sign}'(\Sigma, \Sigma')$, $\mathrm{SEN}'^\circ(f) : \mathrm{SEN}'^\circ(\Sigma') \to \mathrm{SEN}'^\circ(\Sigma)$ is given by
$$\mathrm{SEN}'^\circ(f)(Y) = \mathrm{SEN}'(f)^{-1}(Y), \text{ for all } Y \in \mathcal{X}'_{\Sigma'}.$$
- $\mathrm{SEN}'^\circ_s : \mathbf{Sign}' \to \mathcal{P}\mathrm{SEN}'^\circ$ is the subfunctor of $\overleftarrow{\mathcal{P}}\mathrm{SEN}'^\circ$ defined on objects by
$$\mathrm{SEN}'^\circ_s(\Sigma) = \{\{X \in \mathcal{X}'_\Sigma : \phi \in X\} : \phi \in \mathrm{SEN}'(\Sigma)\},$$
for all $\Sigma \in |\mathbf{Sign}'|$, and on morphisms, for all $f \in \mathbf{Sign}'(\Sigma, \Sigma')$, by $\mathrm{SEN}'^\circ_s(f) : \mathrm{SEN}'^\circ_s(\Sigma) \to \mathrm{SEN}'^\circ_s(\Sigma')$ given, for all $\phi \in \mathrm{SEN}'(\Sigma)$, by
$$\mathrm{SEN}'^\circ_s(f)(\{X \in \mathcal{X}'_\Sigma : \phi \in X\}) = \{Y \in \mathcal{X}'_{\Sigma'} : \mathrm{SEN}'(f)(\phi) \in Y\}.$$
- Finally, denoting by $\eta_\Sigma : \mathrm{SEN}'(\Sigma) \to \mathrm{SEN}'^\circ_s(\Sigma)$ the mapping that assigns $\phi \mapsto \{X \in \mathcal{X}'_\Sigma : \phi \in X\}$, we define, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N,
$$\sigma'^\circ_\Sigma(\eta_\Sigma(\phi_0), \ldots, \eta_\Sigma(\phi_{n-1})) = \eta_\Sigma(\sigma'_\Sigma(\phi_0, \ldots, \phi_{n-1})),$$
for all $\Sigma \in |\mathbf{Sign}'|, \phi_0, \ldots, \phi_{n-1} \in \mathrm{SEN}'(\Sigma)$.
Let $N'^\circ_s = \{\sigma'^\circ : \sigma \text{ in } N\}$ and define $F'^\circ : N \to N'^\circ_s;\ \sigma \mapsto \sigma'^\circ$.

In the next lemma it is shown that this construction yields in fact an N-referential algebraic system.

Lemma 25. *Let* SEN : **Sign** $\to$ **Set** *be a functor, with* N *a category of natural transformations on* SEN. *If* $\mathcal{A}' = \langle \mathbf{A}', \mathcal{X}' \rangle$ *is a congruential* N*-atlas system, with* $\mathbf{A}' = \langle \mathrm{SEN}', \langle N', F' \rangle \rangle$, *where* $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, *then* $\mathcal{A}'^\circ = \langle \mathrm{SEN}'^\circ, \mathrm{SEN}'^\circ_s, \langle N'^\circ_s, F'^\circ \rangle \rangle$ *is an* N*-referential algebraic system.*

PROOF. First, it is shown that SEN'° is a functor. Let $\Sigma, \Sigma', \Sigma'' \in |\mathbf{Sign}'|$, $f \in \mathbf{Sign}'(\Sigma, \Sigma')$, $g \in \mathbf{Sign}'(\Sigma', \Sigma'')$ and $Z \in \mathcal{X}'_{\Sigma''}$. Then

$$\begin{array}{rcl}\mathrm{SEN}'^{\circ}(f)(\mathrm{SEN}'^{\circ}(g)(Z)) & = & \mathrm{SEN}'(f)^{-1}(\mathrm{SEN}'(g)^{-1}(Z)) \\ & = & (\mathrm{SEN}'(g)\mathrm{SEN}'(f))^{-1}(Z) \\ & = & \mathrm{SEN}'(g \circ f)^{-1}(Z) \\ & = & \mathrm{SEN}'^{\circ}(g \circ f)(Z).\end{array}$$

Next, to show that $\mathrm{SEN}'^{\circ}_s : \mathbf{Sign}' \to \mathbf{Set}$ is a subfunctor of $\overleftarrow{\mathcal{P}}\mathrm{SEN}'^{\circ} : \mathbf{Sign}' \to \mathbf{Set}$, we must show that, for all $\Sigma, \Sigma' \in |\mathbf{Sign}'|$ and all $f \in \mathbf{Sign}'(\Sigma, \Sigma')$,

$$\{Y \in \mathcal{X}'_{\Sigma'} : \mathrm{SEN}'(f)(\phi) \in Y\} = \mathrm{SEN}'^{\circ}(f)^{-1}(\{X \in \mathcal{X}'_{\Sigma} : \phi \in X\}).$$

Indeed, we have, for all $Y \in \mathrm{SEN}'^{\circ}(\Sigma')$,

$$\begin{array}{ll} Y \in \mathrm{SEN}'^{\circ}(f)^{-1}(\{X \in \mathcal{X}'_{\Sigma} : \phi \in X\}) & \\ \quad \text{iff} & \mathrm{SEN}'^{\circ}(f)(Y) \in \{X \in \mathcal{X}'_{\Sigma} : \phi \in X\} \\ \quad \text{iff} & \phi \in \mathrm{SEN}'(f)^{-1}(Y) \\ \quad \text{iff} & \mathrm{SEN}'(f)(\phi) \in Y.\end{array}$$

Since $\mathcal{A}'$ is congruential, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, $\sigma'^{\circ} : (\mathrm{SEN}'^{\circ}_s)^n \to \mathrm{SEN}'^{\circ}_s$ is well-defined. So it suffices to show that it is a natural transformation. Let $\Sigma, \Sigma' \in |\mathbf{Sign}'|$, $f \in \mathbf{Sign}'(\Sigma, \Sigma')$ and $\phi_0, \ldots, \phi_{n-1} \in \mathrm{SEN}'(\Sigma)$. Then

$$\begin{tikzcd} \mathrm{SEN}'^{\circ}_s(\Sigma)^n \arrow[r, "\sigma'^{\circ}_{\Sigma}"] \arrow[d, "\mathrm{SEN}'^{\circ}_s(f)^n"'] & \mathrm{SEN}'^{\circ}_s(\Sigma) \arrow[d, "\mathrm{SEN}'^{\circ}_s(f)"] \\ \mathrm{SEN}'^{\circ}_s(\Sigma')^n \arrow[r, "\sigma'^{\circ}_{\Sigma'}"'] & \mathrm{SEN}'^{\circ}_s(\Sigma') \end{tikzcd}$$

$$\begin{array}{l} \sigma'^{\circ}_{\Sigma'}(\mathrm{SEN}'^{\circ}_s(f)(\eta_{\Sigma}(\phi_0)), \ldots, \mathrm{SEN}'^{\circ}_s(f)(\eta_{\Sigma}(\phi_{n-1}))) \\ \qquad = \sigma'^{\circ}_{\Sigma'}(\eta_{\Sigma'}(\mathrm{SEN}'(f)(\phi_0)), \ldots, \eta_{\Sigma'}(\mathrm{SEN}'(f)(\phi_{n-1}))) \\ \qquad = \eta_{\Sigma'}(\sigma'_{\Sigma'}(\mathrm{SEN}'(f)(\phi_0), \ldots, \mathrm{SEN}'(f)(\phi_{n-1}))) \\ \qquad = \eta_{\Sigma'}(\mathrm{SEN}'(f)(\sigma'_{\Sigma}(\phi_0, \ldots, \phi_{n-1}))) \\ \qquad = \mathrm{SEN}'^{\circ}_s(f)(\eta_{\Sigma}(\sigma'_{\Sigma}(\phi_0, \ldots, \phi_{n-1}))) \\ \qquad = \mathrm{SEN}'^{\circ}_s(f)(\sigma'^{\circ}_{\Sigma}(\eta_{\Sigma}(\phi_0), \ldots, \eta_{\Sigma}(\phi_{n-1}))).\end{array}$$

■

Lemma 26. *Let* SEN : **Sign** $\to$ **Set** *be a functor, with* N *a category of natural transformations on* SEN. *Let* $\mathcal{A}' = \langle \mathbf{A}', \mathcal{X}' \rangle$ *be a congruential* N*-atlas system, with* $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F' \rangle \rangle$, *where* $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$. *Then the pair* $\langle \mathrm{I}_{\mathbf{Sign}'}, \eta \rangle : \langle \mathrm{SEN}', \langle N', F' \rangle \rangle \to \langle \mathrm{SEN}'^{\circ}_{s}, \langle N'^{\circ}_{s}, F'^{\circ} \rangle \rangle$ *is an* N*-algebraic system morphism.*

Proof. It must be proved that $\eta : \mathrm{SEN}' \to \mathrm{SEN}'^{\circ}_{s}$ is a natural transformation and that it is (N', N'°_{s})-epimorphic. Suppose that $\Sigma, \Sigma' \in |\mathbf{Sign}'|$, $f \in \mathbf{Sign}'(\Sigma, \Sigma')$ and $\phi \in \mathrm{SEN}'(\Sigma)$. Then

$$\begin{tikzcd}
\mathrm{SEN}'(\Sigma) \arrow[r, "\eta_\Sigma"] \arrow[d, "\mathrm{SEN}'(f)"'] & \mathrm{SEN}'^{\circ}_{s}(\Sigma) \arrow[d, "\mathrm{SEN}'^{\circ}_{s}(f)"] \\
\mathrm{SEN}'(\Sigma') \arrow[r, "\eta_{\Sigma'}"'] & \mathrm{SEN}'^{\circ}_{s}(\Sigma')
\end{tikzcd}$$

$$\begin{array}{rcl}
\eta_{\Sigma'}(\mathrm{SEN}'(f)(\phi)) & = & \{Y \in \mathcal{X}'_{\Sigma'} : \mathrm{SEN}'(f)(\phi) \in Y\} \\
& = & \mathrm{SEN}'^{\circ}_{s}(f)(\{X \in \mathcal{X}'_{\Sigma} : \phi \in X\}) \\
& = & \mathrm{SEN}'^{\circ}_{s}(f)(\eta_\Sigma(\phi)).
\end{array}$$

Finally, for all $\Sigma \in |\mathbf{Sign}'|$ and all $\phi_0, \ldots, \phi_{n-1} \in \mathrm{SEN}'(\Sigma)$,

$$\eta_\Sigma(\sigma'_\Sigma(\phi_0, \ldots, \phi_{n-1})) = \sigma'^{\circ}_{\Sigma}(\eta_\Sigma(\phi_0), \ldots, \eta_\Sigma(\phi_{n-1})),$$

for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, follows from the definition of σ'°. ■

In the following proposition some of the properties of the operator $^{\circ}$, as applied on congruential N-atlas systems, are presented. Proposition 27 forms an analog of Proposition 4.1 of [18].

Proposition 27. *Let* **Sign** *be a category and* SEN : **Sign** $\to$ **Set** *a functor, with* N *a category of natural transformations on* SEN. *Let, also,* $\mathcal{A}' = \langle \mathbf{A}', \mathcal{X}' \rangle$ *be a congruential* N*-atlas system, with* $\mathbf{A}' = \langle \mathrm{SEN}', \langle N', F' \rangle \rangle$.

1. $\mathcal{A}'^{\circ} := \langle \mathrm{SEN}'^{\circ}, \mathrm{SEN}'^{\circ}_{s}, \langle N'^{\circ}_{s}, F'^{\circ} \rangle \rangle$ *is a reduced* N*-referential algebraic system.*
2. *If* $\mathfrak{A}' = \langle \mathcal{A}', \langle F, \alpha \rangle \rangle$ *is an augmented* N*-atlas system, then* $\mathfrak{A}'^{\circ} = \langle \mathcal{A}'^{\circ}, \langle F, \eta \circ \alpha \rangle \rangle$ *is an augmented* N*-referential algebraic system, such that* $\mathcal{I}^{\mathfrak{A}'} = \mathcal{I}^{\mathfrak{A}'^{\circ}}$.
3. *For every* π*-institution* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, $\mathfrak{A}'$ *is an augmented* $\mathcal{I}$-N*-atlas system if and only if* $\mathfrak{A}'^{\circ}$ *is an augmented* $\mathcal{I}$-N*-referential algebraic system.*

4. *If $\mathcal{A}'$ is reduced, then* $\langle \mathrm{I}_{\mathbf{Sign}'}, \eta\rangle : \langle \mathrm{SEN}', \langle N', F'\rangle\rangle \to \langle \mathrm{SEN}'^{\circ}_s, \langle N'^{\circ}_s, F'^{\circ}\rangle\rangle$ *is an isomorphism of N-algebraic systems.*

Proof. 1. Suppose $\Sigma \in |\mathbf{Sign}'|$, $X, X' \in \mathrm{SEN}'^{\circ}(\Sigma)$, such that $\langle X, X'\rangle \in R^{\mathcal{A}'^{\circ}}_{\Sigma}$. Then, for all $\mathcal{X} \in \mathrm{SEN}'^{\circ}_s(\Sigma)$, $X \in \mathcal{X}$ iff $X' \in \mathcal{X}$. But, then, by the definition of $\mathrm{SEN}'^{\circ}_s(\Sigma)$, for all $\phi \in \mathrm{SEN}'(\Sigma)$, $\phi \in X$ iff $\phi \in X'$, which implies $X = X'$. Therefore, $\mathcal{A}'^{\circ}$ is reduced.

2. By Lemma 26 $\langle \mathrm{I}_{\mathbf{Sign}'}, \eta\rangle : \langle \mathrm{SEN}', \langle N', F'\rangle\rangle \to \langle \mathrm{SEN}'^{\circ}_s, \langle N'^{\circ}_s, F'^{\circ}\rangle\rangle$ is an N-algebraic system morphism and, therefore, the composite $\langle F, \eta \circ \alpha\rangle = \langle \mathrm{I}_{\mathbf{Sign}'}, \eta\rangle \circ \langle F, \alpha\rangle$ is also an (N, N'°_s)-epimorphic translation.
   Finally, note that, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\Phi \cup \{\psi\} \subseteq \mathrm{SEN}(\Sigma)$, $X \in \mathcal{X}'_{F(\Sigma')}$, the implication $\alpha_{\Sigma'}(\mathrm{SEN}'(f)(\Phi)) \subseteq X$ implies $\alpha_{\Sigma'}(\mathrm{SEN}'(f)(\psi)) \in X$ is equivalent to
$$\bigcap_{\phi \in \Phi} \eta_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi))) \subseteq \eta_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}'(f)(\psi))).$$

3. This follows directly from Part 2.

4. Since $\langle \mathrm{I}_{\mathbf{Sign}'}, \eta\rangle$ is clearly surjective, it suffices to show that it is also injective. Let $\Sigma \in |\mathbf{Sign}'|$ and $\phi, \psi \in \mathrm{SEN}'(\Sigma)$, such that $\eta_\Sigma(\phi) = \eta_\Sigma(\psi)$. This means that $\{X \in \mathcal{X}'_\Sigma : \phi \in X\} = \{X \in \mathcal{X}'_\Sigma : \psi \in X\}$. Since $\mathcal{A}'$ is reduced, we get that $\phi = \psi$, showing that $\langle \mathrm{I}_{\mathbf{Sign}'}, \eta\rangle$ is an isomorphism of N-algebraic systems. ■

Let **Sign** be a category and SEN : **Sign** $\to$ **Set** a functor, with N a category of natural transformations on SEN. Let, also, $\mathcal{A}' = \langle \mathbf{A}', \mathcal{X}'\rangle$, $\mathcal{A}'' = \langle \mathbf{A}'', \mathcal{X}''\rangle$ be congruential N-atlas systems, with $\mathbf{A}' = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, $\mathbf{A}'' = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$. If $\langle F, \alpha\rangle : \mathcal{A}' \to \mathcal{A}''$ is a morphism of N-atlas systems, with F an isomorphism, then $\alpha^{-1}(\mathcal{X}'') \leq \mathcal{X}'$. Thus, for all $\Sigma \in |\mathbf{Sign}''|$, the map $\alpha^{\circ}_\Sigma : \mathrm{SEN}''^{\circ}(\Sigma) \to \mathrm{SEN}'^{\circ}(F^{-1}(\Sigma))$, defined by $X \mapsto \alpha^{-1}_{F^{-1}(\Sigma)}(X)$ is well-defined. Let us also set $F^{\circ} = F^{-1} : \mathbf{Sign}'' \to \mathbf{Sign}'$.

Proposition 28. *Let* **Sign** *be a category and* SEN : **Sign** $\to$ **Set** *a functor, with N a category of natural transformations on* SEN. *Let, also,* $\mathcal{A}' = \langle \mathbf{A}', \mathcal{X}'\rangle$, $\mathcal{A}'' = \langle \mathbf{A}'', \mathcal{X}''\rangle$ *be congruential N-atlas systems, with* $\mathbf{A}' = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, $\mathbf{A}'' = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$. *If* $\langle F, \alpha\rangle : \mathcal{A}' \to \mathcal{A}''$ *is a morphism of N-atlas systems, with F an isomorphism, then* $\langle F^{\circ}, \alpha^{\circ}\rangle : \mathcal{A}''^{\circ} \to \mathcal{A}'^{\circ}$ *is a morphism of N-referential algebraic systems.*

Proof. First, it must be shown that $\langle F^{\circ}, \alpha^{\circ}\rangle : \mathrm{SEN}''^{\circ} \to \mathrm{SEN}'^{\circ}$ is a translation, i.e., that $\alpha^{\circ} : \mathrm{SEN}''^{\circ} \to \mathrm{SEN}'^{\circ} \circ F^{\circ}$ is a natural transformation. To this end, let $\Sigma, \Sigma' \in |\mathbf{Sign}''|$, $f \in \mathbf{Sign}''(\Sigma, \Sigma')$ and $Y \in \mathrm{SEN}''^{\circ}(\Sigma)$. Then, we have

$$\begin{tikzcd}
\mathrm{SEN}''^{\circ}(\Sigma) \arrow[r, "\alpha^{\circ}_{\Sigma}"] \arrow[d, "\mathrm{SEN}''^{\circ}(f)"'] & \mathrm{SEN}'^{\circ}(F^{\circ}(\Sigma)) \arrow[d, "\mathrm{SEN}'^{\circ}(F^{\circ}(f))"] \\
\mathrm{SEN}''^{\circ}(\Sigma') \arrow[r, "\alpha^{\circ}_{\Sigma'}"'] & \mathrm{SEN}'^{\circ}(F^{\circ}(\Sigma'))
\end{tikzcd}$$

$$\begin{array}{rcl}
\alpha^{\circ}_{\Sigma'}(\mathrm{SEN}''^{\circ}(f)(Y)) & = & \alpha^{\circ}_{\Sigma'}(\mathrm{SEN}''(f)^{-1}(Y)) \\
& = & \alpha^{-1}_{F^{-1}(\Sigma')}(\mathrm{SEN}''(f)^{-1}(Y)) \\
& = & \mathrm{SEN}'(F^{-1}(f))^{-1}(\alpha^{-1}_{F^{-1}(\Sigma)}(Y)) \\
& = & \mathrm{SEN}'^{\circ}(F^{-1}(f))(\alpha^{-1}_{F^{-1}(\Sigma)}(Y)) \\
& = & \mathrm{SEN}'^{\circ}(F^{\circ}(f))(\alpha^{\circ}_{\Sigma}(Y)).
\end{array}$$

Next, it must be shown that $\langle F, \alpha^{\circ^{-1}}\rangle : \mathrm{SEN}'^{\circ}_s \to \mathrm{SEN}''^{\circ}_s$ is a an $(N'^{\circ}_s, N''^{\circ}_s)$-epimorphic translation. First, note that it is well-defined, since, for all $\Sigma \in |\mathbf{Sign}'|$ and all $\mathcal{X} \in \mathrm{SEN}'^{\circ}_s(\Sigma)$, there exists $\phi \in \mathrm{SEN}'(\Sigma)$, such that $\mathcal{X} = \{X \in \mathcal{X}'_{\Sigma} : \phi \in X\}$, whence

$$\begin{array}{rcl}
\alpha^{\circ^{-1}}_{F(\Sigma)}(\mathcal{X}) & = & \alpha^{\circ^{-1}}_{F(\Sigma)}(\{X \in \mathcal{X}'_{\Sigma} : \phi \in X\}) \\
& = & \{Y \in \mathcal{X}''_{F(\Sigma)} : \alpha^{-1}_{\Sigma}(Y) \in \{X \in \mathcal{X}'_{\Sigma} : \phi \in X\}\} \\
& = & \{Y \in \mathcal{X}''_{F(\Sigma)} : \phi \in \alpha^{-1}_{\Sigma}(Y)\} \\
& = & \{Y \in \mathcal{X}''_{F(\Sigma)} : \alpha_{\Sigma}(\phi) \in Y\} \\
& \in & \mathrm{SEN}''^{\circ}_s(F(\Sigma)).
\end{array}$$

Note that this string of equalities has actually shown that, for all $\Sigma \in |\mathbf{Sign}'|$ and all $\phi \in \mathrm{SEN}'(\Sigma)$,

$$\alpha^{\circ^{-1}}_{F(\Sigma)}(\eta'_{\Sigma}(\phi)) = \eta''_{F(\Sigma)}(\alpha_{\Sigma}(\phi)).$$

Let us show now that $\alpha^{\circ^{-1}} : \mathrm{SEN}'^{\circ}_s \to \mathrm{SEN}''^{\circ}_s \circ F$ is a natural transformation. Let $\Sigma, \Sigma' \in |\mathbf{Sign}'|$, $f \in \mathbf{Sign}'(\Sigma, \Sigma')$ and $\mathcal{X} = \{X \in \mathcal{X}'_{\Sigma} : \phi \in X\} = \eta'_{\Sigma}(\phi) \in \mathrm{SEN}'^{\circ}_s(\Sigma)$. Then

$$\begin{tikzcd}
\mathrm{SEN}'^{\circ}_s(\Sigma) \arrow[r, "\alpha^{\circ^{-1}}_{F(\Sigma)}"] \arrow[d, "\mathrm{SEN}'^{\circ}_s(f)"'] & \mathrm{SEN}''^{\circ}_s(F(\Sigma)) \arrow[d, "\mathrm{SEN}''^{\circ}_s(F(f))"] \\
\mathrm{SEN}'^{\circ}_s(\Sigma') \arrow[r, "\alpha^{\circ^{-1}}_{F(\Sigma')}"'] & \mathrm{SEN}''^{\circ}_s(F(\Sigma'))
\end{tikzcd}$$

$$\begin{aligned}
\alpha^{\circ^{-1}}_{F(\Sigma')}(\mathrm{SEN}'^{\circ}_s(f)(\eta'_\Sigma(\phi))) &= \alpha^{\circ^{-1}}_{F(\Sigma')}(\mathrm{SEN}'^{\circ}(f)^{-1}(\eta'_\Sigma(\phi)))\\
&= \alpha^{\circ^{-1}}_{F(\Sigma')}(\eta'_{\Sigma'}(\mathrm{SEN}'(f)(\phi)))\\
&= \eta''_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}'(f)(\phi)))\\
&= \eta''_{F(\Sigma')}(\mathrm{SEN}''(F(f))(\alpha_\Sigma(\phi)))\\
&= \mathrm{SEN}''^{\circ}(F(f))^{-1}(\eta''_{F(\Sigma)}(\alpha_\Sigma(\phi)))\\
&= \mathrm{SEN}''^{\circ}_s(F(f))(\alpha^{\circ^{-1}}_{F(\Sigma)}(\eta'_\Sigma(\phi))).
\end{aligned}$$

Finally, it will be shown that $\langle F, \alpha^{\circ^{-1}}\rangle : \mathrm{SEN}'^{\circ}_s \to \mathrm{SEN}''^{\circ}_s$ is $(N'^{\circ}_s, N''^{\circ}_s)$-epimorphic. To this end, let $\Sigma \in |\mathbf{Sign}'|$, $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N and $\phi_0, \ldots, \phi_{n-1} \in \mathrm{SEN}'(\Sigma)$. Then

$$\begin{aligned}
&\alpha^{\circ^{-1}}_{F(\Sigma)}(\sigma'^{\circ}_\Sigma(\eta'_\Sigma(\phi_0), \ldots, \eta'_\Sigma(\phi_{n-1})))\\
&\quad = \alpha^{\circ^{-1}}_{F(\Sigma)}(\eta'_\Sigma(\sigma'_\Sigma(\phi_0, \ldots, \phi_{n-1}))\\
&\quad = \eta''_{F(\Sigma)}(\alpha_\Sigma(\sigma'_\Sigma(\phi_0, \ldots, \phi_{n-1})))\\
&\quad = \eta''_{F(\Sigma)}(\sigma''_{F(\Sigma)}(\alpha_\Sigma(\phi_0), \ldots, \alpha_\Sigma(\phi_{n-1})))\\
&\quad = \sigma''_{F(\Sigma)}(\eta''_{F(\Sigma)}(\alpha_\Sigma(\phi_0)), \ldots, \eta''_{F(\Sigma)}(\alpha_\Sigma(\phi_{n-1})))\\
&\quad = \sigma''^{\circ}_{F(\Sigma)}(\alpha^{\circ^{-1}}_{F(\Sigma)}(\eta'_\Sigma(\phi_0)), \ldots, \alpha^{\circ^{-1}}_{F(\Sigma)}(\eta'_\Sigma(\phi_{n-1}))).
\end{aligned}$$

■

It is shown next that, if the given morphism of atlas systems is surjective, then the corresponding morphism of reduced referential algebraic systems is strict and that, if the given morphism of atlas systems is strict, then the corresponding morphism of referential algebraic systems is surjective.

PROPOSITION 29. *Let* $\mathbf{Sign}$ *be a category and* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *a functor, with* N *a category of natural transformations on* SEN. *Let, also,* $\mathcal{A}' = \langle \mathbf{A}', \mathcal{X}'\rangle$, $\mathcal{A}'' = \langle \mathbf{A}'', \mathcal{X}''\rangle$ *be congruential* N*-atlas systems, with* $\mathbf{A}' = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, $\mathbf{A}'' = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$, *and* $\langle F, \alpha\rangle : \mathcal{A}' \to \mathcal{A}''$ *a morphism of* N*-atlas systems, with* F *an isomorphism.*

1. *If* $\langle F, \alpha\rangle$ *is surjective, then* $\langle F^\circ, \alpha^\circ\rangle : \mathcal{A}''^{\circ} \to \mathcal{A}'^{\circ}$ *is a strict morphism of* N*-referential algebraic systems.*
2. *If* $\langle F, \alpha\rangle$ *is strict, then* $\langle F^\circ, \alpha^\circ\rangle : \mathcal{A}''^{\circ} \to \mathcal{A}'^{\circ}$ *is a surjective morphism of* N*-referential algebraic systems.*

PROOF. 1. Let $\Sigma \in |\mathbf{Sign}'|$ and $\mathcal{Y} \in \mathrm{SEN}''^{\circ}_s(F(\Sigma))$. By definition of $\mathrm{SEN}''^{\circ}_s(F(\Sigma))$, there exists $\psi \in \mathrm{SEN}''(F(\Sigma))$, such that $\mathcal{Y} = \eta''_{F(\Sigma)}(\psi)$.

Since $\langle F, \alpha\rangle$ is surjective, there exists $\phi \in \mathrm{SEN}'(\Sigma)$, such that $\alpha_\Sigma(\phi) = \psi$. Define $\mathcal{X} = \eta'_\Sigma(\phi) \in \mathrm{SEN}'^{\circ}_s(\Sigma)$. Then we have

$$\alpha^{\circ^{-1}}_{F(\Sigma)}(\mathcal{X}) = \alpha^{\circ^{-1}}_{F(\Sigma)}(\eta'_\Sigma(\phi)) = \eta''_{F(\Sigma)}(\alpha_\Sigma(\phi)) = \eta''_{F(\Sigma)}(\psi) = \mathcal{Y}.$$

Thus, $\langle F^\circ, \alpha^\circ\rangle$ is strict.

2. Suppose $\Sigma \in |\mathbf{Sign}'|$ and $X \in \mathrm{SEN}'^{\circ}(\Sigma)$. Since $\langle F, \alpha\rangle$ is strict, there exists $Y \in \mathrm{SEN}''^{\circ}(F(\Sigma))$, such that $\alpha_\Sigma^{-1}(Y) = X$, i.e., $\alpha^\circ_{F(\Sigma)}(Y) = X$. Thus, $\langle F^\circ, \alpha^\circ\rangle$ is surjective. ∎

Corollary 30. *Let* **Sign** *be a category and* SEN : **Sign** → **Set** *a functor, with N a category of natural transformations on* SEN. *Let, also, $\mathcal{A}' = \langle \mathbf{A}', \mathcal{X}'\rangle$, $\mathcal{A}'' = \langle \mathbf{A}'', \mathcal{X}''\rangle$ be congruential N-atlas systems, with $\mathbf{A}' = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, $\mathbf{A}'' = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$, and $\langle F, \alpha\rangle : \mathcal{A}' \to \mathcal{A}''$ a morphism of N-atlas systems, with F an isomorphism. If $\langle F, \alpha\rangle$ is surjective, then $\langle F^\circ, \alpha^\circ\rangle : \mathcal{A}''^{\circ} \to \mathcal{A}'^{\circ}$ is injective.*

Proof. By Proposition 29, $\langle F^\circ, \alpha^\circ\rangle$ is strict. Moreover, by Proposition 27, $\mathcal{A}''^{\circ}$ is a reduced N-referential algebraic system. Thus, by Proposition 10, $\langle F^\circ, \alpha^\circ\rangle$ is injective. ∎

Corollary 31. *Let* **Sign** *be a category and* SEN : **Sign** → **Set** *a functor, with N a category of natural transformations on* SEN. *For every congruential N-atlas system $\mathcal{A}'$, the N-referential algebraic systems $\mathcal{A}'^{\circ}$ and $\mathcal{A}'^{*\circ}$ are isomorphic.*

Proof. The statement follows from Proposition 29 and the fact that the canonical projection morphism of N-atlas systems $\langle \mathrm{I}, \pi\rangle : \mathcal{A}' \to \mathcal{A}'^{*}$ is strict and surjective. ∎

Corollary 31 implies that the functor $(\ )^\circ : \mathbf{CAS}_N \to \mathbf{RAS}_N$ and its restriction to $\mathbf{CAS}^*_N$ have the same range up to isomorphism.

9. From Referential Algebraic to Atlas Systems

Let **Sign** be a category and SEN : **Sign** → **Set** a functor, with N a category of natural transformations on SEN. Let, also, $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F'\rangle\rangle$ be an N-referential algebraic system. Construct the tuple $\mathcal{F}'^{+} = \langle \mathbf{A}'^{+}, \mathcal{X}'^{+}\rangle$ by setting $\mathbf{A}'^{+} = \langle \mathrm{SEN}'_s, \langle N'_s, F'\rangle\rangle$ and, for every $\Sigma \in |\mathbf{Sign}'|$,

$$\mathcal{X}'^{+}_\Sigma = \{\epsilon_\Sigma(\phi) : \phi \in \mathrm{SEN}'(\Sigma)\}, \text{ with } \epsilon_\Sigma(\phi) = \{X \in \mathrm{SEN}'_s(\Sigma) : \phi \in X\}.$$

Moreover, if $\langle F, \alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'_s$ is an interpretation, then $\langle F^+, \alpha^+\rangle = \langle F, \alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'_s$ is an (N, N'_s)-epimorphic translation.

Proposition 32. *Let* **Sign** *be a category and* SEN : **Sign** $\to$ **Set** *be a functor, with* N *a category of natural transformations on* SEN*.*

1. *If* $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F' \rangle \rangle$ *is an* N*-referential algebraic system, then* $\mathcal{F}'^{+} = \langle \mathbf{A}'^{+}, \mathcal{X}'^{+} \rangle$ *is a Frege-reduced* N*-atlas system.*
2. *Moreover, if* $\mathfrak{F}' = \langle \mathcal{F}', \langle F, \alpha \rangle \rangle$ *is an augmented* N*-referential algebraic system, then* $\mathfrak{F}'^{+} = \langle \mathcal{F}'^{+}, \langle F^{+}, \alpha^{+} \rangle \rangle$ *is an augmented* N*-atlas system.*

Proof. It is not difficult to verify that $\mathcal{F}'^{+} = \langle \mathbf{A}'^{+}, \mathcal{X}'^{+} \rangle$ is a well-defined N-atlas system. In fact, this follows from the fact that, for all $\Sigma, \Sigma' \in |\mathbf{Sign}'|$, all $f \in \mathbf{Sign}'(\Sigma, \Sigma')$ and all $\phi \in \mathrm{SEN}'(\Sigma')$,

$$\mathrm{SEN}'_s(f)^{-1}(\epsilon_{\Sigma'}(\phi)) = \epsilon_\Sigma(\mathrm{SEN}'(f)(\phi)).$$

To see this, let $X \in \mathrm{SEN}'_s(\Sigma)$. Then

$$\begin{array}{lll} X \in \mathrm{SEN}'_s(f)^{-1}(\epsilon_{\Sigma'}(\phi)) & \text{iff} & \mathrm{SEN}'_s(f)(X) \in \epsilon_{\Sigma'}(\phi) \\ & \text{iff} & \phi \in \mathrm{SEN}'(f)^{-1}(X) \\ & \text{iff} & \mathrm{SEN}'(f)(\phi) \in X \\ & \text{iff} & X \in \epsilon_\Sigma(\mathrm{SEN}'(f)(\phi)). \end{array}$$

To see that $\mathcal{F}'^{+}$ is Frege reduced, suppose $\Sigma \in |\mathbf{Sign}'|$ and $X, Y \in \mathrm{SEN}'_s(\Sigma)$, such that $\langle X, Y \rangle \in \Lambda_\Sigma(\mathcal{F}'^{+})$. Then, for all $\phi \in \mathrm{SEN}'(\Sigma)$, $X \in \epsilon_\Sigma(\phi)$ iff $Y \in \epsilon_\Sigma(\phi)$. This is equivalent to $\phi \in X$ iff $\phi \in Y$, for all $\phi \in \mathrm{SEN}'(\Sigma)$. Therefore, $X = Y$ and $\mathcal{F}'^{+}$ is Frege-reduced.

The second statement is obvious in view of the fact that $\langle F^{+}, \alpha^{+} \rangle$: SEN $\to \mathrm{SEN}'_s$ and $\langle F, \alpha \rangle$: SEN $\to \mathrm{SEN}'_s$ are identical. ■

Proposition 33. *Let* **Sign** *be a category and* SEN : **Sign** $\to$ **Set** *be a functor, with* N *a category of natural transformations on* SEN*. Let* $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F' \rangle \rangle$ *be an* N*-referential algebraic system and* $\mathfrak{F}' = \langle \mathcal{F}', \langle F, \alpha \rangle \rangle$ *an augmented* N*-referential algebraic system. Then*

1. $\mathcal{I}^{\mathfrak{F}'} = \mathcal{I}^{\mathfrak{F}'^{+}}$*;*
2. $\mathfrak{F}'$ *is an* $\mathcal{I}$*-*N*-referential algebraic system iff* $\mathfrak{F}'^{+}$ *is an* $\mathcal{I}$*-*N*-atlas system;*
3. *If* $\mathcal{F}'$ *is reduced, then, for all* $\Sigma \in |\mathbf{Sign}'|$, $\epsilon_\Sigma : \mathrm{SEN}'(\Sigma) \to \mathcal{X}'^{+}_\Sigma$ *is a bijection.*

Proof. 1. Suppose, first, that $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\psi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\psi \in C^{\mathfrak{F}'}_\Sigma(\Phi)$. By definition, this means that, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, we have

$$\bigcap_{\phi \in \Phi} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \subseteq \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi)). \tag{11}$$

Let $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\chi \in \mathrm{SEN}'(F(\Sigma'))$, such that $\alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq \epsilon_{F(\Sigma')}(\chi)$. This means that $\chi \in \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi))$, for all $\phi \in \Phi$. Thus, $\chi \in \bigcap_{\phi \in \Phi} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi))$. Therefore, by (11), $\chi \in \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi))$, i.e., $\alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi)) \in \epsilon_{F(\Sigma')}(\chi)$. This shows that $\psi \in C_\Sigma^{\mathfrak{F}'^+}(\Phi)$.

Assume, conversely, that $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\psi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\psi \in C_\Sigma^{\mathfrak{F}'^+}(\Phi)$. This means that, for all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\chi \in \mathrm{SEN}'(F(\Sigma'))$,

$$\alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq \epsilon_{F(\Sigma')}(\chi) \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi)) \in \epsilon_{F(\Sigma')}(\chi). \tag{12}$$

Let $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $X \in \mathrm{SEN}'_s(F(\Sigma'))$, such that $X \in \bigcap_{\phi \in \Phi} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi))$. This implies that, for all $\phi \in \Phi$ and all $\chi \in X$, we have $\chi \in \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi))$, i.e., that, for all $\phi \in \Phi$ and all $\chi \in X$, we have $\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \in \epsilon_{F(\Sigma')}(\chi)$. By (12), $\alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi)) \in \epsilon_{F(\Sigma')}(\chi)$, for all $\chi \in X$, and this is equivalent to $X \in \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi))$. Therefore $\bigcap_{\phi \in \Phi} \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \subseteq \alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi))$, for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, showing that $\psi \in C_\Sigma^{\mathfrak{F}'}(\Phi)$.

2. This follows directly by Part 1.
3. If $\mathcal{F}'$ is reduced, then, for all $\Sigma \in |\mathbf{Sign}'|$ and all $\phi, \psi \in \mathrm{SEN}'(\Sigma)$, if, for all $X \in \mathrm{SEN}'_s(\Sigma)$, $\phi \in X$ iff $\psi \in X$, then we have $\phi = \psi$. Thus, if $\Sigma \in |\mathbf{Sign}'|$ and $\phi, \psi \in \mathrm{SEN}'(\Sigma)$, such that $\epsilon_\Sigma(\phi) = \epsilon_\Sigma(\psi)$, we have that, for all $X \in \mathrm{SEN}'_s(\Sigma)$, $\phi \in X$ iff $\psi \in X$, and, therefore, $\phi = \psi$, i.e., ϵ_Σ is injective. ■

Let **Sign** be a category and SEN : **Sign** → **Set** a functor, with N a category of natural transformations on SEN. Let, also, $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F' \rangle \rangle$ and $\mathcal{F}'' = \langle \mathrm{SEN}'', \mathrm{SEN}''_s, \langle N''_s, F'' \rangle \rangle$ be referential N-algebraic systems and $\langle F, \alpha \rangle : \mathcal{F}' \to \mathcal{F}''$ a morphism of referential algebraic systems. By definition, the functor F is an isomorphism and $\langle F^{-1}, \alpha^{-1} \rangle : \mathrm{SEN}''_s \to \mathrm{SEN}'_s$ is an (N''_s, N'_s)-epimorphic translation. We define $\langle F^+, \alpha^+ \rangle : \mathcal{F}''^+ \to \mathcal{F}'^+$ by setting

$$\langle F^+, \alpha^+ \rangle := \langle F^{-1}, \alpha^{-1} \rangle.$$

Proposition 34. *Let* **Sign** *be a category and* SEN : **Sign** → **Set** *a functor, with N a category of natural transformations on* SEN. *Let, also,* $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F' \rangle \rangle$ *and* $\mathcal{F}'' = \langle \mathrm{SEN}'', \mathrm{SEN}''_s, \langle N''_s, F'' \rangle \rangle$ *be referential N-algebraic systems and* $\langle F, \alpha \rangle : \mathcal{F}' \to \mathcal{F}''$ *a morphism of referential algebraic systems.*

1. *The mapping* $\langle F^+, \alpha^+\rangle : \mathcal{F}''^+ \to \mathcal{F}'^+$ *is a morphism of Frege-reduced* N*-atlas systems;*
2. *If* $\langle F, \alpha\rangle : \mathcal{F}' \to \mathcal{F}''$ *is surjective, then* $\langle F^+, \alpha^+\rangle : \mathcal{F}''^+ \to \mathcal{F}'^+$ *is strict;*
3. *If* $\langle F, \alpha\rangle : \mathcal{F}' \to \mathcal{F}''$ *is strict, then* $\langle F^+, \alpha^+\rangle : \mathcal{F}''^+ \to \mathcal{F}'^+$ *is surjective.*

PROOF. 1. Since $\langle F^{-1}, \alpha^{-1}\rangle : \langle \mathrm{SEN}''_s, \langle N''_s, F''\rangle\rangle \to \langle \mathrm{SEN}'_s, \langle N'_s, F'\rangle\rangle$ is, by definition an (N''_s, N'_s)-epimorphic translation, it suffices to show that, for all $\Sigma \in |\mathbf{Sign}'|$, $(\alpha^{-1}_\Sigma)^{-1} : \mathcal{X}'^+_\Sigma \to \mathcal{X}''^+_{F(\Sigma)}$ is well-defined. In fact, it is shown that, for all $\Sigma \in |\mathbf{Sign}'|$ and all $\phi \in \mathrm{SEN}'(\Sigma)$,

$$(\alpha^{-1}_\Sigma)^{-1}(\epsilon'_\Sigma(\phi)) = \epsilon''_{F(\Sigma)}(\alpha_\Sigma(\phi)).$$

We have

$$\begin{array}{rcl}
(\alpha^{-1}_\Sigma)^{-1}(\epsilon'_\Sigma(\phi)) & = & (\alpha^{-1}_\Sigma)^{-1}(\{X \in \mathrm{SEN}'_s(\Sigma) : \phi \in X\}) \\
& = & \{Y \in \mathrm{SEN}''_s(F(\Sigma)) : \\
& & \qquad \alpha^{-1}_\Sigma(Y) \in \{X \in \mathrm{SEN}'_s(\Sigma) : \phi \in X\}\} \\
& = & \{Y \in \mathrm{SEN}''_s(F(\Sigma)) : \phi \in \alpha^{-1}_\Sigma(Y)\} \\
& = & \{Y \in \mathrm{SEN}''_s(F(\Sigma)) : \alpha_\Sigma(\phi) \in Y\} \\
& = & \epsilon''_{F(\Sigma)}(\alpha_\Sigma(\phi)).
\end{array}$$

2. Suppose that $\langle F, \alpha\rangle$ is surjective. Let $\Sigma \in |\mathbf{Sign}'|$ and $Y \in \mathcal{X}''^+_{F(\Sigma)}$. Then, there exists $\psi \in \mathrm{SEN}''(F(\Sigma))$, such that $Y = \epsilon''_{F(\Sigma)}(\psi)$. Since $\langle F, \alpha\rangle$ is surjective, there exists $\phi \in \mathrm{SEN}'(\Sigma)$, such that $\psi = \alpha_\Sigma(\phi)$. Set $X = \epsilon'_\Sigma(\phi) \in \mathcal{X}'^+_\Sigma$. Then

$$\alpha^{+^{-1}}_{F(\Sigma)}(X) = (\alpha^{-1}_\Sigma)^{-1}(\epsilon'_\Sigma(\phi)) = \epsilon''_{F(\Sigma)}(\alpha_\Sigma(\phi)) = \epsilon''_{F(\Sigma)}(\psi) = Y.$$

Therefore $\langle F^+, \alpha^+\rangle$ is strict.

3. If $\langle F, \alpha\rangle : \mathcal{F}' \to \mathcal{F}''$ is strict, then $\alpha^{-1} : \mathrm{SEN}''_s \circ F \to \mathrm{SEN}'_s$ is surjective, by definition. Thus, $\langle F^+, \alpha^+\rangle : \mathcal{F}''^+ \to \mathcal{F}'^+$ is surjective. ∎

COROLLARY 35. *Let* **Sign** *be a category and* SEN : **Sign** → **Set** *a functor, with* N *a category of natural transformations on* SEN. *Let, also,* $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F'\rangle\rangle$ *and* $\mathcal{F}'' = \langle \mathrm{SEN}'', \mathrm{SEN}''_s, \langle N''_s, F''\rangle\rangle$ *be* N*-referential algebraic systems and* $\langle F, \alpha\rangle : \mathcal{F}' \to \mathcal{F}''$ *a morphism of referential algebraic systems. If* $\langle F, \alpha\rangle$ *is surjective, then* $\langle F^+, \alpha^+\rangle : \mathcal{F}''^+ \to \mathcal{F}'^+$ *is injective.*

PROOF. If $\langle F, \alpha\rangle$ is surjective then, by Proposition 34, Part (2), $\langle F^+, \alpha^+\rangle$ is a strict morphism of reduced N-atlas systems. Thus, by Proposition 19, $\langle F^+, \alpha^+\rangle$ is injective. ∎

Corollary 36. *Let* **Sign** *be a category and* SEN : **Sign** $\to$ **Set** *a functor, with* N *a category of natural transformations on* SEN. *For every* N*-referential algebraic system* $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F'\rangle\rangle$, $\mathcal{F}'^{+}$ *and* $\mathcal{F}'^{*+}$ *are isomorphic reduced congruential* N*-atlas systems.*

Proof. Consider the natural quotient morphism of N-referential algebraic systems $\langle \mathrm{I}, \pi\rangle : \mathcal{F}' \to \mathcal{F}'^{*}$. It is strict and surjective, whence the morphism $\langle \mathrm{I}, \pi^{+}\rangle : \mathcal{F}'^{*+} \to \mathcal{F}'^{+}$ is a strict, surjective and injective morphism of N-atlas systems, i.e., an isomorphism. ■

Corollary 36 shows that the functor $(\ )^{+} : \mathbf{RAS}_N \to \mathbf{CAS}_N$ and its restriction to $\mathbf{RAS}^{*}_N$ have the same range up to isomorphism of N-atlas systems.

10. Duality

In this section, it will be shown that the category $\mathbf{CAS}^{*}_N$ of reduced congruential N-atlas systems is dually equivalent to the category $\mathbf{RAS}^{*}_N$ of reduced N-referential algebraic systems. The two functors $(\ )^{\circ} : \mathbf{CAS}_N \to \mathbf{RAS}_N$ and $(\ )^{+} : \mathbf{RAS}_N \to \mathbf{CAS}_N$ that were defined in the preceding section will be employed. More precisely, their restrictions to the respective subcategories of reduced objects will establish the desired duality between $\mathbf{CAS}^{*}_N$ and $\mathbf{RAS}^{*}_N$.

Let us denote these two functors, i.e., the restriction of $(\ )^{\circ} : \mathbf{CAS}_N \to \mathbf{RAS}_N$ to $\mathbf{CAS}^{*}_N$ and the restriction of $(\ )^{+} : \mathbf{RAS}_N \to \mathbf{CAS}_N$ to $\mathbf{RAS}^{*}_N$ by the same letters. Thus, we have $^{\circ} : \mathbf{CAS}^{*}_N \to \mathbf{RAS}^{*}_N$ and $^{+} : \mathbf{RAS}^{*}_N \to \mathbf{CAS}^{*}_N$. To establish the duality between $\mathbf{CAS}^{*}_N$ and $\mathbf{RAS}^{*}_N$, the unit and the counit of the relevant adjunction will be defined and the fact that they are isomorphisms will be proven.

First, suppose that $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F'\rangle\rangle$ is a reduced N-referential algebraic system. Let us denote $\mathcal{F}'^{+} = \langle\langle \mathrm{SEN}'_s, \langle N'_s, F'\rangle\rangle, \mathcal{X}'^{+}\rangle$ and $\mathcal{F}'^{+\circ} = \langle \mathrm{SEN}'^{\circ}_s, \mathrm{SEN}'^{\circ}_{ss}, \langle N'^{\circ}_{ss}, F'^{\circ}\rangle\rangle$. Define the pair $\langle \mathrm{I}_{\mathbf{Sign}'}, \epsilon^{\mathcal{F}'}\rangle : \mathcal{F}'^{+\circ} \to \mathcal{F}'$ by letting, for all $\Sigma \in |\mathbf{Sign}'|$ $\epsilon^{\mathcal{F}'}_{\Sigma} : \mathrm{SEN}'(\Sigma) \to \mathrm{SEN}'^{\circ}_s(\Sigma)$, be given, for all $\phi \in \mathrm{SEN}'(\Sigma)$, by

$$\epsilon^{\mathcal{F}'}_{\Sigma}(\phi) = \{X \in \mathrm{SEN}'_s(\Sigma) : \phi \in X\}.$$

Moreover, given a reduced N-atlas system $\mathcal{A}' = \langle\langle \mathrm{SEN}', \langle N', F'\rangle\rangle, \mathcal{X}'\rangle$, denote $\mathcal{A}'^{\circ} = \langle \mathrm{SEN}'^{\circ}, \mathrm{SEN}'^{\circ}_s, \langle N'^{\circ}_s, F'^{\circ}\rangle\rangle$ and $\mathcal{A}'^{\circ+} = \langle\langle \mathrm{SEN}'^{\circ}_s, \langle N'^{\circ}_s, F'^{\circ}\rangle\rangle, \mathcal{X}'^{\circ+}\rangle$. Define $\langle \mathbf{I}_{\mathbf{Sign}'}, \eta^{\mathcal{A}'}\rangle : \mathcal{A}' \to \mathcal{A}'^{\circ+}$ by letting, for all $\Sigma \in |\mathbf{Sign}'|$,

$\eta^{\mathcal{A}'}_\Sigma : \mathrm{SEN}'(\Sigma) \to \mathrm{SEN}'^{\circ}_s(\Sigma)$ be given, for all $\phi \in \mathrm{SEN}'(\Sigma)$, by

$$\eta^{\mathcal{A}'}_\Sigma(\phi) = \{X \in \mathcal{X}'_\Sigma : \phi \in X\}.$$

Let $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F'\rangle\rangle$ be a reduced N-referential algebraic system. Recall that $\mathcal{F}'^{+} = \langle\langle \mathrm{SEN}'_s, \langle N'_s, F'\rangle\rangle, \mathcal{X}'^{+}\rangle$, where

$$\mathcal{X}'^{+}_\Sigma = \{\epsilon^{\mathcal{F}'}_\Sigma(\phi) : \phi \in \mathrm{SEN}'(\Sigma)\},$$

for all $\Sigma \in |\mathbf{Sign}'|$, and $\mathcal{F}'^{+\circ} = \langle \mathrm{SEN}'^{\circ}_s, \mathrm{SEN}'^{\circ}_{ss}, \langle N'^{\circ}_{ss}, F'^{\circ}\rangle\rangle$, where

$$\mathrm{SEN}'^{\circ}_s(\Sigma) = \mathcal{X}'^{+}_\Sigma = \{\epsilon^{\mathcal{F}'}_\Sigma(\phi) : \phi \in \mathrm{SEN}'(\Sigma)\},$$

for all $\Sigma \in |\mathbf{Sign}'|$.

Proposition 37. *Let* **Sign** *be a category and* SEN : **Sign** $\to$ **Set** *a functor, with* N *a category of natural transformations on* SEN. *For every reduced referential* N*-algebraic system* $\mathcal{F}' = \langle \mathrm{SEN}', \mathrm{SEN}'_s, \langle N'_s, F'\rangle\rangle$, $\langle \mathrm{I}_{\mathbf{Sign}'}, \epsilon^{\mathcal{F}'}\rangle : \mathcal{F}'^{+\circ} \to \mathcal{F}'$ *is a strict and bijective morphism of referential* N*-algebraic systems. Thus,* $\mathcal{F}'$ *and* $\mathcal{F}'^{+\circ}$ *are isomorphic.*

Proof. Since, taking into account that $\mathcal{F}'$ is reduced, it is clear from the relevant definitions that $\langle \mathrm{I}_{\mathbf{Sign}'}, \epsilon^{\mathcal{F}'}\rangle : \mathcal{F}'^{+\circ} \to \mathcal{F}'$ is bijective, we only show that it is a valid morphism of referential N-algebraic systems and is also strict.

For the first assertion, we must show that, for all $\Sigma \in |\mathbf{Sign}'|$ and all $\mathfrak{X} \in \mathrm{SEN}'^{\circ}_{ss}(\Sigma)$, $(\epsilon^{\mathcal{F}'}_\Sigma)^{-1}(\mathfrak{X}) \in \mathrm{SEN}'_s(\Sigma)$. By the definition of $\mathrm{SEN}'^{\circ}_{ss}(\Sigma)$, there exists $X \in \mathrm{SEN}'_s(\Sigma)$, such that $\mathfrak{X} = \{\mathcal{X} \in \mathcal{X}'^{+}_\Sigma : X \in \mathcal{X}\}$. Thus, we have

$$\begin{array}{rcl}
(\epsilon^{\mathcal{F}'}_\Sigma)^{-1}(\mathfrak{X}) & = & \{\phi \in \mathrm{SEN}'(\Sigma) : \epsilon^{\mathcal{F}'}_\Sigma(\phi) \in \mathfrak{X}\}\\
& = & \{\phi \in \mathrm{SEN}'(\Sigma) : \epsilon^{\mathcal{F}'}_\Sigma(\phi) \in \{\mathcal{X} \in \mathcal{X}'^{+}_\Sigma : X \in \mathcal{X}\}\}\\
& = & \{\phi \in \mathrm{SEN}'(\Sigma) : X \in \epsilon^{\mathcal{F}'}_\Sigma(\phi)\}\\
& = & \{\phi \in \mathrm{SEN}'(\Sigma) : \phi \in X\}\\
& = & X \in \mathrm{SEN}'_s(\Sigma).
\end{array}$$

This calculation also shows that $\langle \mathrm{I}_{\mathbf{Sign}'}, \epsilon^{\mathcal{F}'}\rangle : \mathcal{F}'^{+\circ} \to \mathcal{F}'$ is strict, since, given $\Sigma \in |\mathbf{Sign}'|$ and $X \in \mathrm{SEN}'_s(\Sigma)$, we have that $\mathfrak{X} = \{\mathcal{X} \in \mathcal{X}'^{+}_\Sigma : X \in \mathcal{X}\} \in \mathrm{SEN}'^{\circ}_{ss}(\Sigma)$ and $(\epsilon^{\mathcal{F}'}_\Sigma)^{-1}(\mathfrak{X}) = X$. ■

Proposition 38. *Let* **Sign** *be a category and* SEN : **Sign** $\to$ **Set** *a functor, with* N *a category of natural transformations on* SEN. *For every reduced* N*-atlas system* $\mathcal{A}' = \langle\langle \mathrm{SEN}', \langle N', F'\rangle\rangle, \mathcal{X}'\rangle$, $\langle \mathrm{I}_{\mathbf{Sign}'}, \eta^{\mathcal{A}'}\rangle : \mathcal{A}' \to \mathcal{A}'^{\circ+}$ *is a strict and bijective morphism of* N*-atlas systems. Thus,* $\mathcal{A}'$ *and* $\mathcal{A}'^{\circ+}$ *are isomorphic.*

Proof. It is clear from the relevant definitions that $\langle \mathrm{I}_{\mathbf{Sign}'}, \eta^{\mathcal{A}'} \rangle : \mathcal{A}' \to \mathcal{A}'^{\circ +}$ is bijective. So it will only be shown that it is a valid morphism of N-atlas systems and strict.

For the first assertion, we must show that, for all $\Sigma \in |\mathbf{Sign}'|$ and all $\mathfrak{X} \in \mathcal{X}'^{\circ +}_{\Sigma}$, $(\eta^{\mathcal{A}'}_{\Sigma})^{-1}(\mathfrak{X}) \in \mathcal{X}'_{\Sigma}$. By the definition of $\mathcal{X}'^{\circ +}_{\Sigma}$, there exists $X \in \mathrm{SEN}'^{\circ}(\Sigma) = \mathcal{X}'_{\Sigma}$, such that $\mathfrak{X} = \{\mathcal{X} \in \mathrm{SEN}'^{\circ}_{s}(\Sigma) : X \in \mathcal{X}\}$. Thus, we have

$$\begin{array}{rcl}
(\eta^{\mathcal{A}'}_{\Sigma})^{-1}(\mathfrak{X}) & = & (\eta^{\mathcal{A}'}_{\Sigma})^{-1}(\{\mathcal{X} \in \mathrm{SEN}'^{\circ}_{s}(\Sigma) : X \in \mathcal{X}\}) \\
& = & \{\phi \in \mathrm{SEN}'(\Sigma) : \eta^{\mathcal{A}'}_{\Sigma}(\phi) \in \{\mathcal{X} \in \mathrm{SEN}'^{\circ}_{s}(\Sigma) : X \in \mathcal{X}\}\} \\
& = & \{\phi \in \mathrm{SEN}'(\Sigma) : X \in \eta^{\mathcal{A}'}_{\Sigma}(\phi)\} \\
& = & \{\phi \in \mathrm{SEN}'(\Sigma) : \phi \in X\} \\
& = & X \in \mathcal{X}'_{\Sigma}.
\end{array}$$

This calculation also shows that $\langle \mathrm{I}_{\mathbf{Sign}'}, \eta^{\mathcal{A}'} \rangle : \mathcal{A}' \to \mathcal{A}'^{\circ +}$ is strict, since, given $\Sigma \in |\mathbf{Sign}'|$ and $X \in \mathcal{X}'_{\Sigma}$, we have that $\mathfrak{X} = \{\mathcal{X} \in \mathrm{SEN}'^{\circ}_{s}(\Sigma) : X \in \mathcal{X}\} \in \mathcal{X}'^{\circ +}_{\Sigma}$ and $(\eta^{\mathcal{A}'}_{\Sigma})^{-1}(\mathfrak{X}) = X$. ∎

Proposition 39. *Let* **Sign** *be a category and* SEN : **Sign** → **Set** *a functor, with N a category of natural transformations on* SEN. *Then* $\epsilon : (\ )^{+\circ} \to \mathrm{I}_{\mathbf{RAS}^*_N}$, *with* $\epsilon = \{\langle \mathrm{I}_{\mathbf{Sign}'}, \epsilon^{\mathcal{F}'} \rangle : \mathcal{F}' \in |\mathbf{RAS}^*_N|\}$, *and* $\eta : \mathrm{I}_{\mathbf{CAS}^*_N} \to (\ )^{\circ +}$, *with* $\eta = \{\langle \mathrm{I}_{\mathbf{Sign}'}, \eta^{\mathcal{A}'} \rangle : \mathcal{A}' \in |\mathbf{CAS}^*_N|\}$, *are natural transformations.*

Proof. We must show that the following two rectangles commute, for all reduced N-referential algebraic systems $\mathcal{F}', \mathcal{F}''$, all morphisms $\langle F, \alpha \rangle : \mathcal{F}' \to \mathcal{F}''$ of N-referential algebraic systems, all Frege reduced N-atlas systems $\mathcal{A}'$, $\mathcal{A}''$ and all morphisms of N-atlas systems $\langle G, \beta \rangle : \mathcal{A}' \to \mathcal{A}''$.

$$\begin{tikzcd}
\mathcal{F}'^{+\circ} \arrow[r, "\langle \mathrm{I}_{\mathbf{Sign}'}, \epsilon^{\mathcal{F}'} \rangle"] \arrow[d, "\langle F, \alpha^{+\circ} \rangle"'] & \mathcal{F}' \arrow[d, "\langle F, \alpha \rangle"] \\
\mathcal{F}''^{+\circ} \arrow[r, "\langle \mathrm{I}_{\mathbf{Sign}''}, \epsilon^{\mathcal{F}''} \rangle"'] & \mathcal{F}''
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A}' \arrow[r, "\langle \mathrm{I}_{\mathbf{Sign}'}, \eta^{\mathcal{A}'} \rangle"] \arrow[d, "\langle G, \beta \rangle"'] & \mathcal{A}'^{\circ +} \arrow[d, "\langle G, \beta^{\circ +} \rangle"] \\
\mathcal{A}'' \arrow[r, "\langle \mathrm{I}_{\mathbf{Sign}''}, \eta^{\mathcal{A}''} \rangle"'] & \mathcal{A}''^{\circ +}
\end{tikzcd}$$

We only demonstrate the commutativity of the first square, since that of the second may be shown similarly. Let $\Sigma \in |\mathbf{Sign}'|$ and $\phi \in \mathrm{SEN}'(\Sigma')$ and recall that $\epsilon^{\mathcal{F}'}_{\Sigma} : \mathrm{SEN}'(\Sigma) \to \mathrm{SEN}'^{\circ}_{s}(\Sigma)$. We have

$$\begin{array}{rcl}
\alpha_\Sigma^{+\circ}(\epsilon_\Sigma^{\mathcal{F}'}(\phi)) & = & \alpha_\Sigma^{+\circ}(\{X \in \mathrm{SEN}'_s(\Sigma) : \phi \in X\}) \\
& = & (\alpha_\Sigma^{+})^{-1}(\{X \in \mathrm{SEN}'_s(\Sigma) : \phi \in X\}) \\
& = & \{Y \in \mathrm{SEN}'_s(F(\Sigma)) : \alpha_\Sigma^{+}(Y) \in \{X \in \mathrm{SEN}'_s(\Sigma) : \phi \in X\}\} \\
& = & \{Y \in \mathrm{SEN}'_s(F(\Sigma)) : \phi \in \alpha_\Sigma^{-1}(Y)\} \\
& = & \{Y \in \mathrm{SEN}'_s(F(\Sigma)) : \alpha_\Sigma(\phi) \in Y\} \\
& = & \epsilon_{F(\Sigma)}^{\mathcal{F}''}(\alpha_\Sigma(\phi)).
\end{array}$$
■

THEOREM 40. *Let* **Sign** *be a category and* SEN : **Sign** $\to$ **Set** *a functor, with* N *a category of natural transformations on* SEN*. The categories* $\mathbf{CAS}_N^*$ *and* $\mathbf{RAS}_N^*$ *are dually equivalent categories through* $(\ )^\circ : \mathbf{CAS}_N^* \to \mathbf{RAS}_N^*$ *and* $(\ )^+ : \mathbf{RAS}_N^* \to \mathbf{CAS}_N^*$*.*

PROOF. This follows from Propositions 37, 38 and 39. ■

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The present line of work is far from over. In additional research, currently in progress, the very general duality presented in this paper will be used, along the lines of the sentential paradigm, to obtain a characterization of classes of fully self-extensional π-institutions inside corresponding classes of self-extensional π-institutions. This characterization requires, on the one hand, the extension of the present duality to one between referential algebraic models of a given fixed π-institution $\mathcal{I}$, with a given category of natural transformations on its sentence functor, and corresponding atlas system models, and, on the other, a careful treatment of the class $\mathsf{Alg}\mathcal{I}$ of the algebraic systems naturally associated with $\mathcal{I}$, which consists of the algebraic system reducts of the Tarski-reduced full models of $\mathcal{I}$.

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