Categorical Abstract Algebraic Logic:
Operations on Classes of Models

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Abstract

Constructions on π-institutions, such as taking products and filtered products, are introduced. Based on these constructions, operators on classes of π-institution models of a given π-institution are defined. It is shown that the class of π-institution models of a given π-institution is closed under some of these π-institution model class operators. Moreover, given a collection of models of a π-institution $\mathcal{I}$, strongly adequate for $\mathcal{I}$, a closure operator generating the entire class of all π-institution models of $\mathcal{I}$ out of the given collection of models is provided.

1 Introduction

One of the greatest achievements of the theory of abstract algebraic logic, as developed by Czelakowski, Blok and Pigozzi and Font and Jansana, among others, is the classification of sentential logics in an abstract algebraic hierarchy whose steps provide a measure of the "algebraizability" of the logic, see, e.g., [8, 7, 9]. For instance, logics in the lowest step of this hierarchy, known as the protoalgebraic logics [3], may be studied algebraically, but their ties with their algebraic counterparts are very weak. At the other end of the spectrum are the finitely algebraizable logics [4], whose ties with their equivalent algebraic semantics are very strong. Practically all metalogical properties of the class of finitely algebraizable sentential logics may be translated and studied in the algebraic domain, using existing powerful methods of universal algebra. A major role in the study of the classes in this hierarchy has traditionally been played by the logical matrix models of sentential logics and closure properties that classes of these logical matrix models of a given logic may or may not possess. This tradition of considering closure properties of classes of matrices goes back to the well-known characterizations of varieties and of quasi-varieties of universal algebras by Birkhoff [2] (see also Theorem 4.131 in [12]) and by Mal'cev [10] (see also Theorem V.2.25

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of [6]), respectively. Many important results in abstract algebraic logic follow these two paradigms. Some of the most important and interesting examples of results along these lines will be given as background information in this section. No attempt is made to give appropriate historical credit to the colleagues that originally formulated these results but the reader is referred to Czelakowski's book [7] and to the review article by Font, Jansana and Pigozzi [9] for more references and better placing of these results in the general landscape of abstract algebraic logic.

Suppose that $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ is a **sentential logic**, i.e., $\mathcal{L}$ is an algebraic language type and $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)) \times \mathrm{Fm}_{\mathcal{L}}(V)$ is a structural (but not necessarily finitary) consequence operator on the collection $\mathrm{Fm}_{\mathcal{L}}(V)$ of all formulas of type $\mathcal{L}$ built out of a denumerable set of propositional variables V. A **matrix** of type $\mathcal{L}$ is a pair $\mathfrak{A} = \langle \mathbf{A}, F \rangle$, where $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$ is an $\mathcal{L}$-algebra and $F \subseteq A$ is a set of designated elements, or a filter, on the carrier A of the algebra $\mathbf{A}$. $\mathcal{A}$ is said to be an **$\mathcal{S}$-matrix** if, for all $\Phi \cup \{\phi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$, and every homomorphism $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$,

$$\text{if} \quad \Phi \vdash_{\mathcal{S}} \phi \quad \text{and} \quad h(\Phi) \subseteq F, \quad \text{then} \quad h(\phi) \in F.$$

The greatest $\mathcal{L}$-congruence on the algebra $\mathbf{A}$ that is compatible with the filter F, i.e., does not paste together elements inside with elements outside of F, is called the **Leibniz congruence** associated with F and denoted by $\Omega_{\mathbf{A}}(F)$. If $\Omega_{\mathbf{A}}(F) = \Delta_{\mathbf{A}}$, i.e., to the identity congruence on $\mathbf{A}$, then the matrix $\mathcal{A}$ is said to be **reduced**. The class of all reduced $\mathcal{S}$-matrices is denoted by $\mathbf{Mod}^*(\mathcal{S})$. Closure properties of this class with respect to operations on matrices, such as taking products and ultraproducts of members of a given class of matrices, serve to characterize all major classes of sentential logics in the aforementioned abstract algebraic hierarchy. We repeat here Theorem 3.15 of [9] as an illustration of this phenomenon.

Theorem 1 (Theorem 3.15 of [9]) *Let $\mathcal{S}$ be a sentential logic. Then:*

1. *$\mathcal{S}$ is protoalgebraic iff $\mathbf{Mod}^*(\mathcal{S})$ is closed under subdirect products.*
2. *$\mathcal{S}$ is equivalential iff $\mathbf{Mod}^*(\mathcal{S})$ is closed under submatrices and direct products.*
3. *$\mathcal{S}$ is finitely equivalential iff $\mathbf{Mod}^*(\mathcal{S})$ is closed under submatrices, direct products and ultraproducts.*
4. *$\mathcal{S}$ is weakly algebraizable iff it is protoalgebraic and for every $\langle \mathbf{A}, F \rangle \in \mathbf{Mod}^*(\mathcal{S})$, F is the least $\mathcal{S}$-filter on $\mathbf{A}$.*
5. *$\mathcal{S}$ is algebraizable iff it is equivalential and for every matrix $\langle \mathbf{A}, F \rangle \in \mathbf{Mod}^*(\mathcal{S})$, F is the least $\mathcal{S}$-filter on $\mathbf{A}$.*
6. *$\mathcal{S}$ is finitely algebraizable iff it is finitely equivalential and for every matrix $\langle \mathbf{A}, F \rangle \in \mathbf{Mod}^*(\mathcal{S})$, F is the least $\mathcal{S}$-filter on $\mathbf{A}$.*

The author has developed a categorical theory [16]-[24] (see also [13, 14, 15] for the origins of these developments and for more explanations on the motivation) paralleling the theory of algebraizability of sentential logics [8]. In that theory, π-institutions replace sentential logics as the underlying logical structures. It has also been clear from these developments that π-institution models [17, 19] have an important role to play in analogy with matrix models in the universal algebraic theory. It is reasonable to expect that closure properties of classes of π-institution models may play in this theory a role analogous to the closure properties of classes of logical matrix models. There is, however, one significant difference between matrix models and π-institution models. Roughly speaking, a π-institution model $\mathcal{I}'$ of a given π-institution $\mathcal{I}$ is accompanied by a specific translation from $\mathcal{I}$ to $\mathcal{I}'$, whereas a matrix $\langle \mathbf{A}, F\rangle$ is a matrix model of a sentential logic $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}}\rangle$ if it satisfies interpretability under all possible homomorphisms from $\mathbf{Fm}_{\mathcal{L}}(V)$ into $\mathbf{A}$. Therefore, in introducing operations on classes of π-institution models, besides the transformation of the π-institutions involved, the translations have to also be transformed accordingly, to accompany the newly constructed models.

In the present work, this project of defining transformations on classes of π-institution models is initiated. The hope is that these closure properties will serve in future work in characterizing different properties of classes of π-institutions as related to their algebraizability. The operations that are studied in this work consist of taking sub-institutions, taking direct institution products, taking images and pre-images under semi-interpretations and interpretations and, finally, taking filtered institution products. For each of these operations, preservation results are also formulated and proved. This culminates in the last section in the formulation of general results about the closure of the class of π-institution models of a given π-institution under a variety of operators.

For general background on abstract algebraic logic, the reader is referred to the book [7] and the monograph [8]. For all unexplained categorical notation any of [1], [5] or [11] may be consulted.

2 SubFunctors, SubInstitutions and SubModels

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a functor. A functor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ is a **subfunctor** of SEN, if

- $\mathbf{Sign}'$ is a subcategory of $\mathbf{Sign}$,
- $\mathrm{SEN}'(\Sigma') \subseteq \mathrm{SEN}(\Sigma')$, for all $\Sigma' \in |\mathbf{Sign}'|$, and
- $\mathrm{SEN}'(f)(\phi) = \mathrm{SEN}(f)(\phi)$, for all $f \in \mathbf{Sign}'(\Sigma, \Sigma'), \phi \in \mathrm{SEN}'(\Sigma)$.

If N is a category of natural transformations on SEN, such that, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, all $\Sigma' \in |\mathbf{Sign}'|$ and all $\vec{\phi}' \in \mathrm{SEN}'(\Sigma')^n$, $\sigma_{\Sigma'}(\vec{\phi}') \in \mathrm{SEN}'(\Sigma')$, then SEN' will be said to be an N-**subfunctor** of SEN. If $\mathrm{SEN}' : \mathbf{Sign} \to \mathbf{Set}$ is a subfunctor of $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, with the same domain category, then SEN' is said to be a **simple subfunctor** of SEN.

In what follows, it will always be assumed that, when a tuple $\vec{\phi} \in \mathrm{SEN}(\Sigma)^n$ is under consideration, by ϕ_i will be denoted the i-th component of $\vec{\phi}$, for all $i < n$, i.e., that $\vec{\phi} = \langle \phi_0, \ldots, \phi_{n-1}\rangle$. Given a functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ and a sub-functor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, we also use the notation $\langle J, j\rangle : \mathrm{SEN}' \to \mathrm{SEN}$ to denote the inclusion singleton translation from SEN' to SEN. Note that, if SEN' is an N-subfunctor of SEN, then $\langle J, j\rangle : \mathrm{SEN}' \to \mathrm{SEN}$ is, by definition, an N-morphism from SEN' to SEN. Also, if SEN' is a simple sub-functor of SEN, then, automatically, $J = I_{\mathbf{Sign}}$.

Consider two π-institutions $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ and $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$.
The π-institution $\mathcal{I}'$ is a **sub-institution** of the π-institution $\mathcal{I}$ if

- SEN' is a subfunctor of SEN and
- $C'_{\Sigma'}(\Phi') = C_{\Sigma'}(\Phi') \cap \mathrm{SEN}'(\Sigma')$, for all $\Sigma' \in |\mathbf{Sign}'|, \Phi' \subseteq \mathrm{SEN}'(\Sigma')$.

In case SEN' is a simple subfunctor of SEN, then $\mathcal{I}'$ is said to be a **simple subinstitution** of $\mathcal{I}$ and, in case SEN' is an N-subfunctor of SEN, $\mathcal{I}'$ is said to be an N-**subinstitution** of $\mathcal{I}$.

It is not difficult to see that, if $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ is a subinstitution of $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, then $\langle J, j\rangle : \mathcal{I}' \vdash^s \mathcal{I}$ is a singleton interpretation and if, in addition, SEN' is an N-subfunctor of SEN, then $\langle J, j\rangle : \mathcal{I}' \vdash^s \mathcal{I}$ is an N-interpretation, i.e., a strong (N, N)-logical morphism.

We turn now to the models of given π-institutions, as defined in [17]. Recall that a π-institution $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ is a **model** of a given π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ if there exists a semi-interpretation $\langle F, \alpha\rangle : \mathcal{I} \succ\!\!- \mathcal{I}'$. In case $\langle F, \alpha\rangle$ is an N-semi-interpretation, i.e., an (N, N')-logical morphism, then $\mathcal{I}'$ is said to be an (N, N')-**model** of $\mathcal{I}$ via $\langle F, \alpha\rangle$.

It is shown now that subinstitutions of models via appropriately factorable semi-interpretations via the subinstitution inclusions are also models.

Proposition 2 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution and $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ a model of $\mathcal{I}$ via the singleton semi-interpretation $\langle F, \alpha\rangle : \mathcal{I} \succ\!\!-^s \mathcal{I}'$. Suppose that $\mathcal{I}'' = \langle \mathbf{Sign}'', \mathrm{SEN}'', C''\rangle$ is a subinstitution of $\mathcal{I}'$, with $\langle J, j\rangle : \mathcal{I}'' \vdash^s \mathcal{I}'$ the inclusion, and that $\langle F, \alpha\rangle$ factors through $\langle J, j\rangle$,*

$$\begin{tikzcd} \mathcal{I} \arrow[rr, "\langle F,\alpha\rangle"] \arrow[dr, "\langle G,\beta\rangle"'] & & \mathcal{I}' \\ & \mathcal{I}'' \arrow[ur, "\langle J,j\rangle"'] & \end{tikzcd}$$

i.e., there exists a singleton translation $\langle G, \beta\rangle : \mathcal{I} \rightarrow^s \mathcal{I}''$, such that $\langle F, \alpha\rangle = \langle J, j\rangle \circ \langle G, \beta\rangle$. Then $\mathcal{I}''$ is also a model of $\mathcal{I}$ via $\langle G, \beta\rangle$.

Proof:
We have, for all $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\begin{array}{rll} \phi \in C_\Sigma(\Phi) & \text{implies} & \alpha_\Sigma(\phi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi)) \quad (\text{since } \langle F, \alpha\rangle : \mathcal{I} \succ\!\!-^s \mathcal{I}') \\ & \text{iff} & j_{G(\Sigma)}(\beta_\Sigma(\phi)) \in C'_{G(\Sigma)}(j_{G(\Sigma)}(\beta_\Sigma(\Phi))) \\ & & \qquad\qquad (\text{since } \langle F, \alpha\rangle = \langle J, j\rangle \circ \langle G, \beta\rangle) \\ & \text{iff} & \beta_\Sigma(\phi) \in C'_{G(\Sigma)}(\beta_\Sigma(\Phi)) \quad (\text{since } \langle J, j\rangle : \mathcal{I}'' \vdash^s \mathcal{I}'). \end{array}$$

Thus $\langle G, \beta\rangle : \mathcal{I} \succ\!\!-^s \mathcal{I}''$ is a semi-interpretation and, therefore, $\mathcal{I}''$ is also a model of $\mathcal{I}$ via $\langle G, \beta\rangle$. ■

The π-institution $\mathcal{I}''$ of Proposition 2 together with the semi-interpretation $\langle G, \beta\rangle : \mathcal{I} \succ\!\!-^s \mathcal{I}''$ is said to be a **subinstitution model factor** of the π-institution model $\mathcal{I}'$ via the semi-interpretation $\langle F, \alpha\rangle : \mathcal{I} \succ\!\!-^s \mathcal{I}'$.

The following corollary follows easily from Proposition 2 by taking all the translations involved to respect categories of natural transformations.

Corollary 3 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN, *is a π-institution and $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$, with N' a category of natural transformations on* SEN$'$, *an (N,N')-model of $\mathcal{I}$ via the (N,N')-logical morphism $\langle F,\alpha\rangle : \mathcal{I} \succ^s \mathcal{I}'$. Suppose that $\mathcal{I}'' = \langle \mathbf{Sign}'', \mathrm{SEN}'', C''\rangle$ is an N'-subinstitution of $\mathcal{I}'$, with $\langle J,j\rangle : \mathcal{I}'' \vdash^s \mathcal{I}'$ the inclusion, and that $\langle F,\alpha\rangle$ factors through $\langle J,j\rangle$, i.e., there exists an N-morphism $\langle G,\beta\rangle : \mathcal{I} \to^s \mathcal{I}''$, such that $\langle F,\alpha\rangle = \langle J,j\rangle \circ \langle G,\beta\rangle$. Then $\mathcal{I}''$ is also an (N,N')-model of $\mathcal{I}$ via $\langle G,\beta\rangle$.*

The π-institution $\mathcal{I}''$ of Corollary 3 together with the (N,N')-logical morphism $\langle G,\beta\rangle : \mathcal{I} \succ^s \mathcal{I}''$ is said to be an N-**subinstitution model factor** of the π-institution model $\mathcal{I}'$ via the (N,N')-logical morphism $\langle F,\alpha\rangle : \mathcal{I} \succ^s \mathcal{I}'$.

Finally, it is shown that superinstitutions of models are also models via the composition of the original model semi-interpretations with the inclusion interpretations.

Proposition 4 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution and $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ a model of $\mathcal{I}$ via the singleton semi-interpretation $\langle F,\alpha\rangle : \mathcal{I} \succ^s \mathcal{I}'$. Suppose that $\mathcal{I}'$ is a subinstitution of $\mathcal{I}'' = \langle \mathbf{Sign}'', \mathrm{SEN}'', C''\rangle$, with $\langle J,j\rangle : \mathcal{I}' \vdash^s \mathcal{I}''$ the inclusion.*

$$\mathcal{I} \xrightarrow{\langle F,\alpha\rangle} \mathcal{I}' \xrightarrow{\langle J,j\rangle} \mathcal{I}''$$

Then $\mathcal{I}''$ is also a model of $\mathcal{I}$ via $\langle JF, j_F\alpha\rangle$.

Proof:

We have, for all $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll} \phi \in C_\Sigma(\Phi) & \text{implies} & \alpha_\Sigma(\phi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi)) \quad (\text{since } \langle F,\alpha\rangle : \mathcal{I} \succ^s \mathcal{I}') \\ & \text{iff} & j_{F(\Sigma)}(\alpha_\Sigma(\phi)) \in C''_{F(\Sigma)}(j_{F(\Sigma)}(\alpha_\Sigma(\Phi))) \\ & & \qquad (\text{since } \langle J,j\rangle : \mathcal{I}' \vdash^s \mathcal{I}''). \end{array}$$

Thus $\langle JF, j_F\alpha\rangle : \mathcal{I} \succ^s \mathcal{I}''$ is a semi-interpretation and, therefore, $\mathcal{I}''$ is also a model of $\mathcal{I}$ via $\langle JF, j_F\alpha\rangle$. ■

The π-institution $\mathcal{I}''$ of Proposition 4 together with the semi-interpretation $\langle JF, j_F\alpha\rangle : \mathcal{I} \succ^s \mathcal{I}''$ is said to be a **superinstitution model** of the π-institution model $\mathcal{I}'$ via the semi-interpretation $\langle F,\alpha\rangle : \mathcal{I} \succ^s \mathcal{I}'$.

Similarly with Proposition 2, by stipulating preservation of categories of natural transformations, we obtain the following corollary.

Corollary 5 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN, *is a π-institution and $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$, with N' a category of natural transformations on* SEN$'$, *a model of $\mathcal{I}$ via the (N,N')-logical morphism $\langle F,\alpha\rangle : \mathcal{I} \succ^s \mathcal{I}'$. Suppose that $\mathcal{I}'$ is an N'-subinstitution of $\mathcal{I}'' = \langle \mathbf{Sign}'', \mathrm{SEN}'', C''\rangle$, with $\langle J,j\rangle : \mathcal{I}' \vdash^s \mathcal{I}''$ the inclusion. Then $\mathcal{I}''$ is also an (N,N')-model of $\mathcal{I}$ via $\langle JF, j_F\alpha\rangle$.*

The π-institution $\mathcal{I}''$ of Corollary 5 together with the (N,N')-logical morphism $\langle JF, j_F\alpha\rangle : \mathcal{I} \succ^s \mathcal{I}''$ is said to be an N-**superinstitution model** of the π-institution model $\mathcal{I}'$ via the (N,N')-logical morphism $\langle F,\alpha\rangle : \mathcal{I} \succ^s \mathcal{I}'$.

Given a π-institution $\mathcal{I}$ and a class $\mathfrak{I}$ of π-institution models of $\mathcal{I}$, by $\mathbf{Sf}(\mathfrak{I})$ will be denoted the class of all isomorphic copies of all subinstitution model factors of members of $\mathfrak{I}$ and by $\mathbf{Sp}(\mathfrak{I})$ the class of all isomorphic copies of all superinstitution models of members of $\mathfrak{I}$. By $\mathbf{Sf}^N(\mathfrak{I})$ will be denoted the class of all isomorphic copies of all N-subinstitution model factors of members of $\mathfrak{I}$ and by $\mathbf{Sp}^N(\mathfrak{I})$ the class of all isomorphic copies of all N-superinstitution models of members of $\mathfrak{I}$.

3 Products of Functors, Institutions and Models

Given a collection of categories $\mathbf{Sign}^i, i \in I$, by $\prod_{i\in I}\mathbf{Sign}^i$ will be denoted the product category of the $\mathbf{Sign}^i, i \in I$, and by $P^j : \prod_{i\in I}\mathbf{Sign}^i \to \mathbf{Sign}^j$ the associated j-th projection functor, $j \in I$. We use either of the notations $\langle \Sigma_i : i \in I\rangle$ or $\prod_{i\in I}\Sigma_i$ for the tuple in $|\prod_{i\in I}\mathbf{Sign}^i|$ of the elements $\Sigma_i \in |\mathbf{Sign}^i|, i \in I$, and an analogous notation for tuples of morphisms in this product category.

Given a collection $\mathrm{SEN}^i : \mathbf{Sign}^i \to \mathbf{Set}$, $i \in I$, of functors, the **product functor** $\prod_{i\in I}\mathrm{SEN}^i : \prod_{i\in I}\mathbf{Sign}^i \to \mathbf{Set}$ is defined as the functor, such that, for all $\langle \Sigma_i : i \in I\rangle \in |\prod_{i\in I}\mathbf{Sign}^i|$,

$$\prod_{i\in I}\mathrm{SEN}^i(\langle \Sigma_i : i \in I\rangle) = \prod_{i\in I}\mathrm{SEN}^i(\Sigma_i),$$

and, for all $\langle f_i : i \in I\rangle \in \prod_{i\in I}\mathbf{Sign}^i(\prod_{i\in I}\Sigma_i, \prod_{i\in I}\Sigma'_i)$, by

$$\prod_{i\in I}\mathrm{SEN}^i(\langle f_i : i \in I\rangle)(\vec{\phi}) = \langle \mathrm{SEN}^i(f_i)(\phi_i) : i \in I\rangle,$$

for all $\vec{\phi} \in \prod_{i\in I}\mathrm{SEN}^i(\prod_{i\in I}\Sigma_i)$.

Of course, there exist natural projection translations

$$\langle P^j, \pi^j\rangle : \prod_{i\in I}\mathrm{SEN}^i \to \mathrm{SEN}^j,$$

such that

$$P^j(\prod_{i\in I}\Sigma_i) = \Sigma_j, \quad \text{for all } \prod_{i\in I}\Sigma_i \in |\prod_{i\in I}\mathbf{Sign}^i|,$$

and, similarly for morphisms, and

$$\pi^j_{\prod_{i\in I}\Sigma_i}(\vec{\phi}) = \phi_j, \quad \text{for all } \vec{\phi} \in \prod_{i\in I}\mathrm{SEN}^i(\Sigma_i).$$

Moreover, $\prod_{i\in I}\mathrm{SEN}^i$ has the usual categorical universal property of products:

Lemma 6 *Let* $\mathrm{SEN}^i : \mathbf{Sign}^i \to \mathbf{Set}, i \in I$, *be a collection of functors,* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *a functor and* $\langle F^i, \alpha^i\rangle : \mathrm{SEN} \to^s \mathrm{SEN}^i$, $i \in I$, *singleton translations. Then*

$$\begin{tikzcd}
\prod_{i\in I}\mathrm{SEN}^i \arrow[r, "\langle P^i, \pi^i\rangle"] & \mathrm{SEN}^i \\
\mathrm{SEN} \arrow[u, "\langle G, \beta\rangle"] \arrow[ur, "\langle F^i, \alpha^i\rangle"'] &
\end{tikzcd}$$

there exists a unique singleton translation $\langle G,\beta\rangle : \mathrm{SEN} \to \prod_{i\in I}\mathrm{SEN}^i$, *such that* $\langle P^i,\pi^i\rangle \circ \langle G,\beta\rangle = \langle F^i,\alpha^i\rangle$, *for all* $i \in I$.

Proof:

The functor $G : \mathbf{Sign} \to \prod_{i\in I}\mathbf{Sign}^i$ is given, for all $\Sigma \in |\mathbf{Sign}|$, by $G(\Sigma) = \langle F^i(\Sigma) : i \in I\rangle$, and, for all $f \in \mathbf{Sign}(\Sigma_1,\Sigma_2)$, by $G(f) = \langle F^i(f) : i \in I\rangle$. The natural transformation $\beta : \mathrm{SEN} \to \prod_{i\in I}\mathrm{SEN}^i$ is defined, for all $\Sigma \in |\mathbf{Sign}|$, by

$$\beta_\Sigma(\phi) = \langle \alpha^i_\Sigma(\phi) : i \in I\rangle, \quad \text{for all} \quad \phi \in \mathrm{SEN}(\Sigma).$$

It is left to the reader the easy task to verify that, with these definitions, $\langle G,\beta\rangle$ becomes a singleton translation satisfying the commutativity of the given triangle. ■

$\langle G,\beta\rangle$, as defined in Lemma 6, will be denoted by $\prod_{i\in I}\langle F^i,\alpha^i\rangle$.

Suppose, next, that N^i is a category of natural transformations on $\mathrm{SEN}^i : \mathbf{Sign}^i \to \mathbf{Set}$, such that, for every $i \in I$, $\langle \mathbf{Sign}^i, \mathrm{SEN}^i, N^i\rangle$ is an N-algebraic system. We follow custom in denoting by $\sigma^i : (\mathrm{SEN}^i)^n \to \mathrm{SEN}^i$ the natural transformation on SEN^i corresponding to σ in N. Then one may define a category $\prod_{i\in I}N^i$ of natural transformations on $\prod_{i\in I}\mathrm{SEN}^i$ (the notation is not intended to suggest that $\prod_{i\in I}N^i$ is some product in the usual set theoretic or categorical sense), with $\prod\sigma$ in $\prod_{i\in I}N^i$ denoting the natural transformation corresponding to σ in N, by setting, for all $\prod\sigma : (\prod_{i\in I}\mathrm{SEN}^i)^n \to \prod_{i\in I}\mathrm{SEN}^i$, all $\Sigma_i \in |\mathbf{Sign}^i|, i \in I, \vec{\phi}^0,\ldots,\vec{\phi}^{n-1} \in \prod_{i\in I}\mathrm{SEN}^i(\prod_{i\in I}\Sigma_i)$,

$$\prod\sigma_{\prod_{i\in I}\Sigma_i}(\vec{\phi}^0,\ldots,\vec{\phi}^{n-1}) = \langle \sigma^i_{\Sigma_i}(\phi^0_i,\ldots,\phi^{n-1}_i) : i \in I\rangle.$$

To assert that, for all $i \in I$, the N^i endows the functor SEN^i with an N-algebraic system structure, we sometimes use the terminology that there exist **compatible categories of natural transformations** on $\mathrm{SEN}^i, i \in I$, or that the $\langle \mathbf{Sign}^i, \mathrm{SEN}^i, N^i\rangle$ are **similar**.

In this case, the product $\prod_{i\in I}\mathrm{SEN}^i$ will be referred to as an N-**product** and, it is not difficult to see that $\langle P^j,\pi^j\rangle : \prod_{i\in I}\mathrm{SEN}^i \to \mathrm{SEN}^j$ is an N-translation, for all $j \in I$.

Given π-institutions $\mathcal{I}^i = \langle \mathbf{Sign}^i, \mathrm{SEN}^i, C^i\rangle$, $i \in I$, the **institution product** $\prod_{i\in I}\mathcal{I}^i = \langle \prod_{i\in I}\mathbf{Sign}^i, \prod_{i\in I}\mathrm{SEN}^i, \prod_{i\in I}C^i\rangle$ consists of

- the product functor $\prod_{i\in I}\mathrm{SEN}^i : \prod_{i\in I}\mathbf{Sign}^i \to \mathbf{Set}$ and
- the closure system $\prod_{i\in I}C^i$ on $\prod_{i\in I}\mathrm{SEN}^i$, defined by

$$\prod_{i\in I}C^i_{\prod_{i\in I}\Sigma_i}(\Phi) = \prod_{i\in I}C^i_{\Sigma_i}(\pi^i_{\prod_{i\in I}\Sigma_i}(\Phi)),$$

  for all $\Sigma_i \in |\mathbf{Sign}^i|, i \in I, \Phi \subseteq \prod_{i\in I}\mathrm{SEN}^i(\prod_{i\in I}\Sigma_i)$.

It is now shown that, given a collection of π-institutions $\mathcal{I}^i, i \in I$, the triple $\prod_{i\in I}\mathcal{I}^i$, as defined above, is in fact a π-institution.

Proposition 7 *Suppose that* $\mathcal{I}^i = \langle \mathbf{Sign}^i, \mathrm{SEN}^i, C^i\rangle, i \in I$, *is a collection of* π-*institutions. Then* $\prod_{i\in I}\mathcal{I}^i$ *is a* π-*institution.*

Proof:

Since from the work done so far we have that $\prod_{i\in I}\mathbf{Sign}^i$ is a category and $\prod_{i\in I}\mathrm{SEN}^i : \prod_{i\in I}\mathbf{Sign}^i \to \mathbf{Set}$ is a functor, it suffices to show that $\prod_{i\in I} C^i$, as defined above, is a closure system on $\prod_{i\in I}\mathrm{SEN}^i$. All properties of a closure system follow relatively easily from corresponding properties of the C^i's.

1. For inflation, suppose that $\Sigma_i \in |\mathbf{Sign}^i|$, for all $i \in I$, and $\Phi \subseteq \prod_{i\in I}\mathrm{SEN}^i(\prod_{i\in I}\Sigma_i)$. Then, if $\vec{\phi} \in \Phi$, then $\phi_i \in \pi^i_{\prod_{i\in I}\Sigma_i}(\Phi)$, whence, since $C^i_{\Sigma_i}$ is inflationary, $\phi_i \in C^i_{\Sigma_i}(\pi^i_{\prod_{i\in I}\Sigma_i}(\Phi))$ and, hence, we obtain $\vec{\phi} \in \prod_{i\in I} C^i_{\Sigma_i}(\pi^i_{\prod_{i\in I}\Sigma_i}(\Phi))$, i.e., by the definition of $\prod_{i\in I} C^i$, we get that $\vec{\phi} \in \prod_{i\in I} C^i_{\prod_{i\in I}\Sigma_i}(\Phi)$ and $\prod_{i\in I} C^i$ is inflationary.

2. Proof of monotonicity of $\prod_{i\in I} C^i$ is similar to that for inflation and will not be presented in detail.

3. For idempotency, suppose that $\Sigma_i \in |\mathbf{Sign}^i|, i \in I$, and that $\Phi \subseteq \prod_{i\in I}\mathrm{SEN}^i(\prod_{i\in I}\Sigma_i)$ and $\vec{\phi} \in \prod_{i\in I} C^i_{\prod_{i\in I}\Sigma_i}(\prod_{i\in I} C^i_{\prod_{i\in I}\Sigma_i}(\Phi))$. Then, we get, by unraveling the definition of $\prod_{i\in I} C^i$, that
$$\phi_i \in \prod_{i\in I} C^i_{\Sigma_i}(C^i_{\Sigma_i}(\pi^i_{\prod_{i\in I}\Sigma_i}(\Phi))) = \prod_{i\in I} C^i_{\Sigma_i}(\pi^i_{\prod_{i\in I}\Sigma_i}(\Phi)),$$
for all $i \in I$, by the idempotency of $C^i, i \in I$. This, again by the definition of $\prod_{i\in I} C^i$, gives that $\vec{\phi} \in \prod_{i\in I} C^i_{\prod_{i\in I}\Sigma_i}(\Phi)$. Thus $\prod_{i\in I} C^i$ is also idempotent.

4. Finally, for structurality, suppose that $\Sigma_i, \Sigma_i' \in |\mathbf{Sign}^i|$, that $f_i \in \mathbf{Sign}^i(\Sigma_i, \Sigma_i'), i \in I$, and that $\Phi \subseteq \prod_{i\in I}\mathrm{SEN}^i(\prod_{i\in I}\Sigma_i)$. Then, if
$$\vec{\phi} \in \prod_{i\in I}\mathrm{SEN}^i(\prod_{i\in I} f_i)(\prod_{i\in I} C^i_{\prod_{i\in I}\Sigma_i}(\Phi)),$$
then $\phi_i \in \mathrm{SEN}^i(f_i)(C^i_{\Sigma_i}(\pi^i_{\prod_{i\in I}\Sigma_i}(\Phi)))$, for all $i \in I$, whence, by the structurality of C^i, $\phi_i \in C^i_{\Sigma_i'}(\mathrm{SEN}^i(f_i)(\pi^i_{\prod_{i\in I}\Sigma_i}(\Phi)))$, for all $i \in I$, and, therefore,
$$\begin{array}{rcl}\vec{\phi} & \in & \prod_{i\in I} C^i_{\Sigma_i'}(\mathrm{SEN}^i(f_i)(\pi^i_{\prod_{i\in I}\Sigma_i}(\Phi)))\\ & = & \prod_{i\in I} C^i_{\Sigma_i'}(\pi^i_{\prod_{i\in I}\Sigma_i'}(\prod_{i\in I}\mathrm{SEN}^i(\prod_{i\in I} f_i)(\Phi)))\\ & = & \prod_{i\in I} C^i_{\prod_{i\in I}\Sigma_i'}(\prod_{i\in I}\mathrm{SEN}^i(\prod_{i\in I} f_i)(\Phi)),\end{array}$$
and $\prod_{i\in I} C^i$ is also structural.

■

Moreover, it is shown that $\langle P^j, \pi^j\rangle : \prod_{i\in I}\mathcal{I}^i \succ^s \mathcal{I}^j$ is a surjective singleton semi-interpretation from the product π-institution to the j-th factor, for all $j \in I$.

Proposition 8 *Suppose that $\mathcal{I}^i = \langle \mathbf{Sign}^i, \mathrm{SEN}^i, C^i\rangle, i \in I$, is a collection of π-institutions. Then the surjective singleton translation $\langle P^j, \pi^j\rangle : \prod_{i\in I}\mathrm{SEN}^i \to^s \mathrm{SEN}^j$ is a singleton semi-interpretation $\langle P^j, \pi^j\rangle : \prod_{i\in I}\mathcal{I}^i \succ^s \mathcal{I}^j$, for all $j \in I$.*

Proof:
Suppose $\Sigma_i \in |\mathbf{Sign}^i|, i \in I$, and $\Phi \cup \{\vec{\phi}\} \subseteq \prod_{i\in I} \mathrm{SEN}^i(\prod_{i\in I} \Sigma_i)$. Then, if $\vec{\phi} \in \prod_{i\in I} C^i_{\prod_{i\in I}\Sigma_i}(\Phi)$, then $\phi_i \in C^i_{\Sigma_i}(\pi^i_{\prod_{i\in I}\Sigma_i}(\Phi))$, for all $i \in I$. Each of these conditions is equivalent to

$$\pi^i_{\prod_{i\in I}\Sigma_i}(\vec{\phi}) \in C^i_{P^i(\prod_{i\in I}\Sigma_i)}(\pi^i_{\prod_{i\in I}\Sigma_i}(\Phi)).$$

Thus, for all $i \in I$, $\langle P^i, \pi^i\rangle : \prod_{i\in I} \mathrm{SEN}^i \rightarrow^s \mathrm{SEN}^i$ is a singleton semi-interpretation $\langle P^i, \pi^i\rangle : \prod_{i\in I} \mathcal{I}^i \rangle\!\!-^s \mathcal{I}^i$. ■

As a corollary, imposing preservation of natural transformations, we obtain

Corollary 9 *Suppose that $\mathcal{I}^i = \langle \mathbf{Sign}^i, \mathrm{SEN}^i, C^i\rangle$, with N^i a category of natural transformations on SEN^i, $i \in I$, is a collection of π-institutions with compatible categories of natural transformations. Then the surjective singleton translation $\langle P^j, \pi^j\rangle : \prod_{i\in I} \mathrm{SEN}^i \rightarrow^s \mathrm{SEN}^j$ is a surjective $(\prod_{i\in I} N^i,\ N^j)$-logical morphism $\langle P^j, \pi^j\rangle : \prod_{i\in I} \mathcal{I}^i \rangle\!\!-^s \mathcal{I}^j$, for all $j \in I$.*

Lemma 6 will now be adapted to cover the case of singleton semi-interpretations between π-institutions.

Proposition 10 *Suppose that $\mathcal{I}^i = \langle \mathbf{Sign}^i, \mathrm{SEN}^i, C^i\rangle, i \in I$, is a collection of π-institutions, $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ a π-institution and $\langle F^i, \alpha^i\rangle : \mathcal{I} \rangle\!\!-^s \mathcal{I}^i$, $i \in I$, a collection of singleton semi-interpretations. Then, if $\langle G, \beta\rangle = \prod_{i\in I}\langle F^i, \alpha^i\rangle$, for all $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,*

$$\begin{tikzcd}
\prod_{i\in I} \mathcal{I}^i \arrow[r, "\langle P^i, \pi^i\rangle"] & \mathcal{I}^i \\
\mathcal{I} \arrow[u, "\langle G, \beta\rangle"] \arrow[ur, "\langle F^i, \alpha^i\rangle"'] &
\end{tikzcd}$$

$$\beta_\Sigma(\phi) \in \prod_{i\in I} C^i_{G(\Sigma)}(\beta_\Sigma(\Phi)) \quad \textit{iff} \quad \alpha^i_\Sigma(\phi) \in C^i_{F^i(\Sigma)}(\alpha^i_\Sigma(\Phi)),\ \textit{for all } i \in I.$$

Proof:
Note that the left-to-right implication is just a restatement of Proposition 8. So it suffices to show the right-to-left implication. To this end, let $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$ be such that $\alpha^i_\Sigma(\phi) \in C^i_{F^i(\Sigma)}(\alpha^i_\Sigma(\Phi))$, for all $i \in I$. Then, by the definition of a product,

$$\prod_{i\in I} \alpha^i_\Sigma(\phi) \in \prod_{i\in I} C^i_{\prod_{i\in I} F^i(\Sigma)}(\prod_{i\in I} \alpha^i_\Sigma(\Phi_i)).$$

But this is equivalent to $\beta_\Sigma(\phi) \in \prod_{i\in I} C^i_{G(\Sigma)}(\beta_\Sigma(\Phi))$. ■

Next, the possibility of lifting a collection of semi-interpretations from a given π-institution to a collection of π-institution models to a semi-interpretation into the product π-institution is explored.

Proposition 11 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, $\mathcal{I}^i = \langle \mathbf{Sign}^i, \mathrm{SEN}^i, C^i\rangle$, $i \in I$, a collection of π-institutions and $\langle F^i, \alpha^i\rangle : \mathcal{I} \succ\!\!-^s \mathcal{I}^i$, $i \in I$, singleton semi-interpretations from $\mathcal{I}$ into $\mathcal{I}^i, i \in I$.*

$$\begin{tikzcd}
\prod_{i\in I} \mathcal{I}^i \arrow[r, "\langle P^i, \pi^i\rangle"] & \mathcal{I}^i \\
\mathcal{I} \arrow[u, "\langle G, \beta\rangle"] \arrow[ur, "\langle F^i, \alpha^i\rangle"'] &
\end{tikzcd}$$

Then, there exists a singleton semi-interpretation $\langle G, \beta\rangle : \mathcal{I} \succ\!\!-^s \prod_{i\in I} \mathcal{I}^i$, such that the triangle above commutes, i.e., $\langle P^i, \pi^i\rangle \circ \langle G, \beta\rangle = \langle F^i, \alpha^i\rangle$, for all $i \in I$.

Proof:

Lemma 6 provides a singleton translation $\langle G, \beta\rangle : \mathrm{SEN} \to \prod_{i\in I} \mathrm{SEN}^i$, that makes the given triangle commute. So, it suffices to show that $\langle G, \beta\rangle : \mathcal{I} \succ\!\!-^s \prod_{i\in I} \mathcal{I}^i$ is a semi-interpretation. We have, for all $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll}
\phi \in C_\Sigma(\Phi) & \text{implies} & \alpha^i_\Sigma(\phi) \in C^i_{F^i(\Sigma)}(\alpha^i_\Sigma(\Phi)), \text{ for all } i \in I, \\
 & & \qquad\qquad (\text{since } \langle F^i, \alpha^i\rangle : \mathcal{I} \succ\!\!-^s \mathcal{I}^i) \\
 & \text{iff} & \beta_\Sigma(\phi) \in \prod_{i\in I} C^i_{G(\Sigma)}(\beta_\Sigma(\Phi)) \quad (\text{by Proposition 10}).
\end{array}$$

■

As before, there is a version of Proposition 11 dealing with (N, N')-logical morphisms instead of simple semi-interpretations.

Corollary 12 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, N a category of natural transformations on SEN, $\mathcal{I}^i = \langle \mathbf{Sign}^i, \mathrm{SEN}^i, C^i\rangle$, $i \in I$, a collection of π-institutions, with compatible categories of natural transformations N^i on $\mathrm{SEN}^i, i \in I$, and $\langle F^i, \alpha^i\rangle : \mathcal{I} \succ\!\!-^s \mathcal{I}^i, i \in I$, an (N, N^i)-logical morphism from $\mathcal{I}$ into $\mathcal{I}^i, i \in I$. Then, there exists an $(N, \prod_{i\in I} N^i)$-logical morphism $\langle G, \beta\rangle : \mathcal{I} \succ\!\!-^s \prod_{i\in I} \mathcal{I}^i$, such that $\langle P^i, \pi^i\rangle \circ \langle G, \beta\rangle = \langle F^i, \alpha^i\rangle$, for all $i \in I$.*

Given a class $\mathfrak{I}$ of π-institution models of a given π-institution $\mathcal{I}$, by $\mathbf{P}(\mathfrak{I})$ will be denoted the class of all isomorphic copies of institution products of families of members of $\mathfrak{I}$ via the product semi-interpretations. Furthermore, by $\mathbf{P}^N(\mathfrak{I})$ will be denoted the class of all isomorphic copies of institution $(N, \prod_{i\in I} N^i)$-models of families of (N, N^i)-models, $i \in I$, of members of $\mathfrak{I}$ with compatible categories of natural transformations.

4 Homomorphic Images and Pre-Images

Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ and $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ are π-institutions. $\mathcal{I}'$ will be said to be a **homomorphic image** of $\mathcal{I}$ if there exists a singleton semi-interpretation $\langle F, \alpha\rangle : \mathcal{I} \succ\!\!-^s \mathcal{I}'$. In that case $\mathcal{I}$ is said to be a **homomorphic pre-image** of $\mathcal{I}'$. $\mathcal{I}'$ is said to be a **strict homomorphic image** of $\mathcal{I}$ in case there exists a singleton interpretation $\langle F, \alpha\rangle : \mathcal{I} \vdash^s \mathcal{I}'$. In that case $\mathcal{I}$ is said to be a **strict homomorphic pre-image** of $\mathcal{I}'$.

We urge the reader to notice that neither in the preceding definitions nor in the following ones are the semi-interpretations or the interpretations involved required to be surjective. This is a difference between the present context and the well-known corresponding definitions from the context of logical matrices.

Given two π-institutions $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ and categories of natural transformations N, N' on $\mathrm{SEN}, \mathrm{SEN}'$, respectively, $\mathcal{I}'$ is said to be a **logical image** of $\mathcal{I}$ via $\langle F,\alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$ and $\mathcal{I}$ a **logical pre-image** of $\mathcal{I}'$ if $\langle F,\alpha\rangle : \mathcal{I} \succ^s \mathcal{I}'$ is an (N,N')-logical morphism. $\mathcal{I}'$ is a **strict logical image** of $\mathcal{I}$ via $\langle F,\alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$ and $\mathcal{I}$ a **strict logical pre-image** of $\mathcal{I}'$ if $\langle F,\alpha\rangle : \mathcal{I} \vdash^s \mathcal{I}'$ is a strong (N,N')-logical morphism. An injective (N,N')-bilogical morphism is an **isomorphism**. In that case, we write $\mathcal{I} \cong \mathcal{I}'$.

The following propositions describe some preservation properties when one takes homomorphic images and pre-images and strict homomorphic images and pre-images.

Proposition 13 *If $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ a model of $\mathcal{I}$ via the singleton semi-interpretation $\langle F,\alpha\rangle : \mathcal{I} \succ^s \mathcal{I}'$ and $\mathcal{I}'' = \langle \mathbf{Sign}'', \mathrm{SEN}'', C''\rangle$ a homomorphic image of $\mathcal{I}'$ via the singleton semi-interpretation $\langle G,\beta\rangle : \mathcal{I}' \succ^s \mathcal{I}''$,*

$$\mathcal{I} \xrightarrow{\langle F,\alpha\rangle} \mathcal{I}' \xrightarrow{\langle G,\beta\rangle} \mathcal{I}''$$

then $\mathcal{I}''$ is a model of $\mathcal{I}$ via the semi-interpretation $\langle GF, \beta_F\alpha\rangle : \mathcal{I} \succ^s \mathcal{I}''$.

Proof:
Composition of two semi-interpretations is also a semi-interpretation. ■

Preservation of natural transformations now yields

Corollary 14 *If $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is a π-institution, $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$, with N' a category of natural transformations on SEN', a model of $\mathcal{I}$ via the (N,N')-logical morphism $\langle F,\alpha\rangle : \mathcal{I} \succ^s \mathcal{I}'$ and $\mathcal{I}'' = \langle \mathbf{Sign}'', \mathrm{SEN}'', C''\rangle$, with N'' a category of natural transformations on SEN'', a logical image of $\mathcal{I}'$ via the (N',N'')-logical morphism $\langle G,\beta\rangle : \mathcal{I}' \succ^s \mathcal{I}''$, then $\mathcal{I}''$ is a model of $\mathcal{I}$ via the (N,N'')-logical morphism $\langle GF, \beta_F\alpha\rangle : \mathcal{I} \succ^{se} \mathcal{I}''$.*

For strict homomorphic images and pre-images we have the following adaptation of Proposition 13.

Proposition 15 *Suppose that $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ is a π-institution and $\mathcal{I}'' = \langle \mathbf{Sign}'', \mathrm{SEN}'', C''\rangle$ a strict homomorphic image of $\mathcal{I}'$ via the singleton interpretation $\langle G,\beta\rangle : \mathcal{I}' \vdash^s \mathcal{I}''$. Given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, $\mathcal{I}'$ is a model of $\mathcal{I}$ via the singleton semi-interpretation $\langle F,\alpha\rangle : \mathcal{I} \succ^s \mathcal{I}'$*

$$\mathcal{I} \xrightarrow{\langle F,\alpha\rangle} \mathcal{I}' \xrightarrow{\langle G,\beta\rangle} \mathcal{I}''$$

if and only if $\mathcal{I}''$ is a model of $\mathcal{I}$ via the singleton semi-interpretation $\langle GF, \beta_F\alpha\rangle : \mathcal{I} \succ^s \mathcal{I}''$.

Proof:
If $\langle F,\alpha\rangle : \mathcal{I} \succ^s \mathcal{I}'$, then, for all $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll} \phi \in C_\Sigma(\Phi) & \text{implies} & \alpha_\Sigma(\phi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi)) \\ & \text{iff} & \beta_{F(\Sigma)}(\alpha_\Sigma(\phi)) \in C''_{G(F(\Sigma))}(\beta_{F(\Sigma)}(\alpha_\Sigma(\Phi))). \end{array}$$

If, conversely, $\langle GF, \beta_F\alpha\rangle : \mathcal{I} \rangle\!\!-^s \mathcal{I}''$, then,for all $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\begin{array}{rl} \phi \in C_\Sigma(\Phi) \quad \text{implies} & \beta_{F(\Sigma)}(\alpha_\Sigma(\phi)) \in C''_{G(F(\Sigma))}(\beta_{F(\Sigma)}(\alpha_\Sigma(\Phi))) \\ \text{iff} & \alpha_\Sigma(\phi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi)). \end{array}$$

■

And, once more preservation of natural transformations results in

Corollary 16 *Suppose that $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$, with N' a category of natural transformations, is a π-institution and $\mathcal{I}'' = \langle \mathbf{Sign}'', \mathrm{SEN}'', C''\rangle$ a strict logical image of $\mathcal{I}'$ via the strong (N', N'')-logical morphism $\langle G, \beta\rangle : \mathcal{I}' \vdash^s \mathcal{I}''$. Given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, $\mathcal{I}'$ is a model of $\mathcal{I}$ via the (N, N')-logical morphism $\langle F, \alpha\rangle : \mathcal{I} \rangle\!\!-^s \mathcal{I}'$*

$$\mathcal{I} \xrightarrow{\langle F, \alpha\rangle} \mathcal{I}' \xrightarrow{\langle G, \beta\rangle} \mathcal{I}''$$

if and only if $\mathcal{I}''$ is a model of $\mathcal{I}$ via the (N, N'')-logical morphism $\langle GF, \beta_F\alpha\rangle : \mathcal{I} \rangle\!\!-^s \mathcal{I}''$.

Given a class $\mathfrak{I}$ of π-institution models of a given π-institution $\mathcal{I}$, by $\mathbf{H}(\mathfrak{I})$ will be denoted the class of all isomorphic copies of homomorphic images of members of $\mathfrak{I}$, by $\mathbf{H}^{-1}(\mathfrak{I})$ will be denoted the class of all isomorphic copies of homomorphic pre-images of members of $\mathfrak{I}$, by $\mathbf{H}_{\mathrm{S}}(\mathfrak{I})$ the class of all isomorphic copies of strict homomorphic images of members of $\mathfrak{I}$ and, finally, by $\mathbf{H}_{\mathrm{S}}^{-1}(\mathfrak{I})$ the class of all isomorphic copies of strict homomorphic pre-images of members of $\mathfrak{I}$.

We add the superscript ^{N} to all four operators to denote logical images, logical pre-images, strict logical images and strict logical pre-images, respectively. Thus, the classes $\mathbf{H}^N(\mathfrak{I})$, $\mathbf{H}^{N^{-1}}(\mathfrak{I}), \mathbf{H}_{\mathrm{S}}^N(\mathfrak{I})$ and $\mathbf{H}_{\mathrm{S}}^{N^{-1}}(\mathfrak{I})$, respectively, are obtained.

5 Filtered Products

Suppose that $\mathrm{SEN}^i : \mathbf{Sign}^i \to \mathbf{Set}, i \in I$, is a collection of functors. Let F be a filter over the index set I. For all $\Sigma_i \in |\mathbf{Sign}^i|$, and $\phi_i, \psi_i \in \mathrm{SEN}^i(\Sigma_i), i \in I$, define the equivalence relation $\equiv^F_{\prod_{i\in I}\Sigma_i} \subseteq (\prod_{i\in I}\mathrm{SEN}^i(\Sigma_i))^2$, by

$$\langle \phi_i : i \in I\rangle \equiv^F_{\prod_{i\in I}\Sigma_i} \langle \psi_i : i \in I\rangle \quad \text{iff} \quad \{i \in I : \phi_i = \psi_i\} \in F.$$

In this case $\langle \phi_i : i \in I\rangle$ and $\langle \psi_i : i \in I\rangle$ are said to be $\prod_{i\in I}\Sigma_i$**-equivalent modulo** F. Let $[\vec{\phi}]_F$ or $\vec{\phi}/F$ denote the equivalence class of $\vec{\phi}$ modulo the filter F. Then set

$$\prod_{i\in I}\mathrm{SEN}^i(\Sigma_i)/F = \{\vec{\phi}/F : \vec{\phi} \in \prod_{i\in I}\mathrm{SEN}^i(\Sigma_i)\}.$$

The **filtered product** $\prod_F \mathrm{SEN}^i : \prod_{i\in I}\mathbf{Sign}^i \to \mathbf{Set}$ of the functors $\mathrm{SEN}^i : \mathbf{Sign}^i \to \mathbf{Set}, i \in I$, **modulo the filter** F is the functor defined by

$$\prod_F \mathrm{SEN}^i(\prod_{i\in I}\Sigma_i) = \prod_{i\in I}\mathrm{SEN}^i(\Sigma_i)/F,$$

for all $\prod_{i\in I}\Sigma_i \in |\prod_{i\in I}\mathbf{Sign}^i|$, and, given $\prod_{i\in I} f_i \in \prod_{i\in I}\mathbf{Sign}^i(\prod_{i\in I}\Sigma_i, \prod_{i\in I}\Sigma_i')$, by

$$\prod_F \mathrm{SEN}^i(\prod_{i\in I} f_i)(\vec{\phi}/F) = \langle \mathrm{SEN}^i(f_i)(\phi_i) : i\in I\rangle/F,$$

for all $\vec{\phi}\in\prod_{i\in I}\mathrm{SEN}^i(\Sigma_i)$. Note that, because

$$\{i\in I : \mathrm{SEN}^i(f_i)(\phi_i) = \mathrm{SEN}^i(f_i)(\psi_i)\} \supseteq \{i\in I : \phi_i = \psi_i\},$$

the action of $\prod_F \mathrm{SEN}^i$ on morphisms is well-defined.

Now suppose that $N^i, i\in I$, are compatible categories of natural transformations on $\mathrm{SEN}^i, i\in I$. Recall the definition of $\prod_{i\in I} N^i$. Note that, if $\sigma : (\prod_{i\in I}\mathrm{SEN}^i)^n \to \prod_{i\in I}\mathrm{SEN}^i$ is in $\prod_{i\in I} N^i$, then, for all $\prod_{i\in I}\Sigma_i \in |\prod_{i\in I}\mathbf{Sign}^i|$, and all $\vec{\phi}^0,\ldots,\vec{\phi}^{n-1},\vec{\psi}^0,\ldots,\vec{\psi}^{n-1} \in \prod_{i\in I}\mathrm{SEN}^i(\prod_{i\in I}\Sigma_i)$,

$$\{i\in I : \sigma^i_{\Sigma_i}(\phi^0_i,\ldots,\phi^{n-1}_i) = \sigma^i_{\Sigma_i}(\psi^0_i,\ldots,\psi^{n-1}_i)\} \supseteq \bigcap_{j=0}^{n-1}\{i\in I : \phi^j_i = \psi^j_i\},$$

whence, if $\vec{\phi}^i \equiv^F_{\prod_{i\in I}\Sigma_i} \vec{\psi}^i$, for all $i = 1,\ldots,n-1$, then

$$\sigma_{\prod_{i\in I}\Sigma_i}(\vec{\phi}^0,\ldots,\vec{\phi}^{n-1}) \equiv^F_{\prod_{i\in I}\Sigma_i} \sigma_{\prod_{i\in I}\Sigma_i}(\vec{\psi}^0,\ldots,\vec{\psi}^{n-1}).$$

Thus, one may define the category of natural transformations $\prod_F N^i$ and may similarly define the N-filtered product $\prod_F \mathrm{SEN}^i$ of the functors SEN^i, $i\in I$, with $\prod_F N^i$ the natural choice of a category of natural transformations on $\prod_F \mathrm{SEN}^i$.

Next, it is shown that there exists a natural projection translation from the institution product of a collection $\mathrm{SEN}^i, i\in I$, of sentence functors to any of their filtered products.

Lemma 17 *Suppose that* $\mathrm{SEN}^i : \mathbf{Sign}^i \to \mathbf{Set}$, $i\in I$, *is a collection of sentence functors and* F *a filter over* I. *Then, there exists a singleton surjective translation* $\langle I_{\prod_{i\in I}\mathbf{Sign}^i}, \pi^F\rangle : \prod_{i\in I}\mathrm{SEN}^i \to^s \prod_F \mathrm{SEN}^i$.

Proof:
Given $\Sigma_i \in |\mathbf{Sign}^i|, i\in I$, define $\pi^F_{\prod_{i\in I}\Sigma_i} : \prod_{i\in I}\mathrm{SEN}^i(\prod_{i\in I}\Sigma_i) \to \prod_F \mathrm{SEN}^i(\prod_{i\in I}\Sigma_i)$, by

$$\pi^F_{\prod_{i\in I}\Sigma_i}(\vec{\phi}) = \vec{\phi}/F, \quad \text{for all} \quad \vec{\phi}\in\prod_{i\in I}\mathrm{SEN}^i(\Sigma_i).$$

It is not difficult to check that, defined in this way, π^F is a natural transformation and, therefore, that $\langle I_{\prod_{i\in I}\mathbf{Sign}^i}, \pi^F\rangle : \prod_{i\in I}\mathrm{SEN}^i \to^s \prod_F \mathrm{SEN}^i$ is a singleton surjective translation. ■

The translation $\langle I_{\prod_{i\in I}\mathbf{Sign}^i}, \pi^F\rangle$ is called the **natural projection** onto the filtered product of the $\mathrm{SEN}^i, i\in I$. When the subscript of the identity functor $I_{\prod_{i\in I}\mathbf{Sign}^i}$ is clear from context, it will be omitted to shorten the notation. Accordingly, the natural projection will be written $\langle I, \pi^F\rangle$.

Preservation of categories of natural transformations yields the following corollary.

Corollary 18 *Suppose that* $\mathrm{SEN}^i : \mathbf{Sign}^i \to \mathbf{Set}$*,* $i \in I$*, is a collection of sentence functors,* $N^i, i \in I$*, is a collection of compatible categories of natural transformations on* $\mathrm{SEN}^i, i \in I$*, and* F *a filter over* I*. Then, there exists a surjective* N*-morphism* $\langle I_{\prod_{i\in I}\mathbf{Sign}^i}, \pi^F\rangle : \prod_{i\in I}\mathrm{SEN}^i \to^s \prod_F \mathrm{SEN}^i$*.*

Having at hand the definition of a filtered product of functors, it is now possible to define a filtered product of π-institutions. Suppose, to this end, that $\mathcal{I}^i = \langle \mathbf{Sign}^i, \mathrm{SEN}^i, C^i\rangle, i \in I$, is a collection of π-institutions and that F is a filter over the index set I. Define the **filtered product** $\prod_F \mathcal{I}^i = \langle \prod_{i\in I}\mathbf{Sign}^i, \prod_F \mathrm{SEN}^i, \prod_F C^i\rangle$, of the $\mathcal{I}^i, i \in I$, **modulo the filter** F by letting, for all $\prod_{i\in I}\Sigma_i \in |\prod_{i\in I}\mathbf{Sign}^i|$ and all $\Phi \subseteq \prod_F \mathrm{SEN}^i(\Sigma_i)$,

$$\begin{aligned}&\textstyle\prod_F C^i_{\prod_{i\in I}\Sigma_i}(\Phi) =\\ &\qquad \{\vec{\psi}/F \in \textstyle\prod_F \mathrm{SEN}^i(\prod_{i\in I}\Sigma_i) : \{i \in I : \psi_i \in C^i_{\Sigma_i}(\pi^i_{\Sigma_i}(\bigcup\Phi))\} \in F\}.\end{aligned}$$

Again, because

$$\{i \in I : \phi_i \in C^i_{\Sigma_i}(\pi^i_{\Sigma_i}(\cup\Phi)) \text{ iff } \psi_i \in C^i_{\Sigma_i}(\pi^i_{\Sigma_i}(\cup\Phi))\} \supseteq \{i \in I : \phi_i = \psi_i\},$$

$\prod_F C^i_{\prod_{i\in I}\Sigma_i}(\Phi)$ is well-defined.

Recall, now, that, given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ and a cardinal number κ, we write $|C| = \kappa$ if κ is the least infinite cardinal such that, for all $\Sigma \in |\mathbf{Sign}|, \Phi \subseteq \mathrm{SEN}(\Sigma)$,

$$C_\Sigma(\Phi) = \bigcup\{C_\Sigma(\Phi') : \Phi' \subseteq \Phi \text{ and } |\Phi'| < \kappa\}.$$

Moreover, a filter F over I is said to be a κ-**filter** if, for all $D \subseteq F$, with $|D| < \kappa$, $\bigcap D \in F$. It is not difficult to verify that, if, for some cardinal μ, we have $|C^i| \le \mu$, for all $i \in I$, and F is a μ-filter, then $\prod_F C^i$, defined as above, is a closure system on $\prod_F \mathrm{SEN}^i$, whence $\prod_F \mathcal{I}^i$ is indeed a π-institution.

Proposition 19 *Suppose that* $\mathcal{I}^i = \langle \mathbf{Sign}^i, \mathrm{SEN}^i, C^i\rangle$*,* $i \in I$*, is a collection of* π*-institutions, with* $|C^i| \le \mu$*, for all* $i \in I$*, and* F *a* μ*-filter over the index set* I*, for some cardinal number* μ*. Then, the triple* $\prod_F \mathcal{I}^i = \langle \prod_{i\in I}\mathbf{Sign}^i, \prod_F \mathrm{SEN}^i, \prod_F C^i\rangle$ *is a* π*-institution.*

Proof:

It has been seen that $\prod_F \mathrm{SEN}^i : \prod_{i\in I}\mathbf{Sign}^i \to \mathbf{Set}$ is a functor. So, it suffices to show that $\prod_F C^i$ is a closure system on $\prod_F \mathrm{SEN}^i$.

1. For inflation, suppose that $\Sigma_i \in |\mathbf{Sign}^i|, i \in I$, and that $\Phi \cup \{\vec{\phi}\} \subseteq \prod_{i\in I}\mathrm{SEN}^i(\prod_{i\in I}\Sigma_i)$, such that $\vec{\phi}/F \in \Phi/F$ and assume, without loss of generality, that Φ is closed under $\equiv^F_{\prod_{i\in I}\Sigma_i}$. Thus, there exists $\vec{\psi} \in \Phi$, such that $\vec{\phi} \equiv^F_{\prod_{i\in I}\Sigma_i} \vec{\psi}$, i.e., $X = \{i \in I : \phi_i = \psi_i\} \in F$. Then, by inflation for $C^i, i \in I$, we have that $\pi^i_{\prod_{i\in I}\Sigma_i}(\vec{\phi}) \in C^i_{\Sigma_i}(\pi^i_{\prod_{i\in I}\Sigma_i}(\Phi))$, for all $i \in X$, whence $\{i \in I : \pi^i_{\prod_{i\in I}\Sigma_i}(\vec{\phi}) \in C^i_{\Sigma_i}(\pi^i_{\prod_{i\in I}\Sigma_i}(\Phi))\} \supseteq X \in F$ and, therefore $\vec{\phi} \in \prod_F C^i_{\prod_{i\in I}\Sigma_i}(\Phi)$ and $\prod_F C^i$ is inflationary.

2. Proof of the monotonicity of $\prod_F C^i$ is similar to that of inflation and details will be omitted.

3. For idempotency, suppose $\Sigma_i \in |\mathbf{Sign}^i|, i \in I$, and that $\Phi \cup \{\vec{\phi}\} \subseteq \prod_{i\in I} \mathrm{SEN}^i(\prod_{i\in I} \Sigma_i)$, such that
$$\vec{\phi}/F \in \prod_F C^i_{\prod_{i\in I}\Sigma_i}(\prod_F C^i_{\prod_{i\in I}\Sigma_i}(\Phi/F)).$$
To ease the argument, use the notation $\Psi/F = \prod_F C^i_{\prod_{i\in I}\Sigma_i}(\Phi/F)$. So, we also have $\vec{\phi}/F \in \prod_F C^i_{\prod_{i\in I}\Sigma_i}(\Psi/F)$. The second condition gives
$$X = \{i \in I : \phi_i \in C^i_{\Sigma_i}(\pi^i_{\prod_{i\in I}\Sigma_i}(\bigcup \Psi/F))\} \in F. \tag{1}$$
The first condition gives, for all $\vec{\psi} \in \bigcup \Psi/F$,
$$Y_{\vec{\psi}} = \{i \in I : \psi_i \in C^i_{\Sigma_i}(\pi^i_{\prod_{i\in I}\Sigma_i}(\bigcup \Phi/F))\} \in F. \tag{2}$$
Because $|C^i| \leq \mu$, we may assume, without loss of generality, that $|\bigcup \Psi/F| < \mu$. Now, using Conditions (1) and (2), together with idempotency of $C^i, i \in I$, and the fact that F is a μ-filter, we obtain
$$\begin{aligned} &\{i \in I : \phi_i \in C^i_{\Sigma_i}(\pi^i_{\prod_{i\in I}\Sigma_i}(\bigcup \Phi/F))\} \\ &\qquad \supseteq \{i \in I : \phi_i \in C^i_{\Sigma_i}(C^i_{\Sigma_i}(\pi^i_{\prod_{i\in I}\Sigma_i}(\bigcup \Phi/F)))\} \\ &\qquad \supseteq X \cap \textstyle\bigcap_{\vec{\psi}\in\bigcup\Psi/F} Y_{\vec{\psi}} \\ &\qquad \in F. \end{aligned}$$
Therefore $\vec{\phi}/F \in \prod_F C^i_{\prod_{i\in I}\Sigma_i}(\Phi/F)$ and $\prod_F C^i$ is also idempotent.

4. For structurality, suppose that $\Sigma_i, \Sigma'_i \in |\mathbf{Sign}^i|, f \in \mathbf{Sign}^i(\Sigma_i, \Sigma'_i)$, for all $i \in I$, and $\Phi \subseteq \prod_{i\in I} \mathrm{SEN}^i(\prod_{i\in I}\Sigma_i)$, $\vec{\psi} \in \prod_{i\in I} \mathrm{SEN}^i(\prod_{i\in I}\Sigma'_i)$, such that
$$\vec{\psi}/F \in \prod_F \mathrm{SEN}^i(\prod_{i\in I} f_i)(\prod_F C^i_{\prod_{i\in I}\Sigma_i}(\Phi/F)).$$
Then, there exists $\vec{\phi} \in \prod_{i\in I}\mathrm{SEN}^i(\prod_{i\in I}\Sigma_i)$, such that
$$\vec{\phi}/F \in \prod_F C^i_{\prod_{i\in I}\Sigma_i}(\Phi/F) \quad \text{and} \quad \vec{\psi}/F = \prod_F \mathrm{SEN}^i(\prod_{i\in I} f_i)(\vec{\phi}/F).$$
Thus, we obtain $X = \{i \in I : \phi_i \in C^i_{\Sigma_i}(\pi^i_{\prod_{i\in I}\Sigma_i}(\bigcup \Phi/F))\} \in F$ and $Y = \{i \in I : \psi_i = \mathrm{SEN}^i(f_i)(\phi_i)\} \in F$. Therefore
$$\begin{aligned} &\{i \in I : \psi_i \in C^i_{\Sigma_i}(\mathrm{SEN}^i(f_i)(\pi^i_{\prod_{i\in I}\Sigma_i}(\bigcup \Phi/F)))\} \\ &\qquad \supseteq \{i \in I : \psi_i \in \mathrm{SEN}^i(f_i)(C^i_{\Sigma_i}(\pi^i_{\prod_{i\in I}\Sigma_i}(\bigcup \Phi/F)))\} \\ &\qquad \supseteq X \cap Y \\ &\qquad \in F \end{aligned}$$
and, hence $\vec{\psi} \in \prod_F C^i_{\prod_{i\in I}\Sigma_i}(\prod_F \mathrm{SEN}^i(\prod_{i\in I} f_i)(\Phi/F))$, i.e., $\prod_F C^i$ is also structural.

■

From now on when a filtered product $\prod_F \mathcal{I}^i$ of a collection $\mathcal{I}^i, i \in I$, of π-institutions modulo a given filter F is considered, there will be an implicit assumption that all closure systems $C^i, i \in I$, involved, are of a given cardinality $|C^i| \leq \mu, i \in I$, and that the filter is also a μ-filter so that the resulting filtered product is itself a π-institution, according to Proposition 19.

Lemma 17 and Corollary 18 have the following extensions when it comes to filtered products of π-institutions.

Lemma 20 *Let μ be a cardinal number. Suppose that $\mathcal{I}^i = \langle \mathbf{Sign}^i, \mathrm{SEN}^i, C^i \rangle$, $i \in I$, is a collection of π-institutions, such that $|C^i| \leq \mu$, for all $i \in I$, and F is a μ-filter over I. Then, there exists a singleton surjective semi-interpretation $\langle I, \pi^F \rangle : \prod_{i \in I} \mathcal{I}^i \vdash^s \prod_F \mathcal{I}^i$.*

Proof:
We know by Lemma 17 that $\langle I, \pi^F \rangle : \prod_{i \in I} \mathrm{SEN}^i \to^s \prod_F \mathrm{SEN}^i$ is a surjective singleton translation. So it suffices to show that it is a semi-interpretation $\langle I, \pi^F \rangle : \prod_{i \in I} \mathcal{I}^i \vdash^s \prod_F \mathcal{I}^i$. To this end, let $\Sigma_i \in |\mathbf{Sign}^i|, i \in I$, $\Phi \cup \{\vec{\phi}\} \subseteq \prod_{i \in I} \mathrm{SEN}^i(\Sigma_i)$. Then

$$\begin{array}{lcl} \vec{\phi} \in \prod_{i \in I} C^i_{\prod_{i \in I} \Sigma_i}(\Phi) & \text{iff} & \{i \in I : \phi_i \in C^i_{\Sigma_i}(\pi^i_{\prod_{i \in I} \Sigma_i}(\Phi))\} = I \\ & \text{implies} & \vec{\phi}/F \in \prod_F C^i_{\prod_{i \in I} \Sigma_i}(\Phi/F). \end{array}$$

■

As in the case of Lemma 17, requiring preservation of natural transformations yields the following corollary.

Corollary 21 *Let μ be a cardinal number. Suppose that $\mathcal{I}^i = \langle \mathbf{Sign}^i, \mathrm{SEN}^i, C^i \rangle$, $i \in I$, is a collection of π-institutions, such that $|C^i| \leq \mu$, for all $i \in I$, with $N^i, i \in I$, compatible categories of natural transformations on $\mathrm{SEN}^i, i \in I$, and let F be a μ-filter over I. Then, there exists a surjective $(\prod_{i \in I} N^i, \prod_F N^i)$-logical morphism $\langle I, \pi^F \rangle : \prod_{i \in I} \mathcal{I}^i \vdash^s \prod_F \mathcal{I}^i$.*

As is customary in model theory and, more specifically, in the model theory of logical matrices in abstract algebraic logic, if $\mathcal{I}^i = \mathcal{I}$, for all $i \in I$, then a filtered product $\prod_F \mathcal{I}^i$ is said to be a **filtered power** of $\mathcal{I}$ and, if F is an ultrafilter over I, then $\prod_F \mathcal{I}^i$ is said to be an **ultrafiltered product** or, more often, an **ultraproduct** of $\mathcal{I}^i, i \in I$. An ultrafiltered power of $\mathcal{I}$ is also called an **ultrapower** of $\mathcal{I}$.

The following proposition shows that families of singleton translations from a given functor to a family of functors give rise to singleton translations from the given functor to filtered products of the family of functors.

Proposition 22 *Suppose that $\prod_F \mathrm{SEN}^i$ is a filtered product of a family $\mathrm{SEN}^i : \mathbf{Sign}^i \to \mathbf{Set}, i \in I$, of functors and $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ a functor. If there exist singleton translations $\langle F^i, \alpha^i \rangle : \mathrm{SEN} \to^s \mathrm{SEN}^i, i \in I$, then there exists a singleton translation $\langle F, \alpha \rangle : \mathrm{SEN} \to^s \prod_F \mathrm{SEN}^i$, such that, for all $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$,*

$$\alpha_\Sigma(\phi) = \langle \alpha^i_\Sigma(\phi) : i \in I \rangle / F. \tag{3}$$

Proof:

Suppose that there exist singleton translations $\langle F^i, \alpha^i \rangle : \mathrm{SEN} \rightarrow^s \mathrm{SEN}^i$, $i \in I$. Then, by Lemma 6, there exists $\prod_{i \in I} \langle F^i, \alpha^i \rangle : \mathrm{SEN} \rightarrow^s \prod_{i \in I} \mathrm{SEN}^i$, such that $\langle P^i, \pi^i \rangle \circ \prod_{i \in I} \langle F^i, \alpha^i \rangle = \langle F^i, \alpha^i \rangle$.

$$\begin{tikzcd}
\mathrm{SEN} \arrow[r, "\langle F^i, \alpha^i \rangle"] \arrow[dr, "\prod_{i \in I} \langle F^i, \alpha^i \rangle"'] & \mathrm{SEN}^i & \\
& \prod_{i \in I} \mathrm{SEN}^i \arrow[u, "\langle P^i, \pi^i \rangle"'] \arrow[r, "\langle \mathrm{I}, \pi^F \rangle"] & \prod_F \mathrm{SEN}^i
\end{tikzcd}$$

Now compose $\prod_{i \in I} \langle F^i, \alpha^i \rangle$ with $\langle I, \pi^F \rangle : \prod_{i \in I} \mathrm{SEN}^i \rightarrow^s \prod_F \mathrm{SEN}^i$ to get $\langle F, \alpha \rangle : \mathrm{SEN} \rightarrow^s \prod_F \mathrm{SEN}^i$. We have, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\alpha_\Sigma(\phi) = \pi^F_{\prod_{i \in I} F^i(\Sigma)}(\prod_{i \in I} \alpha^i_\Sigma(\phi)) = \prod_{i \in I} \alpha^i_\Sigma(\phi)/F.$$

■

The singleton translation $\langle F, \alpha \rangle : \mathrm{SEN} \rightarrow \prod_F \mathrm{SEN}^i$, displayed in Equation (3), is called the **filtered product** of the singleton translations $\langle F^i, \alpha^i \rangle : \mathrm{SEN} \rightarrow \mathrm{SEN}^i, i \in I$, **modulo** the filter F and will be denoted by $\prod_F \langle F^i, \alpha^i \rangle$ or by $\prod_{i \in I} \langle F^i, \alpha^i \rangle / F$. Thus, an alternative, more informal, way to state the content of Proposition 24 is to say that the filtered product of singleton translations from a given functor to the factors of a filtered product functor is a singleton translation from the functor to the filtered product of the factors.

Proposition 22 has the following corollary.

Corollary 23 *Let* $\mathrm{SEN}^i : \mathbf{Sign}^i \rightarrow \mathbf{Set}, i \in I$, *be a family of functors, with* $N^i, i \in I$, *compatible categories of natural transformations on* $\mathrm{SEN}^i, i \in I$, *and* $\mathrm{SEN} : \mathbf{Sign} \rightarrow \mathbf{Set}$ *a functor, with* N *a category of natural transformations on* SEN. *If there exist* N*-morphisms* $\langle F^i, \alpha^i \rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}^i, i \in I$, *then there exists an* N*-morphism* $\langle F, \alpha \rangle : \mathrm{SEN} \rightarrow^s \prod_F \mathrm{SEN}^i$, *such that, for all* $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$, $\alpha_\Sigma(\phi) = \langle \alpha^i_\Sigma(\phi) : i \in I \rangle / F$.

Finally, an extension of Proposition 22 and an extension of Corollary 23 are presented for the case of π-institutions.

Proposition 24 *Let* μ *be a cardinal number. Suppose that* $\prod_F \mathcal{I}^i$ *is a filtered product of a family* $\mathcal{I}^i = \langle \mathbf{Sign}^i, \mathrm{SEN}^i, C^i \rangle$, $i \in I$, *of* π*-institutions, such that* $|C^i| \leq \mu$, *for all* $i \in I$, *and* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ *a* π*-institution. Let* F *be a* μ*-filter over* I. *If there exist singleton semi-interpretations* $\langle F^i, \alpha^i \rangle : \mathcal{I} \rangle\!\!-^s \mathcal{I}^i, i \in I$, *then there exists a singleton semi-interpretation* $\langle F, \alpha \rangle : \mathcal{I} \rangle\!\!-^s \prod_F \mathcal{I}^i$, *such that, for all* $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$,

$$\alpha_\Sigma(\phi) = \langle \alpha^i_\Sigma(\phi) : i \in I \rangle / F. \tag{4}$$

Proof:

The proof is very similar to the proof of Proposition 22 and it will be omitted. ■

We do obtain in this case as well

Corollary 25 *Let μ be a cardinal number. Suppose $\mathcal{I}^i = \langle \mathbf{Sign}^i, \mathrm{SEN}^i, C^i\rangle$, $i \in I$, is a family of π-institutions, such that $|C^i| \leq \mu$, for all $i \in I$, with $N^i, i \in I$, compatible categories of natural transformations on* $\mathrm{SEN}^i, i \in I$, *and $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ a π-institution, with N a category of natural transformations on* SEN. *Let F be a μ-filter over I. If there exist (N, N^i)-logical morphisms $\langle F^i, \alpha^i\rangle : \mathcal{I} \succ^s \mathcal{I}^i, i \in I$, then there exists an $(N, \prod_F N^i)$-logical morphism $\langle F, \alpha\rangle : \mathcal{I} \succ^s \prod_F \mathcal{I}^i$, such that, for all $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$, $\alpha_\Sigma(\phi) = \langle \alpha^i_\Sigma(\phi) : i \in I\rangle/F$.*

Given a π-institution $\mathcal{I}$ and a class $\mathfrak{I}$ of π-institution models of $\mathcal{I}$, by $\mathbf{P}_{\mathrm{R}}(\mathfrak{I})$ will be denoted the class of all isomorphic copies of filtered products of members of $\mathfrak{I}$ via the filtered product semi-interpretations and by $\mathbf{P}_{\mathrm{U}}(\mathfrak{I})$ the class of all isomorphic copies of ultraproducts of members of $\mathfrak{I}$ via via the analogous ultraproduct semi-interpretations. We add as before the superscript ^{N} to denote that natural transformations are preserved, thus obtaining $\mathbf{P}^N_{\mathrm{R}}(\mathfrak{I})$ and $\mathbf{P}^N_{\mathrm{U}}(\mathfrak{I})$, respectively.

6 Closure Properties of Model Classes

In this final section, some of the results that were proven on sub-institutions, institution products, logical and bilogical images and pre-images and filtered products in the preceding sections are reviewed and recast in model class operator forms. A new result is also proven to the effect that the operator $\mathbf{H}\mathbf{H}^{-1}_{\mathrm{S}}\mathbf{P}$ suffices to generate the entire class of all π-institution models of a given π-institution $\mathcal{I}$ out of any given subclass strongly adequate for $\mathcal{I}$.

For the reader's convenience, the operations on classes of π-institution models that have been introduced so far are summarized in the following table:

Symbol	Brief Description
$\mathbf{Sf}(\mathfrak{I})$	Subinstitution Model Factors
$\mathbf{Sp}(\mathfrak{I})$	Superinstitution Models
$\mathbf{P}(\mathfrak{I})$	Institution Products
$\mathbf{H}(\mathfrak{I})$	Homomorphic Images
$\mathbf{H}^{-1}(\mathfrak{I})$	Homomorphic Pre-Images
$\mathbf{H}_{\mathrm{S}}(\mathfrak{I})$	Strict Homomorphic Images
$\mathbf{H}^{-1}_{\mathrm{S}}(\mathfrak{I})$	Strict Homomorphic Pre-Images
$\mathbf{P}_{\mathrm{R}}(\mathfrak{I})$	Filtered Institution products
$\mathbf{P}_{\mathrm{U}}(\mathfrak{I})$	Institution Ultraproducts

Given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, denote by $\mathrm{Mod}(\mathcal{I})$ the class of all pairs $\langle \mathcal{I}', \langle F, \alpha\rangle\rangle$, consisting of a π-institution model $\mathcal{I}'$ of $\mathcal{I}$, together with a singleton semi-interpretation $\langle F, \alpha\rangle : \mathcal{I} \succ^s \mathcal{I}'$, via which $\mathcal{I}'$ is considered to be a model of $\mathcal{I}$. Similarly, if N is a category of natural transformations on SEN, denote by $\mathrm{Mod}^N(\mathcal{I})$ the class of all pairs $\langle \mathcal{I}', \langle F, \alpha\rangle\rangle$, consisting of a π-institution (N, N')-model $\mathcal{I}'$ of $\mathcal{I}$, together with an (N, N')-logical morphism $\langle F, \alpha\rangle : \mathcal{I} \succ^s \mathcal{I}'$, via which $\mathcal{I}'$ is considered to be a model of $\mathcal{I}$.

The following result now follows by collecting together several of the results that have been proven in the previous sections.

Theorem 26 (Closure Properties of the Class of Models) *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution. Then*

1. *$\mathbf{Sf}(\mathrm{Mod}(\mathcal{I})) \subseteq \mathrm{Mod}(\mathcal{I})$ and $\mathbf{Sp}(\mathrm{Mod}(\mathcal{I})) \subseteq \mathrm{Mod}(\mathcal{I})$.*

2. *$\mathbf{P}(\mathrm{Mod}(\mathcal{I})) \subseteq \mathrm{Mod}(\mathcal{I})$.*

3. *$\mathbf{H}(\mathrm{Mod}(\mathcal{I})) \subseteq \mathrm{Mod}(\mathcal{I})$, $\mathbf{H}_{\mathrm{S}}(\mathrm{Mod}(\mathcal{I})) \subseteq \mathrm{Mod}(\mathcal{I})$ and $\mathbf{H}_{\mathrm{S}}^{-1}(\mathrm{Mod}(\mathcal{I})) \subseteq \mathrm{Mod}(\mathcal{I})$.*

4. *$\mathbf{P}_{\mathrm{R}}(\mathrm{Mod}(\mathcal{I})) \subseteq \mathrm{Mod}(\mathcal{I})$.*

Proof:

1. Combine Propositions 2 and 4.

2. Use Proposition 11.

3. Use Proposition 13 for the first and Proposition 15 for the second and third inclusions.

4. Use Proposition 24.

■

And, of course, requiring preservation of natural transformations from the institution morphisms, we get

Theorem 27 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution and N a category of natural transformations on* SEN. *Then*

1. *$\mathbf{Sf}^N(\mathrm{Mod}^N(\mathcal{I})) \subseteq \mathrm{Mod}^N(\mathcal{I})$ and $\mathbf{Sp}^N(\mathrm{Mod}^N(\mathcal{I})) \subseteq \mathrm{Mod}^N(\mathcal{I})$.*

2. *$\mathbf{P}^N(\mathrm{Mod}^N(\mathcal{I})) \subseteq \mathrm{Mod}^N(\mathcal{I})$.*

3. *$\mathbf{H}^N(\mathrm{Mod}^N(\mathcal{I})) \subseteq \mathrm{Mod}^N(\mathcal{I})$, $\mathbf{H}_{\mathrm{S}}^N(\mathrm{Mod}^N(\mathcal{I})) \subseteq \mathrm{Mod}^N(\mathcal{I})$ and $\mathbf{H}_{\mathrm{S}}^{N^{-1}}(\mathrm{Mod}^N(\mathcal{I})) \subseteq \mathrm{Mod}^N(\mathcal{I})$.*

4. *$\mathbf{P}_{\mathrm{R}}^N(\mathrm{Mod}^N(\mathcal{I})) \subseteq \mathrm{Mod}^N(\mathcal{I})$.*

Proof:
The Proof is very similar to that of Theorem 26. In this case, for 1, we combine Corollaries 3 and 5. For 2, we use Corollary 12. For 3 Corollary 14 for the first and Corollary 16 for the second and third inclusions. Finally, for the last part, use Corollary 25. ■

Finally, a result in the spirit of Theorem 0.6.1 of [7], a fundamental general result for the study of the structure of the model classes of sentential logics, is provided for π-institutions. It depicts the kind of results, like Theorem 3.15 of [9], given in the Introduction, that it is hoped that the current line of research will motivate in the framework of π-institutions.

Theorem 28 (Characterization of $\mathrm{Mod}(\mathcal{I})$) *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution and $\mathfrak{K}$ a class of π-institution models of $\mathcal{I}$, that is strongly adequate for $\mathcal{I}$. Then*

$$\mathrm{Mod}(\mathcal{I}) = \mathbf{H}\mathbf{H}_{\mathrm{S}}^{-1}\mathbf{P}(\mathfrak{K}).$$

Proof:
Since $\mathfrak{K} \subseteq \mathrm{Mod}(\mathcal{I})$, Theorem 26 yields that

$$\mathbf{HH}_{\mathrm{S}}^{-1}\mathbf{P}(\mathfrak{K}) \subseteq \mathbf{HH}_{\mathrm{S}}^{-1}\mathbf{P}(\mathrm{Mod}(\mathcal{I})) \subseteq \mathrm{Mod}(\mathcal{I}).$$

Suppose, conversely, that $\langle \mathcal{I}', \langle F, \alpha \rangle \rangle \in \mathrm{Mod}(\mathcal{I})$ and assume that $\mathfrak{K} = \{\langle \mathcal{I}^i, \langle F^i, \alpha^i \rangle \rangle : i \in I\}$. Then, it is not difficult to see, using the following diagram, that

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "\langle F^i, \alpha^i \rangle"] \arrow[d, "\langle \mathrm{I}_{\mathbf{Sign}}, \iota \rangle"'] \arrow[dr, "\prod_{i \in I} \langle F^i, \alpha^i \rangle"] & \mathcal{I}^i \arrow[d, "\langle P^i, \pi^i \rangle"] \\
\mathcal{I} \arrow[r, "\prod_{i \in I} \langle F^i, \alpha^i \rangle"'] \arrow[d, "\langle F, \alpha \rangle"] & \prod_{i \in I} \mathcal{I}^i \\
\mathcal{I}' &
\end{tikzcd}$$

$\langle \mathcal{I}, \langle I_{\mathbf{Sign}}, \iota \rangle \rangle \in \mathbf{H}_{\mathrm{S}}^{-1}\mathbf{P}(\mathrm{Mod}(\mathfrak{K}))$, whence, $\langle \mathcal{I}', \langle F, \alpha \rangle \rangle \in \mathbf{HH}_{\mathrm{S}}^{-1}\mathbf{P}(\mathfrak{K})$.
Therefore, we obtain $\mathrm{Mod}(\mathcal{I}) = \mathbf{HH}_{\mathrm{S}}^{-1}\mathbf{P}(\mathfrak{K})$. ■

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