Categorical Abstract Algebraic Logic:
Compatibility Operators and the Leibniz Hierarchy

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To **Don Pigozzi** this work is dedicated
on the occasion of his 80th Birthday.

Abstract

A unified treatment of the operator approach to categorical abstract algebraic logic (CAAL) was recently presented by the author using as tools the notions of compatibility operator of Czelakowski, of coherent compatibility operator of Albuquerque, Font and Jansana and exploiting an abstract Galois connection established via the use of these operators. The approach encompasses previous work by the author, but it also enriches the semantic, i.e., operator-based, side of the categorical Leibniz hierarchy with many new results. In this paper, we continue the work by providing, inter alia, characterizations of the categorical analogs of the classes of the Leibniz hierarchy based on full generalized matrix systems and on various properties of the categorical Leibniz and Suszko operators.

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1 Introduction: Compatibility Operators

The present paper is a continuation of the work presented in [31], which is, in turn, based on the work of Albuquerque, Font and Jansana [1]. Thus, to avoid repetition and to get as quickly as possible to new results, not as yet covered in [31], we open with a very brief overview of the work in [1] and present in Section 2 only the most basic notions of CAAL that were used in [31] and are also needed in the present work. For additional notions and results, as needed, we rely heavily on the predecessor paper [31], freely referring to its contents, albeit at the expense of self-sufficiency and at the risk of causing a, hopefully, minor inconvenience. For brevity Theorem P.x refers to Theorem x of [31] (the *P*redecessor paper) and the same holds for lemmas, propositions etc.

Let $\mathcal{S} = \langle \mathcal{L}, C_{\mathcal{S}} \rangle$ be a sentential logic. An *$\mathcal{S}$-compatibility operator* $\nabla^{\boldsymbol{A}}$ maps an $\mathcal{S}$-filter F in the collection $\mathrm{Fi}_{\mathcal{S}}(\boldsymbol{A})$ of $\mathcal{S}$-filters on an $\mathcal{L}$-algebra $\boldsymbol{A}$ to a congruence $\nabla^{\boldsymbol{A}}(F)$ on $\boldsymbol{A}$ that is compatible with the filter. The Leibniz operator [4] $\Omega^{\boldsymbol{A}}$ is the largest $\mathcal{S}$-compatibility operator and the Suszko operator [9] $\widetilde{\Omega}^{\boldsymbol{A}}$ is the largest order-preserving $\mathcal{S}$-compatibility operator.

Given an $\mathcal{S}$-compatibility operator $\nabla^{\boldsymbol{A}}$, the *lifting* $\widetilde{\nabla}^{\boldsymbol{A}}$ associates with an arbitrary collection of $\mathcal{S}$-filters on $\boldsymbol{A}$ the largest congruence on $\boldsymbol{A}$ that is compatible with all filters in the collection. Moreover, the *relativization* $\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}$ associates with an $\mathcal{S}$-filter on $\boldsymbol{A}$ the largest congruence on $\boldsymbol{A}$ that is compatible with all $\mathcal{S}$-filters on $\boldsymbol{A}$ containing the given filter. The Tarski operator [11] is the lifting of the Leibniz operator and the Suszko operator is its relativization. These three operators constitute the prototypical examples of operators. They motivated the general theory and they form the cornerstones on which both the work in [1] and the present work are based.

Given an $\mathcal{S}$-compatibility operator $\nabla^{\boldsymbol{A}}$, and a congruence θ on $\boldsymbol{A}$, let $\nabla^{\boldsymbol{A}^{-1}}(\theta) = \{F \in \mathrm{Fi}_{\mathcal{S}}(\boldsymbol{A}) : \theta \subseteq \nabla^{\boldsymbol{A}}(F)\}$. The springboard of the theory in [1] is the observation that $\widetilde{\nabla}^{\boldsymbol{A}}$ and $\nabla^{\boldsymbol{A}^{-1}}$ form a Galois connection: $\mathcal{P}(\mathrm{Fi}_{\mathcal{S}}(\boldsymbol{A})) \underset{\nabla^{\boldsymbol{A}^{-1}}}{\overset{\widetilde{\nabla}^{\boldsymbol{A}}}{\rightleftarrows}} \mathrm{Con}(\boldsymbol{A})$. The fixed points are the so-called $\nabla^{\boldsymbol{A}}$-*full sets of $\mathcal{S}$-filters* and the $\nabla^{\boldsymbol{A}}$-*full congruences*.

For a given $\mathcal{S}$-filter $F \in \mathrm{Fi}_{\mathcal{S}}(\boldsymbol{A})$, the collection of all $\mathcal{S}$-filters on $\boldsymbol{A}$ with which $\nabla^{\boldsymbol{A}}(F)$ is compatible constitutes the $\nabla^{\boldsymbol{A}}$-*class* $[\![F]\!]^{\nabla^{\boldsymbol{A}}}$ of F (Definition 3.14 of [1]), which forms a complete lattice. The smallest element of this class is denoted $F^{\nabla^{\boldsymbol{A}}} = \bigcap [\![F]\!]^{\nabla^{\boldsymbol{A}}}$. A filter F is termed a $\nabla^{\boldsymbol{A}}$-*filter* in [1] if $F = F^{\nabla^{\boldsymbol{A}}}$, i.e., if it is the smallest filter that is compatible with its $\nabla^{\boldsymbol{A}}$-associated congruence (see, also, [12, 13] and [19]).

A *family* $\nabla = \{\nabla^{\boldsymbol{A}}\}_{\boldsymbol{A} \in \mathsf{Alg}(\mathcal{L})}$ *of* $\mathcal{S}$*-compatibility operators* is formed when an $\mathcal{S}$-compatibility operator $\nabla^{\boldsymbol{A}}$ is defined, for every $\mathcal{L}$-algebra $\boldsymbol{A}$. To relate the members of ∇ the increasing in strength notions of coherence, commutativity with inverse images of surjective homomorphisms and commutativity with inverse images of arbitrary homomorphisms are introduced in Definitions 4.5 and 4.7 of [1]. The first is novel in [1] whereas the latter two are well known in traditional abstract algebraic logic (AAL) and play a critical role in the theory of protoalgebraic [2], equivalential [6, 7] and algebraizable [4, 16] logics (see also [8, 14]).

Coherence is used in establishing a General Correspondence Theorem (Theorem 4.15 of [1]) that encompasses several well-known isomorphism theorems from the theory of protoalgebraic logics and beyond, including results of Blok and Pigozzi [4, 5], of Czelakowski [9] and of Font and Jansana [12]. Moreover, a parallel categorical theory led to the formulation of analogs of these Correspondence Theorems in the categorical context (Theorems 40 and 43 of [31]), which comprise some previously known correspondence theorems from CAAL, e.g., Theorem 13 of [24] and Theorem 5.9 of [28].

Using an abstract family ∇ of $\mathcal{S}$-compatibility operators, Albuquerque, Font and Jansana define in Subsection 4.2 of [1] classes of algebras consisting of algebras that are reduced with respect to corresponding types of congruences. These parallel the well-known classes $\mathsf{Alg}^*\mathcal{S}$ of (Leibniz-) reduced, $\mathsf{Alg}\mathcal{S}$ of Tarski-reduced and $\mathsf{Alg}^{\mathsf{Su}}\mathcal{S}$ of Suszko-reduced algebras from the classical operator theory of AAL. The hypotheses of coherence and commutativity with inverse images of surjective homomorphisms imply various relationships between these classes, analogous to those established in the traditional context between $\mathsf{Alg}^*\mathcal{S}, \mathsf{Alg}\mathcal{S}$ and $\mathsf{Alg}^{\mathsf{Su}}\mathcal{S}$.

Using the concepts of full generalized matrix models, of the Leibniz operator, of the Suszko operator and of the aforementioned classes of algebras associated with $\mathcal{S}$, a wealth of characterizations of the classes in the AAL hierarchy is obtained in Section 6 of [1]. Some of these have been well-known in the AAL literature, some less well-known and some are new. What is remarkable, however, and motivated the present exposition, is the fact that they are all obtained as consequences of the treatment of abstract $\mathcal{S}$-compatibility operators and the basic Galois connection, as specialized in the context of the three main operators of AAL, essentially the Leibniz operator, since it is the fundamental among the three, whose lifting and relativization are the Tarski and Suszko operators, respectively.

2 The Categorical Compatibility Operators

In [31] the work of Albuquerque, Font and Jansana [1], outlined in Section 1 was adapted to the context of logics formalized as π-institutions, forming the fundamental objects of study in CAAL. The work of [31] constitutes the first part in the study along these lines and it is continued in the present paper. Thus, the present work relies heavily both on results presented in [1] and their more abstract versions established in [31]. In this section we review very briefly preliminary concepts and results from [31] that will be needed for subsequent developments.

Let **Sign** be a category, referred to as a **category of signatures**. Let, also, $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a set-valued functor from the category of signatures, referred to as a **sentence functor**. A collection $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, with $T_\Sigma \subseteq \mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$, is called a **sentence family** of SEN.

Consider a **category** N **of natural transformations** on SEN in the sense of, e.g., Section 2 of [25]. The triple $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ is called an **algebraic system**. An equivalence family $\theta = \{\theta_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ on SEN, i.e., a $|\mathbf{Sign}|$-indexed family of equivalence relations, is called a **congruence family** on $\boldsymbol{A}$ if it is invariant under N-morphisms, i.e., if, for all $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi_0, \psi_0, \ldots, \varphi_{k-1}, \psi_{k-1} \in \mathrm{SEN}(\Sigma)$,

$$\langle \varphi_i, \psi_i\rangle \in \theta_\Sigma,\ i < k,\ \text{imply}\ \langle \sigma_\Sigma(\varphi_0, \ldots, \varphi_{k-1}), \sigma_\Sigma(\psi_0, \ldots, \psi_{k-1})\rangle \in \theta_\Sigma.$$

A **congruence system** is an equivalence family that is invariant under both **Sign**-morphisms and N-morphisms. The collection of all congruence systems on $\boldsymbol{A}$ is denoted by $\mathrm{ConSys}(\boldsymbol{A})$. Ordered by signature-wise inclusion $\leq$, it forms a complete lattice denoted by $\mathbf{ConSys}(\boldsymbol{A}) = \langle \mathrm{ConSys}(\boldsymbol{A}), \leq\rangle$.

Let $\boldsymbol{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be a fixed algebraic system, termed the **base algebraic system**. An algebraic system $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ is called an N-**algebraic system** if there exists a surjective functor $\ ' : N \to N'$ that preserves all projection natural transformations and, therefore, preserves also the arities of all natural transformations in N. We write σ' in N' to indicate the image in N' of a σ in N under the functor $\ '$. Given two N-algebraic systems $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ and $\boldsymbol{B} = \langle \mathbf{Sign}'', \mathrm{SEN}'', N''\rangle$, an N-(**algebraic system**) **morphism** $\langle H, \gamma\rangle : \boldsymbol{A} \to \boldsymbol{B}$ consists of

- a functor $H : \mathbf{Sign}' \to \mathbf{Sign}''$ and
- a natural transformation $\gamma : \mathrm{SEN}' \to \mathrm{SEN}'' \circ H$, such that, for all $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, all $\Sigma \in |\mathbf{Sign}'|$ and all $\varphi_0, \ldots, \varphi_{k-1} \in \mathrm{SEN}'(\Sigma)$,

$$\gamma_\Sigma(\sigma'_\Sigma(\varphi_0, \ldots, \varphi_{k-1})) = \sigma''_{H(\Sigma)}(\gamma_\Sigma(\varphi_0), \ldots, \gamma_\Sigma(\varphi_{k-1})).$$

Given an N-morphism $\langle H,\gamma\rangle : \boldsymbol{A} \to \boldsymbol{B}$, the **kernel** of $\langle H,\gamma\rangle$ is the congruence system $\mathrm{Ker}(\langle H,\gamma\rangle) = \{\mathrm{Ker}_\Sigma(\langle H,\gamma\rangle)\}_{\Sigma\in|\mathbf{Sign}'|}$ on $\boldsymbol{A}$, defined, for all $\Sigma \in |\mathbf{Sign}'|$, by

$$\mathrm{Ker}_\Sigma(\langle H,\gamma\rangle) = \{\langle\varphi,\psi\rangle \in \mathrm{SEN}'(\Sigma)^2 : \gamma_\Sigma(\varphi) = \gamma_\Sigma(\psi)\}.$$

Given an algebraic system $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ and a congruence system θ on $\boldsymbol{A}$, one can define the **quotient algebraic system** $\boldsymbol{A}/\theta = \langle \mathbf{Sign}, \mathrm{SEN}^\theta, N^\theta\rangle$ of $\boldsymbol{A}$ by θ (see, e.g., [22]). In this case $\langle I_{\mathbf{Sign}}, \pi^\theta\rangle : \boldsymbol{A} \to \boldsymbol{A}/\theta$ denotes the projection morphism from $\boldsymbol{A}$ onto $\boldsymbol{A}/\theta$. Thus, given a class K of algebraic systems, it makes sense to consider the **K-relative congruence systems on $\boldsymbol{A}$**, i.e., those $\theta \in \mathrm{ConSys}(\boldsymbol{A})$, such that $\boldsymbol{A}/\theta \in \mathsf{K}$. The class of all relative K-congruence systems on $\boldsymbol{A}$ is denoted by $\mathrm{ConSys}_{\mathsf{K}}(\boldsymbol{A})$.

Let $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ be an algebraic system and $T = \{T_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ a sentence family of SEN. A congruence system $\theta = \{\theta_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ on $\boldsymbol{A}$ is **compatible with** T, denoted T comp θ, if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi,\psi \in \mathrm{SEN}(\Sigma)$,

$$\langle\varphi,\psi\rangle \in \theta_\Sigma \quad \text{and} \quad \varphi \in T_\Sigma \quad \text{imply} \quad \psi \in T_\Sigma.$$

Consider, again, an algebraic system $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$. Given a sentence family T of SEN there always exists a largest congruence system on $\boldsymbol{A}$ that is compatible with T (Proposition 2.2. of [25]). It is called the **Leibniz congruence system** of T on $\boldsymbol{A}$ and denoted $\Omega^{\boldsymbol{A}}(T) = \{\Omega^{\boldsymbol{A}}_\Sigma(T)\}_{\Sigma\in|\mathbf{Sign}|}$.

Given a collection $\mathcal{T}$ of sentence families of SEN, there always exists a largest congruence system on $\boldsymbol{A}$ that is compatible with every $T \in \mathcal{T}$. This is termed the **Tarski congruence system** of $\mathcal{T}$ on $\boldsymbol{A}$ and denoted by $\widetilde{\Omega}^{\boldsymbol{A}}(\mathcal{T})$.

A **π-institution**[1] $\mathcal{I} = \langle \boldsymbol{A}, C\rangle$ consists of

- an algebraic system $\boldsymbol{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$ and
- a **closure system** C on SEN, i.e., a family of closure operators $C = \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ that satisfy, for all $\Sigma,\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma,\Sigma')$,

  $$\mathrm{SEN}(f)(C_\Sigma(\Phi)) \subseteq C_{\Sigma'}(\mathrm{SEN}(f)(\Phi)), \text{ for all } \Phi \subseteq \mathrm{SEN}(\Sigma),$$

  a condition known as **structurality**.

Given a π-institution $\mathcal{I} = \langle \boldsymbol{A}, C\rangle$, a sentence family (system) $T = \{T_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ of SEN is called a **theory family** (**system**) if each $T_\Sigma \subseteq \mathrm{SEN}(\Sigma)$ is a Σ-**theory**, i.e., a closed set under C: $C_\Sigma(T_\Sigma) = T_\Sigma$. The collection of all

[1] This is the same as a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, augmented with a category N of natural transformations on its sentence functor SEN, in traditional CAAL.

theory families of $\mathcal{I}$ is denoted by $\mathrm{ThFam}(\mathcal{I})$. Ordered by signature wise inclusion $\le$, the collection of all theory families forms a complete lattice that is denoted by $\mathbf{ThFam}(\mathcal{I}) = \langle \mathrm{ThFam}(\mathcal{I}), \le \rangle$.

Let $\mathcal{I} = \langle \boldsymbol{A}, C \rangle$ be a π-institution. As a special case of the definition of the Tarski congruence system of a collection of sentence families, we obtain the **Tarski congruence system of** $\mathcal{I}$, i.e., the largest congruence system that is compatible with every theory family $T \in \mathrm{ThFam}(\mathcal{I})$. Ordinarily, instead of the notation $\widetilde{\Omega}^{\boldsymbol{A}}(\mathrm{ThFam}(\mathcal{I}))$, we use the notation $\widetilde{\Omega}^{\boldsymbol{A}}(C)$ or $\widetilde{\Omega}(\mathcal{I})$ for this congruence system.

Consider, again, a π-institution $\mathcal{I} = \langle \boldsymbol{A}, C \rangle$ and a theory family $T \in \mathrm{ThFam}(\mathcal{I})$. The **Suszko congruence system of** T **in** $\mathcal{I}$, denoted $\widetilde{\Omega}^{\mathcal{I}}(T)$, is the largest congruence system that is compatible with all $T' \in \mathrm{ThFam}(\mathcal{I})$, such that $T \le T'$. This set is usually denoted $(\mathrm{ThFam}(\mathcal{I}))^T = \{T' \in \mathrm{ThFam}(\mathcal{I}) : T \le T'\}$. Thus, $\widetilde{\Omega}^{\mathcal{I}}(T) = \widetilde{\Omega}^{\boldsymbol{A}}((\mathrm{ThFam}(\mathcal{I}))^T)$.

In summary, the three congruence systems $\Omega^{\boldsymbol{A}}(T)$, $\widetilde{\Omega}^{\mathcal{I}}(T)$ and $\widetilde{\Omega}^{\boldsymbol{A}}(C)$ are related by $\widetilde{\Omega}^{\mathcal{I}}(T) = \bigcap\{\Omega^{\boldsymbol{A}}(T') : T' \in \mathrm{ThFam}(\mathcal{I}), T \le T'\}$ and $\widetilde{\Omega}(\mathcal{I}) = \bigcap\{\Omega^{\boldsymbol{A}}(T) : T \in \mathrm{ThFam}(\mathcal{I})\}$.

Let $\mathbf{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ be a base algebraic system and $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ an N-algebraic system. A pair $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha \rangle \rangle$ is an (**interpreted**) N-**algebraic system**[2] if $\boldsymbol{A}$ is an N-algebraic system and $\langle F, \alpha \rangle : \mathrm{SEN} \to \mathrm{SEN}'$ is an N-morphism.

Let $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha \rangle \rangle$ and $\mathcal{B} = \langle \boldsymbol{B}, \langle G, \beta \rangle \rangle$ be two interpreted N-algebraic systems. An N-morphism $\langle H, \gamma \rangle : \boldsymbol{A} \to \boldsymbol{B}$ is called an N-**morphism from** $\mathcal{A}$ **to** $\mathcal{B}$, denoted $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$, if the following triangle commutes:

$$\begin{tikzcd}
 & \mathrm{SEN} \arrow[dl, "{\langle F, \alpha \rangle}"'] \arrow[dr, "{\langle G, \beta \rangle}"] & \\
\mathrm{SEN}' \arrow[rr, "{\langle H, \gamma \rangle}"'] & & \mathrm{SEN}''
\end{tikzcd}$$

Such an N-morphism is said to be **surjective** if both $H : \mathbf{Sign}' \to \mathbf{Sign}''$ and all $\gamma_{\Sigma'} : \mathrm{SEN}'(\Sigma') \to \mathrm{SEN}''(H(\Sigma'))$, $\Sigma' \in |\mathbf{Sign}'|$, are surjective.

An N-**matrix system** $\mathfrak{A} = \langle \boldsymbol{A}, T' \rangle$ is a pair consisting of an N-algebraic system $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ and a sentence family $T' = \{T'_\Sigma\}_{\Sigma \in |\mathbf{Sign}'|}$ of SEN'. An (**interpreted**) N-**matrix system**[2] $\mathfrak{A} = \langle \mathcal{A}, T' \rangle$ is a pair consisting of an interpreted N-algebraic system $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha \rangle \rangle$ and a sentence family $T' = \{T'_\Sigma\}_{\Sigma \in |\mathbf{Sign}'|}$ of SEN'.

[2] Hopefully, the overloading of terminology will not cause any confusion.

Fix a base algebraic system $\mathbf{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ and a π-institution $\mathcal{I} = \langle \mathbf{F}, C \rangle$, referred to as the **base π-institution**.[3] Then an interpreted N-matrix system $\mathfrak{A} = \langle \mathcal{A}, T' \rangle$ is called an **$\mathcal{I}$-matrix system** if T' is an **$\mathcal{I}$-filter family** of $\mathcal{A}$, i.e., for all $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\varphi \in C_\Sigma(\Phi)$, and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq T'_{\Sigma'} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) \in T'_{\Sigma'}.$$

We denote by $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ the collection of all $\mathcal{I}$-filter families of $\mathcal{A}$. Ordered by signature-wise inclusion $\leq$, this set becomes a complete lattice, denoted by $\mathbf{FiFam}^{\mathcal{I}}(\mathcal{A}) = \langle \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}), \leq \rangle$. We set $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^{T'} = \{T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : T' \leq T''\}$. The following lemma (Lemma P.3) provides some preservation properties of $\mathcal{I}$-filter families under the application of N-morphisms between the underlying N-algebraic systems.

Lemma 1 *Let $\mathcal{I} = \langle \mathbf{F}, C \rangle$ be a π-institution, $\mathcal{A}, \mathcal{B}$ be N-algebraic systems, $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$ an N-morphism and T'' a sentence family of $\mathcal{B}$.*

1. *If $T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, then $\gamma^{-1}(T'') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.*
2. *If $\gamma^{-1}(T'') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, then $T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$.*
3. *If $\langle H, \gamma \rangle$ is such that H is an isomorphism, and $\mathrm{Ker}(\langle H, \gamma \rangle)$ is compatible with $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, then $\gamma(T') \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$.*

Similar concepts and terminology may be applied to the so-called generalized matrix systems or gmatrix systems for short. An **N-gmatrix system** $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{T}' \rangle$ is a pair consisting of an N-algebraic system $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}', N' \rangle$ and a collection of sentence families $\mathcal{T}'$ of SEN'. An **(interpreted) N-gmatrix system**[2] $\mathbb{A} = \langle \mathcal{A}, \mathcal{T}' \rangle$ is a pair consisting of an interpreted N-algebraic system $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha \rangle \rangle$ and a collection of sentence families $\mathcal{T}'$ of SEN'. An **$\mathcal{I}$-gmatrix system** $\mathbb{A} = \langle \mathcal{A}, \mathcal{T}' \rangle$ is a tuple, such that every sentence family in $\mathcal{T}'$ is an $\mathcal{I}$-filter family of $\mathcal{A}$.

Note that, given an interpreted N-algebraic system $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha \rangle \rangle$, the pair $\mathcal{I}' = \langle \boldsymbol{A}, \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \rangle$ is also a π-institution (in closure system form). In accordance, we define the **Suszko congruence of** $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, denoted $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T')$ by

$$\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T') = \widetilde{\Omega}^{\mathcal{I}'}(T') = \bigcap \{\Omega^{\boldsymbol{A}}(T'') : T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}), T' \leq T''\}.$$

[3]The qualifying "base" is omitted whenever $\mathcal{I}$ is considered fixed in a specific context.

We also extend the notation $\Omega^{\boldsymbol{A}}(T')$ and $\Omega^{\boldsymbol{A}}(\mathcal{T}')$ to interpreted N-algebraic systems, writing $\Omega^{\mathcal{A}}(T')$ and $\Omega^{\mathcal{A}}(\mathcal{T}')$, with the meaning that these are identical to those applied to the underlying N-algebraic system $\boldsymbol{A}$ of $\mathcal{A}$. The restriction of $\Omega^{\mathcal{A}}$ to $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is the **Leibniz operator on** $\mathcal{A}$. The restriction of $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}$ to $\mathrm{ThFam}^{\mathcal{I}}(\mathcal{A})$ is the **Suszko operator on** $\mathcal{A}$ and the restriction of $\widetilde{\Omega}^{\mathcal{A}}$ on $\mathcal{P}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$ is the **Tarski operator on** $\mathcal{A}$. The families

$$\begin{array}{rcl} \Omega & = & \{\Omega^{\mathcal{A}} : \mathcal{A} \text{ an } N\text{-algebraic system}\} \\ \widetilde{\Omega}^{\mathcal{I}} := \widetilde{\Omega}^{\bullet,\mathcal{I}} & = & \{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}} : \mathcal{A} \text{ an } N\text{-algebraic system}\} \\ \widetilde{\Omega} & = & \{\widetilde{\Omega}^{\mathcal{A}} : \mathcal{A} \text{ an } N\text{-algebraic system}\} \end{array}$$

are termed the **Leibniz**, the **Suszko** and the **Tarski operator**, respectively. Saying that one of those **has a property** P **globally** means that property P holds for every member of the family. E.g., the Leibniz operator is globally order preserving if $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}(\boldsymbol{A})$ is order preserving, for every N-algebraic system $\mathcal{A}$. Proposition P.4 asserts some properties of these operators:

Proposition 2 *Let $\mathcal{I}$ be a π-institution, $\mathcal{A},\mathcal{B}$ two N-algebraic systems and $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$ a surjective N-morphism. For all $\mathcal{T}'' \cup \{T''\} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$,*

1. $\gamma^{-1}(\Omega^{\mathcal{B}}(T'')) = \Omega^{\mathcal{A}}(\gamma^{-1}(T''))$;
2. $\gamma^{-1}(\widetilde{\Omega}^{\mathcal{B}}(\mathcal{T}'')) = \widetilde{\Omega}^{\mathcal{A}}(\gamma^{-1}(\mathcal{T}''))$.
3. $\gamma^{-1}(\widetilde{\Omega}^{\mathcal{B},\mathcal{I}}(T'')) = \widetilde{\Omega}^{\mathcal{A},\mathcal{I}}((\gamma^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})))^{\gamma^{-1}(T'')})$.

The original definition of a full model in AAL was given by Font and Jansana in [11] and, it was, subsequently, adapted in CAAL in [23].

Let $\mathcal{I} = \langle \mathbf{F}, C\rangle$, with $\mathbf{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, be a π-institution and $\mathcal{A} = \langle \boldsymbol{A}, \langle F,\alpha\rangle\rangle$, with $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$, an N-algebraic system. A collection $\mathcal{T}' \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is **full** if $\mathcal{T}' = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}') \le \Omega^{\mathcal{A}}(T')\}$, i.e., $\mathcal{T}'$ consists of all $\mathcal{I}$-filter families on $\mathcal{A}$ with which the Tarski congruence system $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}')$ of $\mathcal{T}'$ is compatible.

If $\mathcal{T}'$ is full, then $\mathcal{T}'$ is a closure system on $\mathcal{A}$, whence the pair $\mathcal{I}' = \langle \boldsymbol{A}, \mathcal{T}'\rangle$ is a π-institution. We use the terminology **full $\mathcal{I}$-gmatrix system** for $\mathbb{A} = \langle \mathcal{A}, \mathcal{T}'\rangle$ when $\mathcal{T}'$ is a full collection of $\mathcal{I}$-filter families. Full $\mathcal{I}$-gmatrix systems were characterized in Proposition P.5 (see also Proposition 2.7 of [1]).

Proposition 3 *Let $\mathcal{A}$ be an N-algebraic system, let $\mathcal{T}' \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and $\langle I_{\mathbf{Sign}'}, \pi\rangle : \boldsymbol{A} \to \boldsymbol{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}')$ be the canonical projection N-morphism. Then the following conditions are equivalent:*

(i) $\mathcal{T}'$ *is full.*

(ii) $\pi(\mathcal{T}') = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}'))$.

(iii) $\mathcal{T}' = \pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}')))$.

(iv) $\mathcal{T}' = \gamma^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}))$ *for some N-algebraic system $\mathcal{B}$ and some surjective N-morphism $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$, with H an isomorphism.*

Given two N-matrix systems $\mathfrak{A} = \langle \boldsymbol{A}, T'\rangle$ and $\mathfrak{B} = \langle \boldsymbol{B}, T''\rangle$, an N-**matrix system morphism** $\langle H,\gamma\rangle : \mathfrak{A} \to \mathfrak{B}$ is a N-morphism $\langle H,\gamma\rangle : \boldsymbol{A} \to \boldsymbol{B}$, such that $\gamma^{-1}(T'') \le T'$. It is called **strict** if $\gamma^{-1}(T'') = T'$. These definitions extend to interpreted systems with the proviso that N-morphisms must be replaced by morphisms between interpreted systems, i.e., algebraic morphisms commuting with the interpretations.

A N-matrix system $\mathfrak{A} = \langle \boldsymbol{A}, T'\rangle$, with $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$ is said to be **Leibniz reduced** or simply **reduced** if $\Omega^{\boldsymbol{A}}(T') = \Delta^{\mathrm{SEN}'}$, where $\Delta^{\mathrm{SEN}'}$ is the identity congruence system on $\boldsymbol{A}$. A gmatrix system $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{T}'\rangle$ is **Tarski reduced** or simply **reduced** if $\widetilde{\Omega}^{\boldsymbol{A}}(\mathcal{T}') = \Delta^{\mathrm{SEN}'}$. Finally, we call an $\mathcal{I}$-matrix system $\mathbb{A} = \langle \mathcal{A}, T'\rangle$ **Suszko reduced** if $\widetilde{\Omega}^{\boldsymbol{A},\mathcal{I}}(T') = \Delta^{\mathrm{SEN}'}$. This terminology extends to interpreted N-matrix systems and to interpreted N-gmatrix systems.

By analogy with the universal algebraic framework, reduced $\mathcal{I}$-matrix systems, Suszko reduced $\mathcal{I}$-matrix systems and Tarski reduced $\mathcal{I}$-gmatrix systems give rise to natural classes of N-algebraic systems that are associated to a given base π-institution $\mathcal{I}$.

$$\begin{array}{rcl}
\mathsf{AlgSys}^*(\mathcal{I}) & = & \{\mathcal{A} : (\exists T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))(\Omega^{\mathcal{A}}(T') = \Delta^{\mathrm{SEN}'})\} \\
\mathsf{AlgSys}^{\mathsf{Su}}(\mathcal{I}) & = & \{\mathcal{A} : (\exists T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))(\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T') = \Delta^{\mathrm{SEN}'})\} \\
\mathsf{AlgSys}(\mathcal{I}) & = & \{\mathcal{A} : (\exists \mathcal{T}' \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))(\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}') = \Delta^{\mathrm{SEN}'})\} \\
 & = & \{\mathcal{A} : \widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathrm{SEN}'}\}.
\end{array}$$

Analogously with the corresponding AAL classes and accompanying results, established in [4, 9, 11], we may obtain the following characterizations of these classes ($\mathbf{I}$ denotes the isomorphic copies operator for interpreted N-algebraic systems):

Lemma 4 *Let $\mathcal{I}$ be a π-institution.*

1. $\mathsf{AlgSys}^*(\mathcal{I}) = \mathbf{I}(\{\mathcal{A}/\Omega^{\mathcal{A}}(T) : \mathcal{A}$ *N-alg system*, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\})$.

2. $\mathsf{AlgSys}^{\mathsf{Su}}(\mathcal{I}) = \mathbf{I}(\{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T) : \mathcal{A}$ *N-alg system*, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\})$.

3. $\mathsf{AlgSys}(\mathcal{I}) = \mathbf{I}(\{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) : \mathcal{A}\ N\text{-}alg\ system,\ \mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\ full\})$.
4. $\mathsf{AlgSys}(\mathcal{I}) = \mathbf{I}(\{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) : \mathcal{A}\ N\text{-}alg\ system,\ \mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\})$.
5. $\mathsf{AlgSys}(\mathcal{I}) = \mathsf{AlgSys}^{\mathsf{Su}}(\mathcal{I})$.

Finally, we define the main classes of the CAAL hierarchy of π-institutions. We note that in the traditional AAL hierarchy of sentential logics, the most important classes have equivalent semantic and syntactic characterizations. The semantic ones involve properties of the Leibniz and the other compatibility operators, and associated classes of models, whereas the syntactic ones are based on the existence of sets of formulas satisfying specific properties, such as, e.g., reflexivity, the deduction-detachment theorem or the congruence property [14]. In contrast, in the categorical setting, it has been conjectured that the corresponding semantic and syntactic properties may not be equivalent in general. Thus, a π-institution is said to be, e.g., *semantically protoalgebraic* if it satisfies the semantic property and *syntactically protoalgebraic* if it satisfies the corresponding syntactic property and, most likely, these terms are not equivalent. In the present work, we only introduce and employ the definitions of the semantically defined classes, using the categorical compatibility operators, and, hence, we omit the qualification "semantically", even though, as pointed out, it is, strictly speaking, necessary for differentiation purposes.

Definition 5 *Let $\mathcal{I} = \langle \boldsymbol{F}, C\rangle$ be a π-institution.*

- *$\mathcal{I}$ is* **protoalgebraic** ([2] in AAL and [25] in CAAL) *if Ω is globally order-preserving.*
- *$\mathcal{I}$ is* **equivalential** ([6, 7] in AAL and [27] in CAAL) *if Ω is globally order preserving and commutes with inverse N-morphisms.*
- *$\mathcal{I}$ is* **truth-equational** ([20] in AAL and [29] in CAAL) *if Ω is globally completely order reflecting.*
- *$\mathcal{I}$ is* **weakly algebraizable** ([10] in AAL and [30] in CAAL) *if it is protoalgebraic and truth-equational.*
- *$\mathcal{I}$ is* **algebraizable** ([4, 16, 17] in AAL and [21] in CAAL) *if it is equivalential and truth-equational.*

With this definition, we preserve the AAL Leibniz hierarchy:

$$\begin{tikzcd}
 & \text{algebraizable} \arrow[dl] \arrow[dr] & & \\
\text{equivalential} \arrow[dr] & & \text{weakly algebraizable} \arrow[dl] \arrow[dr] & \\
 & \text{protoalgebraic} & & \text{truth-equational}
\end{tikzcd}$$

3 The Leibniz and Suszko Operators

Albuquerque, Font and Jansana defined in [1] arbitrary $\mathcal{S}$-compatibility operators, studied their properties extensively and used the theory to prove a wealth of results pertaining to the Leibniz hierarchy of AAL by specializing to the Leibniz, Suszko and Tarski operators. Taking after their work, in [31], given an arbitrary π-institution $\mathcal{I} = \langle \boldsymbol{F}, C\rangle$ over a base algebraic system $\boldsymbol{F} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, an $\mathcal{I}$-compatibility operator on an interpreted N-algebraic system $\mathcal{A} = \langle \boldsymbol{A}, \langle F, \alpha\rangle\rangle$, with $\boldsymbol{A} = \langle \mathbf{Sign}', \mathrm{SEN}', N'\rangle$, was defined to be a mapping $\nabla^{\mathcal{A}} : \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \to \mathrm{ConSys}(\boldsymbol{A})$, such that, for all $T \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A})$, $\nabla^{\mathcal{A}}(T)$ is compatible with T. The $\nabla^{\mathcal{A}}$**-class** of $T \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A})$ was defined in Definition P.20 by

$$[\![T]\!]^{\nabla^{\mathcal{A}}} = \{T' \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A}) : \nabla^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\}.$$

In other words, the $\nabla^{\mathcal{A}}$-class of a filter family T of $\mathcal{A}$ consists of all those filter families of $\mathcal{A}$ with which the $\nabla^{\mathcal{A}}$-congruence system of T is compatible. The least element of this class is denoted by $T^{\nabla^{\mathcal{A}}} = \bigcap [\![T]\!]^{\nabla^{\mathcal{A}}}$ and T is called a $\nabla^{\mathcal{A}}$**-filter family** if $T = T^{\nabla^{\mathcal{A}}}$. The collection of all $\nabla^{\mathcal{A}}$-filter families on $\mathcal{A}$ is denoted by $\mathrm{FiFam}^{\nabla^{\mathcal{A}}}(\mathcal{A})$.

If a compatibility operator $\nabla^{\mathcal{A}}$ is defined for every N-algebraic system $\mathcal{A}$, the family $\nabla = \{\nabla^{\mathcal{A}} : \mathcal{A} \text{ an } N\text{-algebraic system}\}$ is formed. Such a family ∇ was called (**weakly**) **coherent** in Definition P.32 (see, also Definition 4.7 of [1]) if, for all surjective N-morphisms $\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$ (with H an isomorphism) and all $T'' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, if $\langle H, \gamma\rangle$ is $\nabla^{\mathcal{A}}$-compatible with $\gamma^{-1}(T'')$, then $\nabla^{\mathcal{A}}(\gamma^{-1}(T'')) = \gamma^{-1}(\nabla^{\mathcal{B}}(T''))$. $\nabla^{\mathcal{A}}$**-compatibility of** $\langle H, \gamma\rangle$ **with** $\gamma^{-1}(T'')$ means that $\mathrm{Ker}(\langle H, \gamma\rangle) \leq \nabla^{\mathcal{A}}(\gamma^{-1}(T''))$. Perhaps one of the most important results proven in [31] is the General Correspondence Theorem P.40 (see, also, Theorem 4.15 of [1]).

Theorem 6 (General Correspondence Theorem) *Let ∇ be a weakly coherent family of $\mathcal{I}$-compatibility operators. For every surjective N-morphism $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$, with H an isomorphism, and every $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, if $\langle H,\gamma\rangle$ is $\nabla^{\mathcal{A}}$-compatible with T, then $\langle H,\gamma\rangle$ induces an order isomorphism between $[\![T]\!]^{\nabla^{\mathcal{A}}}$ and $[\![\gamma(T)]\!]^{\nabla^{\mathcal{B}}}$, whose inverse is given by γ^{-1}.*

In the sequel, we view the Leibniz operator $\Omega^{\mathcal{A}}$ on an N-algebraic system $\mathcal{A}$ as a special case of an $\mathcal{I}$-compatibility operator and apply some of the results obtained in [31] to prove various analogs for the CAAL hierarchy of π-institutions of corresponding results established in [1] for sentential logics.

The $\Omega^{\mathcal{A}}$**-class** of $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is defined by specializing Definition P.20 of the $\nabla^{\mathcal{A}}$-class $[\![T]\!]^{\nabla^{\mathcal{A}}}$ of a theory family $T \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A})$ with respect to an arbitrary $\mathcal{I}$-compatibility operator $\nabla^{\mathcal{A}}$:

$$[\![T]\!]^{\Omega^{\mathcal{A}}} = \Omega^{\mathcal{A}^{-1}}(\Omega^{\mathcal{A}}(T)) = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')\}.$$

We call this the **Leibniz class** of T. By analogy to $T^{\nabla^{\mathcal{A}}}$, $T^{\Omega^{\mathcal{A}}}$ is the least element of the Leibniz class of T, called the **Leibniz filter family of** T. The filter family T is called a **Leibniz filter family** if $T = T^{\Omega^{\mathcal{A}}}$. The collection of all Leibniz filter families if $\mathrm{FiFam}^{\Omega^{\mathcal{A}}}(\mathcal{A})$.

The corresponding AAL notions have been developed and explored extensively in the work of the Barcelona School of AAL by, e.g., Font and Jansana [12, 13] and Jansana [19], and, more recently, by Albuquerque, Font and Jansana [1].

Every Leibniz class is a full closure system:

Proposition 7 *For all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, the Leibniz class $[\![T]\!]^{\Omega^{\mathcal{A}}}$ is full, whence it is a closure system on $\mathcal{A}$. It satisfies $\widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^{\Omega^{\mathcal{A}}}) = \Omega^{\mathcal{A}}(T)$. It is the largest $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, and the only full, such that $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \Omega^{\mathcal{A}}(T)$.*

Proof: By Proposition P.21, $[\![T]\!]^{\Omega^{\mathcal{A}}} = \Omega^{\mathcal{A}^{-1}}(\widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^{\Omega^{\mathcal{A}}}))$ and $[\![T]\!]^{\Omega^{\mathcal{A}}}$ is full. By Lemma P.27, $T \in [\![T]\!]^{\Omega^{\mathcal{A}}}$, whence, we get $\widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^{\Omega^{\mathcal{A}}}) \leq \Omega^{\mathcal{A}}(T)$. By Corollary P.11, $\widetilde{\Omega}^{\mathcal{A}} \circ \Omega^{\mathcal{A}^{-1}}$ is a closure on $\mathrm{ConSys}(\boldsymbol{A})$, whence

$$\Omega^{\mathcal{A}}(T) \leq \widetilde{\Omega}^{\mathcal{A}}(\Omega^{\mathcal{A}^{-1}}(\Omega^{\mathcal{A}}(T))) = \widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^{\Omega^{\mathcal{A}}}).$$

Thus, the second claimed equality holds. By Corollary P.18, the Tarski operator is injective on full $\mathcal{I}$-gmatrix systems, proving uniqueness. ■

Borrowing AAL notation, we denote by

$$[T] = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \Omega^{\mathcal{A}}(T') = \Omega^{\mathcal{A}}(T)\}$$

the equivalence class of the kernel of the Leibniz operator determined by a $T \in \mathrm{FiFam}^{\mathcal{A}}(\mathcal{I})$.

It is clear that $[T] \subseteq [\![T]\!]^{\Omega^{\mathcal{A}}}$, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. In the case of protoalgebraic π-institutions, the least elements in the collections of the form $[T]$ and the least elements in corresponding classes $[\![T]\!]^{\Omega^{\mathcal{A}}}$ coincide. Finally, for any π-institution $\mathcal{I}$, a filter family T equals $\bigcap[T]$ if and only if it is a Leibniz filter family.

Lemma 8 *For all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$:*

1. $T^{\Omega^{\mathcal{A}}} \leq \bigcap[T] \leq T$.
2. *If $T = T^{\Omega^{\mathcal{A}}}$, then $T = \bigcap[T]$.*
3. *If $\mathcal{I}$ is protoalgebraic, then $T = T^{\Omega^{\mathcal{A}}}$ iff $T = \bigcap[T]$.*

Proof: Suppose $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

1. Since $[T] \subseteq [\![T]\!]^{\Omega^{\mathcal{A}}}$, we get $T^{\Omega^{\mathcal{A}}} = \bigcap[\![T]\!]^{\Omega^{\mathcal{A}}} \leq \bigcap[T]$. Moreover, since $T \in [T]$, $\bigcap[T] \leq T$.
2. Immediate consequence of Part 1.
3. Suppose that $\mathcal{I}$ is protoalgebraic. Let $T = \bigcap[T]$. Since $T^{\Omega^{\mathcal{A}}} \leq T$, by protoalgebraicity, $\Omega^{\mathcal{A}}(T^{\Omega^{\mathcal{A}}}) \leq \Omega^{\mathcal{A}}(T)$. On the other hand, since $T^{\Omega^{\mathcal{A}}} \in [\![T]\!]^{\Omega^{\mathcal{A}}}$ ($[\![T]\!]^{\Omega^{\mathcal{A}}}$ is full), $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T^{\Omega^{\mathcal{A}}})$. Therefore, $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T^{\Omega^{\mathcal{A}}})$, whence $T^{\Omega^{\mathcal{A}}} \in [T]$ and, thus, $T = \bigcap[T] \leq T^{\Omega^{\mathcal{A}}}$. ■

Recalling that the Leibniz operator is the largest $\mathcal{I}$-compatibility operator, we can easily derive

Lemma 9 *Let $\nabla^{\mathcal{A}}$ be an $\mathcal{I}$-compatibility operator on $\mathcal{A}$. Then, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$:*

1. $[\![T]\!]^{\Omega^{\mathcal{A}}} \subseteq [\![T]\!]^{\nabla^{\mathcal{A}}}$.
2. $T^{\nabla^{\mathcal{A}}} \leq T^{\Omega^{\mathcal{A}}}$.
3. *Every $\nabla^{\mathcal{A}}$-filter family is a Leibniz filter family.*

Proof: For Part 1, if $T' \in [\![T]\!]^{\Omega^{\mathcal{A}}}$, we get $\Omega^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T')$, whence, since $\nabla^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T)$, we get $\nabla^{\mathcal{A}}(T) \le \Omega^{\mathcal{A}}(T')$ and, hence, $T' \in [\![T]\!]^{\nabla^{\mathcal{A}}}$. Part 2 is a direct consequence of Part 1. For Part 3, note, first, that, by Lemma 8, $T^{\Omega^{\mathcal{A}}} \le T$. But we also have

$$\begin{array}{rcll} T & = & T^{\nabla^{\mathcal{A}}} & \text{(by hypothesis)} \\ & \le & T^{\Omega^{\mathcal{A}}}. & \text{(by Part 2)} \end{array}$$

Thus, $T = T^{\Omega^{\mathcal{A}}}$ and T is a Leibniz filter family. ∎

Proposition 10 *For all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $T^{\Omega^{\mathcal{A}}}$ is a Leibniz filter family of $\mathcal{A}$.*

Proof: By Lemma 8, $(T^{\Omega^{\mathcal{A}}})^{\Omega^{\mathcal{A}}} \le T^{\Omega^{\mathcal{A}}}$. For the reverse inclusion, since $T^{\Omega^{\mathcal{A}}} \in [\![T]\!]^{\Omega^{\mathcal{A}}}$ ($[\![T]\!]^{\Omega^{\mathcal{A}}}$ is full), we get $[\![T^{\Omega^{\mathcal{A}}}]\!]^{\Omega^{\mathcal{A}}} \subseteq [\![T]\!]^{\Omega^{\mathcal{A}}}$. Therefore, $T^{\Omega^{\mathcal{A}}} = \bigcap [\![T]\!]^{\Omega^{\mathcal{A}}} \le \bigcap [\![T^{\Omega^{\mathcal{A}}}]\!]^{\Omega^{\mathcal{A}}} = (T^{\Omega^{\mathcal{A}}})^{\Omega^{\mathcal{A}}}$. ∎

One may thus derive the fact that, if $\mathcal{I}$ is a protoalgebraic π-institution, an $\mathcal{I}$-filter family is a Leibniz filter family iff it is the least element of some full $\mathcal{I}$-gmatrix system. Generalizing this to arbitrary π-institutions, we obtain

Theorem 11 *An $\mathcal{I}$-filter family T of $\mathcal{A}$ is a Leibniz filter family iff there exists a full $\mathcal{I}$-gmatrix system $\langle \mathcal{A}, \mathcal{T} \rangle$, such that $T = \bigcap \mathcal{T}$.*

Proof: Suppose that $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is a Leibniz filter family. Thus, T is the least element of its Leibniz class, which, by Proposition 7 is full.

Conversely, assume $T = \bigcap \mathcal{T}$, where $\langle \mathcal{A}, \mathcal{T} \rangle$ is a full $\mathcal{I}$-gmatrix system. Since $T = \bigcap \mathcal{T} \in \mathcal{T}$, we have $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \le \Omega^{\mathcal{A}}(T)$. Therefore, $[\![T]\!]^{\Omega^{\mathcal{A}}} = \Omega^{\mathcal{A}^{-1}}(\Omega^{\mathcal{A}}(T)) \subseteq \Omega^{\mathcal{A}^{-1}}(\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})) = \mathcal{T}$, with the last equality justified by Proposition P.17. Hence, $T = \bigcap \mathcal{T} \le \bigcap [\![T]\!]^{\Omega^{\mathcal{A}}} = T^{\Omega^{\mathcal{A}}}$. The converse inclusion always holds, whence T is in fact a Leibniz filter family. ∎

Corollary P.28 applied to the Leibniz operator yields the next proposition, which generalizes Proposition 5.6 of [1], which, in turn, taking into account Lemma 8, is an abstraction of Proposition 10 of [12].

Proposition 12 *A filter family $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is a Leibniz filter family of $\mathcal{A}$ iff $T/\Omega^{\mathcal{A}}(T)$ is the least $\mathcal{I}$-filter family of $\mathcal{A}/\Omega^{\mathcal{A}}(T)$.*

Taking into account Proposition 2 and instantiating Proposition P.35, we obtain the following

Lemma 13 *The Leibniz operator* $\Omega = \{\Omega^{\mathcal{A}} : \mathcal{A}$ *an* N*-algebraic system*$\}$ *is a coherent family of* $\mathcal{I}$*-compatibility operators.*

With this at hand, we can establish a correspondence theorem along the lines of Theorem 6 for Leibniz classes of algebraic systems connected by surjective N-morphisms (see Theorem 5.8 of [1]).

Theorem 14 (Correspondence for Leibniz Classes) *Let* $\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$ *be a surjective* N*-morphism and* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*. If* H *is an isomorphism and* $\langle H, \gamma\rangle$ *is* $\Omega^{\mathcal{A}}$*-compatible with* T*, then* $\langle H, \gamma\rangle$ *induces an order isomorphism between* $[\![T]\!]^{\Omega^{\mathcal{A}}}$ *and* $[\![\gamma(T)]\!]^{\Omega^{\mathcal{B}}}$*, whose inverse is* γ^{-1}*. Moreover, for every* $T' \in [\![T]\!]^{\Omega^{\mathcal{A}}}$*,* $\langle H, \gamma\rangle$ *induces an order isomorphism between* $[T']$ *and* $[\gamma(T')]$*.*

Proof: The first part follows by Theorem 6, using Lemma 13 to justify its applicability. For the second part, let $T', T'' \in [\![T]\!]^{\Omega^{\mathcal{A}}}$. Note that $T' \in [\![T]\!]^{\Omega^{\mathcal{A}}}$ implies $[T'] \subseteq [\![T]\!]^{\Omega^{\mathcal{A}}}$. By the already establishes first part, we obtain that $\gamma^{-1}(\gamma(T')) = T'$ and $\gamma^{-1}(\gamma(T'')) = T''$. Using Proposition 2 and the surjectivity of $\langle H, \gamma\rangle$, we now get

$$\begin{array}{rll}\Omega^{\mathcal{A}}(T') = \Omega^{\mathcal{A}}(T'') & \text{iff} & \Omega^{\mathcal{A}}(\gamma^{-1}(\gamma(T'))) = \Omega^{\mathcal{A}}(\gamma^{-1}(\gamma(T''))) \\ & \text{iff} & \gamma^{-1}(\Omega^{\mathcal{B}}(\gamma(T'))) = \gamma^{-1}(\Omega^{\mathcal{B}}(\gamma(T''))) \\ & \text{iff} & \Omega^{\mathcal{B}}(\gamma(T')) = \Omega^{\mathcal{B}}(\gamma(T'')).\end{array}$$

This shows that $T'' \in [T']$ iff $\gamma(T'') \in [\gamma(T')]$. Therefore, the order isomorphism between $[\![T]\!]^{\Omega^{\mathcal{A}}}$ and $[\![\gamma(T)]\!]^{\Omega^{\mathcal{B}}}$ restricts to one between $[T']$ and $[\gamma(T')]$. ∎

Theorem 14 partly generalizes to arbitrary π-institutions and to larger collections of filter families a correspondence theorem related to protoalgebraic π-institutions (Theorem 5.9 of [28], see also [3]). It is also an abstraction to the categorical context of the Correspondences established in Corollary 7.7 of [5] and Corollary 9 of [12].

Corollary 15 *For every surjective* N*-morphism* $\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$ *and every* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, if* H *is an isomorphism and* $\langle H, \gamma\rangle$ *is* $\Omega^{\mathcal{A}}$*-compatible with* T*, then* T *is a Leibniz filter family of* $\mathcal{A}$ *if and only if* $\gamma(T)$ *is a Leibniz filter family of* $\mathcal{B}$*.*

Switching to the Suszko operator for a similar study as the one just completed for the Leibniz operator, we call the class

$$[\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}} = \Omega^{\mathcal{A}^{-1}}(\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)) = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T) \leq \Omega^{\mathcal{A}}(T')\}$$

the **Suszko class** of T. Moreover, $T^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}$ is the least element of the class $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}$, and T is said to be a **Suszko filter family** if $T = T^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}$. Finally, $\mathrm{FiFam}^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}(\mathcal{A})$ denotes the collection of all Suszko filter families of $\mathcal{A}$.

Along the lines of Proposition 7, we show the following

Proposition 16 *For all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, the Suszko class $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}$ is full, whence it is a closure system on $\mathcal{A}$. It satisfies $\widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}) = \widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)$ and is the largest $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, and the only full, such that $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)$.*

Proof: By Proposition P.21, $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}} = \Omega^{\mathcal{A}^{-1}}(\widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}))$ and $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}$ is full. Note that, if $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, with $T \leq T'$, then

$$\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T) \leq \widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T') \leq \Omega^{\mathcal{A}}(T').$$

Thus, $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T \subseteq [\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}$. Therefore, $\widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}) \leq \widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)$. On the other hand,

$$\begin{array}{rcl}\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T) & \leq & \bigcap \Omega^{\mathcal{A}}(\{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T) \leq \Omega^{\mathcal{A}}(T')\}) \\ & = & \widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}).\end{array}$$

Thus, the second claimed equality holds. By Corollary P.18, the Tarski operator is injective on full $\mathcal{I}$-gmatrix systems, giving uniqueness. ∎

Taking into account the fact that the Suszko operator is the largest order preserving $\mathcal{I}$-compatibility operator, we also get

Lemma 17 *For $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$:*

1. $T^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}} \leq T^{\Omega^{\mathcal{A}}} \leq T$.
2. *Every Suszko filter family is a Leibniz filter family.*
3. *If $T \leq T'$, then $[\![T']\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}} \subseteq [\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}$ and $T^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}} \leq T'^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}$.*
4. $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T \subseteq [\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}} \subseteq (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^{T^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}}$.
5. $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}} \subseteq [\![T^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}$.
6. $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}} = (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$ *if and only if $T = T^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}$, i.e., iff T is a Suszko filter family.*

Proof: All statements are specializations of results proven previously for arbitrary $\mathcal{I}$-compatibility operators: The inequalities in Part 1 follow from Lemma 9 and Lemma 8, respectively. Part 2 specializes Lemma 9. Parts 3 and 4 specialize Lemma P.24. Part 5 specializes Lemma P.27 and, finally, Part 6 specializes Lemma P.25.

■

Note that for a $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, the filter family $T^{\widetilde{\Omega}^{\mathcal{I},\mathcal{A}}}$ need not be a Suszko filter family. This is illustrated in the sentential context in (counter) Example 5.12 of [1].

Since, according to Lemma 17, Suszko filter families are special cases of Leibniz filter families and the latter are least elements of full $\mathcal{I}$-gmatrix systems, the former are also. In addition, they can be characterized as the least elements of those full $\mathcal{I}$-gmatrix systems that are up-sets. The only such up-sets are the principal ones determined by the Suszko filter families themselves. More precisely, we obtain the following analog of Theorem 5.13 of [1]:

Theorem 18 *For all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, the following are equivalent:*

(i) *T is a Suszko filter family of $\mathcal{A}$.*

(ii) *$\langle \mathcal{A}, (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T \rangle$ is a full $\mathcal{I}$-gmatrix system.*

(iii) *$T = \bigcap \mathcal{T}$, for some full upset $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.*

Moreover, the principal upset $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$ is the only $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ that satisfies Condition (iii).

Proof:

(i)⇒(ii) The hypothesis implies, by Lemma 17, that $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}} = (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$. Therefore, by Proposition 16, we get that $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$ is full.

(ii)⇒(iii) Since $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$ is an up-set and $T = \bigcap(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$, (iii) follows.

(iii)⇒(i) Suppose, now, that $T = \bigcap \mathcal{T}$, for some full upset $\mathcal{T} \subseteq \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

Since $\mathcal{T}$ is full, it is a closure system, whence $T \in \mathcal{T}$. Therefore, since $\mathcal{T}$ is an up-set, $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T \subseteq \mathcal{T}$. But, by hypothesis, $T = \bigcap \mathcal{T}$, whence $\mathcal{T} \subseteq (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$. This shows that $\mathcal{T} = (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$.

Since $\mathcal{T}$ is full,

$$\begin{aligned}\mathcal{T} &= (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T \\ &= \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{A}}((\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T) \leq \Omega^{\mathcal{A}}(T')\}.\end{aligned}$$

But $\widetilde{\Omega}^{\mathcal{A}}((\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T) = \widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)$. Thus, $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T = [\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}$ and, therefore, $T = \bigcap [\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}$ is a Suszko filter family. ∎

It is possible that a Suszko filter family is the least filter family of another full $\mathcal{I}$-gmatrix system without that system being an up-set. In fact, since every Suszko filter family is also a Leibniz filter family, it is also the least filter family of a Leibniz class, which is also a full $\mathcal{I}$-gmatrix system.

Corollary P.28 applied to the Suszko operator gives

Proposition 19 *A filter family $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is a Suszko filter family of $\mathcal{A}$ iff $T/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)$ is the least filter family of $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)$.*

Moreover, Proposition P.42 and Lemma 13 yield

Lemma 20 *The Suszko operator $\widetilde{\Omega}^{\bullet,\mathcal{I}} = \{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}} : \mathcal{A}$ an N-algebraic system$\}$ is a weakly coherent family of $\mathcal{I}$-compatibility operators.*

The weak coherence asserted by Lemma 20 allows using the relativized Correspondence Theorem P.43:

Theorem 21 (Correspondence for Suszko Classes) *For every surjective $\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$ and every $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, if H is an isomorphism and $\langle H, \gamma\rangle$ is $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}$-compatible with T, then $\langle H, \gamma\rangle$ induces an order isomorphism between $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}$ and $[\![\gamma(T)]\!]^{\widetilde{\Omega}^{\mathcal{B},\mathcal{I}}}$, whose inverse is given by γ^{-1}.*

Theorem 21 is a strengthening of Corollary 3.12 of [26], which is an abstraction in the categorical context of Corollary 2.7 of [9]. In that case, we had an isomorphism between $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$ and $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}))^{\gamma(T)}$, under the hypothesis that $\langle H, \gamma\rangle$, with H an isomorphism, is a surjective, deductive $\mathcal{I}$-matrix system morphism from $\langle \mathcal{A}, T\rangle$ to $\langle \mathcal{B}, \gamma(T)\rangle$. The generalization is obtained by observing that the property of being deductive is equivalent to $\langle H, \gamma\rangle$ being $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}$-compatible with T. The present isomorphism extends that result to an isomorphism between the entire Suszko classes $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}$ and $[\![\gamma(T)]\!]^{\widetilde{\Omega}^{\mathcal{B},\mathcal{I}}}$, which contain the up-sets $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$ and $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}))^{\gamma(T)}$, respectively.

Finally, focusing on the least filter families of the complete lattices whose order isomorphism is established in Theorem 21, we obtain:

Corollary 22 *If $\langle H,\gamma\rangle : \mathcal{A} \to \mathcal{B}$ is a surjective N-morphism, with H an isomorphism, and, for $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\langle H,\gamma\rangle$ is $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}$-compatible with T, then T is a Suszko filter family of $\mathcal{A}$ iff $\gamma(T)$ is a Suszko filter family of $\mathcal{B}$.*

4 The Leibniz Hierarchy

We saw that a Leibniz class $[\![T]\!]^{\Omega^{\mathcal{A}}}$ is full, whence $\langle \mathcal{A}, [\![T]\!]^{\Omega^{\mathcal{A}}}\rangle$ is a full $\mathcal{I}$-gmatrix system. The following proposition characterizes those full $\mathcal{I}$-gmatrix systems that are of this form.

Proposition 23 *Let $\langle \mathcal{A}, \mathcal{T}\rangle$ be a full $\mathcal{I}$-gmatrix system. The following are equivalent:*

(i) $\mathcal{T} = [\![T]\!]^{\Omega^{\mathcal{A}}}$, *for some* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

(ii) $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \in \mathsf{AlgSys}^*(\mathcal{I})$.

Proof:

(i)⇒(ii) Assume $\mathcal{T} = [\![T]\!]^{\Omega^{\mathcal{A}}}$, for some $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Then, taking into account Proposition 7, $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \widetilde{\Omega}([\![T]\!]^{\Omega^{\mathcal{A}}}) = \Omega^{\mathcal{A}}(T)$. Thus, $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \in \mathsf{AlgSys}^*(\mathcal{I})$.

(ii)⇒(i) Suppose $\mathcal{B} = \mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \in \mathsf{AlgSys}^*(\mathcal{I})$. Then, there is $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, such that $\Omega^{\mathcal{B}}(T') = \Delta^{\mathrm{SEN}'/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})}$. Thus, $[\![T']\!]^{\Omega^{\mathcal{B}}} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$. Let $\langle I,\pi\rangle := \langle I_{\mathbf{Sign}'}, \pi^{\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})}\rangle : \mathcal{A} \to \mathcal{B}$ be the projection N-morphism. Since $\mathcal{T}$ is full, $\mathcal{T} = \pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}))$. Thus, $\mathcal{T} = \pi^{-1}([\![T']\!]^{\Omega^{\mathcal{B}}})$. Moreover $\mathrm{Ker}(\langle I,\pi\rangle) = \pi^{-1}(\Delta^{\mathrm{SEN}'/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})}) = \pi^{-1}(\Omega^{\mathcal{B}}(T')) = \Omega^{\mathcal{A}}(\pi^{-1}(T'))$, showing that $\langle I,\pi\rangle$ is $\Omega^{\mathcal{A}}$-compatible with $\pi^{-1}(T')$. The Correspondence Theorem 14 now yields that $\mathcal{T} = \pi^{-1}([\![T']\!]^{\Omega^{\mathcal{B}}}) = [\![\pi^{-1}(T')]\!]^{\Omega^{\mathcal{A}}}$.

■

It is not always the case that every full $\mathcal{I}$-gmatrix system is of the form $[\![T]\!]^{\Omega^{\mathcal{A}}}$, for some $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. This reflects the fact that, as in the sentential framework, it may happen that $\mathsf{AlgSys}^*(\mathcal{I}) \subsetneq \mathsf{AlgSys}(\mathcal{I})$. The following proposition provides some equivalent conditions for equality to hold.

Proposition 24 *Let $\mathcal{I}$ be a π-institution. The following are equivalent:*

(i) $\mathsf{AlgSys}(\mathcal{I}) = \mathsf{AlgSys}^*(\mathcal{I})$.

(ii) *For all* $\mathcal{A}$*, the class of full* $\mathcal{I}$*-gmatrix systems on* $\mathcal{A}$ *is* $\{\langle\mathcal{A},[\![T]\!]^{\Omega^{\mathcal{A}}}\rangle : T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})\}$.

(iii) *For all* $\mathcal{A}$ *and all* $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, there exists* $T'\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$*, such that* $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)=\Omega^{\mathcal{A}}(T')$.

Proof:

(i)⇒(ii) Suppose $\mathcal{T}\subseteq\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is full. Then $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})\in\mathsf{AlgSys}(\mathcal{I})$, i.e., by hypothesis, $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T})\in\mathsf{AlgSys}^*(\mathcal{I})$. Thus, by Proposition 23, $\mathcal{T}=[\![T]\!]^{\Omega^{\mathcal{A}}}$, for some $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

(ii)⇒(iii) Recall that all Suszko classes are full. Thus, by hypothesis, for all $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, there exists $T'\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}=[\![T']\!]^{\Omega^{\mathcal{A}}}$. Therefore, we obtain $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)=\widetilde{\Omega}^{\mathcal{A}}([\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}})=\widetilde{\Omega}^{\mathcal{A}}([\![T']\!]^{\Omega^{\mathcal{A}}})=\Omega^{\mathcal{A}}(T')$.

(iii)⇒(i) Recall that $\mathsf{AlgSys}^*(\mathcal{I})\subseteq\mathsf{AlgSys}(\mathcal{I})$ always holds. To prove the reverse inclusion, suppose that $\mathcal{A}\in\mathsf{AlgSys}(\mathcal{I})=\mathsf{AlgSys}^{\mathrm{Su}}(\mathcal{I})$. Thus, there exists $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)=\Delta^{\mathrm{SEN}'}$. By hypothesis, there exists $T'\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)=\Omega^{\mathcal{A}}(T')=\Delta^{\mathrm{SEN}'}$. Thus, $\mathcal{A}\in\mathsf{AlgSys}^*(\mathcal{I})$. ∎

Protoalgebraicity of $\mathcal{I}$ implies that $\mathsf{AlgSys}(\mathcal{I})=\mathsf{AlgSys}^*(\mathcal{I})$. This is a consequence of the following

Proposition 25 *A* π*-institution* $\mathcal{I}$ *is protoalgebraic iff the Leibniz and the Suszko operators coincide, i.e., iff, for all* $\mathcal{A}$ *and all* $T\in\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

$$\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)=\Omega^{\mathcal{A}}(T).$$

Proof: $\mathcal{I}$ is protoalgebraic iff the Leibniz operator is order-preserving iff, since it is, by definition, the largest $\mathcal{I}$-compatibility operator, it is the largest order-preserving $\mathcal{I}$-compatibility operator iff it is equal to the Suszko operator. ∎

It follows that for a protoalgebraic π-institution, Leibniz and Suszko classes, Leibniz and Suszko filter families, associated classes of N-algebraic systems and all other notions associated with those operators coincide. Therefore, in particular, $\mathsf{AlgSys}(\mathcal{I})=\mathsf{AlgSys}^*(\mathcal{I})$. This equality yields a characterization of the full $\mathcal{I}$-gmatrix systems in terms of Leibniz classes of filter families.

Corollary 26 *If a π-institution $\mathcal{I}$ is protoalgebraic,* $\mathsf{AlgSys}^*(\mathcal{I}) = \mathsf{AlgSys}(\mathcal{I})$ *and the full $\mathcal{I}$-gmatrix systems are the $\mathcal{I}$-gmatrix systems of the form* $\langle\mathcal{A}, [\![T]\!]^{\Omega^{\mathcal{A}}}\rangle$*, for some N-algebraic system $\mathcal{A}$ and some* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

In addition, we obtain the following characterizations of protoalgebraicity in terms of full $\mathcal{I}$-gmatrix systems, abstracting Theorem 6.5 of [1], which, in turn, extends Theorem 3.4 of [11]:

Theorem 27 *For a π-institution $\mathcal{I}$, the following are equivalent:*

(i) *$\mathcal{I}$ is protoalgebraic.*

(ii) *Every full collection of $\mathcal{I}$-filter families is an upset, i.e., has form* $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$*, for some $\mathcal{I}$-filter family T on some $\mathcal{A}$.*

(iii) *Every full collection of $\mathcal{I}$-filter families is of the form* $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$*, for some Suszko $\mathcal{I}$-filter family T on some $\mathcal{A}$.*

(iv) $[\![T]\!]^{\Omega^{\mathcal{A}}} = (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^{T^{\Omega^{\mathcal{A}}}}$*, for all $\mathcal{A}$ and all* $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

Proof:

(i)⇒(ii) Suppose $\langle\mathcal{A},\mathcal{T}\rangle$ is a full $\mathcal{I}$-gmatrix system. Then

$$\mathcal{T} = \{T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) : \widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \leq \Omega^{\mathcal{A}}(T')\}.$$

Since, by hypothesis, Ω is order-preserving, $\mathcal{T}$ is an upset. Since $\mathcal{T}$ is a closure system, it must be of the form $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$, for some $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, in fact for $T = \bigcap\mathcal{T}$.

(ii)⇒(iii) This follows by Theorem 18.

(iii)⇒(iv) For all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $[\![T]\!]^{\Omega^{\mathcal{A}}}$ is a full $\mathcal{I}$-gmatrix system. Thus, by hypothesis, $[\![T]\!]^{\Omega^{\mathcal{A}}} = (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^{T'}$, with $T' = \bigcap[\![T]\!]^{\Omega^{\mathcal{A}}} = T^{\Omega^{\mathcal{A}}}$.

(iv)⇒(i) Let $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \leq T'$. Then $T^{\Omega^{\mathcal{A}}} \leq T \leq T'$. By hypothesis, $T' \in [\![T]\!]^{\Omega^{\mathcal{A}}}$. Thus, $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. Since, for all $\mathcal{A}$, $\Omega^{\mathcal{A}}$ is order-preserving on $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\mathcal{I}$ is protoalgebraic. ■

Strengthening Proposition 25, we show that the coincidence of any two of the Leibniz and Suszko corresponding notions characterizes protoalgebraicity:

Proposition 28 *For a π-institution $\mathcal{I}$, the following are equivalent:*

(i) *$\mathcal{I}$ is protoalgebraic.*

(ii) *The full classes of $\mathcal{I}$-filter families coincide with the $\widetilde{\Omega}^{\mathcal{I}}$-full classes.*

(iii) $[\![T]\!]^{\Omega^{\mathcal{A}}} = [\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}$, *for all $\mathcal{A}$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.*

Proof:

(i)⇒(ii(i)) These follow from Proposition 25.

(ii)⇒(i) Suppose that the full classes of $\mathcal{I}$-filter families coincide with the $\widetilde{\Omega}^{\mathcal{I}}$-full classes. By Lemma P.27, every $\widetilde{\Omega}^{\mathcal{I}}$-full class is an upset. Thus, by hypothesis, all full classes are upsets. By Theorem 27, $\mathcal{I}$ is protoalgebraic.

(iii)⇒(i) Suppose $[\![T]\!]^{\Omega^{\mathcal{A}}} = [\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}$, for all $\mathcal{A}$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Let $\mathcal{A}$ be an N-algebraic system and $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \leq T'$. By Lemma 17, $T' \in [\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}} = [\![T]\!]^{\Omega^{\mathcal{A}}}$. Hence $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$. We conclude that $\mathcal{I}$ is protoalgebraic. ■

We note that none of the two conditions

- $T^{\Omega^{\mathcal{A}}} = T^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}$, for all $\mathcal{A}$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, or
- T is a Suszko filter family iff it is a Lebniz filter family, for all $\mathcal{A}$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$,

characterize protoalgebraicity of $\mathcal{I}$. This is because both hold vacuously for all truth-equational π-institutions. For every truth equational π-institution $\mathcal{I}$ and every N-algebraic system $\mathcal{A}$, T is Suszko, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, and, hence, T is also Leibniz (see Theorem 32 and related remarks pertaining to the sentential case following Proposition 6.6 of [1]).

In Theorem 27 a characterization of protoalgebraic π-institutions was provided in terms of the form of full classes of $\mathcal{I}$-filter families and of Leibniz classes. Next, we provide a similar characterization for truth-equational logics.

Given $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, we have $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}} \supseteq (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$. Thus, T is a Suszko filter family when the reverse inclusion holds, i.e., when

$$\text{for all } T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}),\ \widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T) \leq \Omega^{\mathcal{A}}(T') \text{ implies } T \leq T'.$$

This holding for all $T \in \mathrm{ThFam}^{\mathcal{I}}(\mathcal{A})$ is tantamount to $\Omega^{\mathcal{A}}$ being completely order reflecting (due to Raftery [20] in the AAL context):

Lemma 29 *The Leibniz operator $\Omega^{\mathcal{A}}$ is completely order reflecting on the class* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *iff, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, the following holds:*

$$\text{for all } T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}),\ \widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T) \leq \Omega^{\mathcal{A}}(T') \text{ implies } T \leq T'.$$

Proof: Suppose, first, that the Leibniz operator $\Omega^{\mathcal{A}}$ on $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ is completely order reflecting. Assume that $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T) \leq \Omega^{\mathcal{A}}(T')$. Note that

$$\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T) = \widetilde{\Omega}^{\mathcal{A}}((\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T) = \bigcap\{\Omega^{\mathcal{A}}(U) : U \in (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T\}.$$

Therefore, by hypothesis, $T = \bigcap(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T \leq T'$.

Conversely, assume that the displayed formula in the statement holds, for all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Let $T^i, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $i \in I$, be such that $\bigcap_{i \in I} \Omega^{\mathcal{A}}(T^i) \leq \Omega^{\mathcal{A}}(T')$. Then

$$\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(\bigcap_{i \in I} T^i) \leq \bigcap_{i \in I} \widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T^i) \leq \bigcap_{i \in I} \Omega^{\mathcal{A}}(T^i) \leq \Omega^{\mathcal{A}}(T').$$

Now, by hypothesis, $\bigcap_{i \in I} T^i \leq T'$ and $\Omega^{\mathcal{A}}$ is completely order reflecting on $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. ■

Corollary P.29 asserts that, for any $\mathcal{I}$-compatibility operator $\nabla^{\mathcal{A}}$, the condition $\nabla^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$ implies $T \leq T'$, for all $T, T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, is equivalent to the condition that all $\mathcal{I}$-filter families on $\mathcal{A}$ are $\nabla^{\mathcal{A}}$-filter families. Combining Lemma 29, with Corollary P.29, applied to the case of the Suszko operator, we get the following

Proposition 30 *The Leibniz operator $\Omega^{\mathcal{A}}$ is completely order reflecting on* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *iff every $\mathcal{I}$-filter family T of $\mathcal{A}$ is a Suszko filter family.*

Corollary P.29, applied to the case of the Leibniz operator, also gives

Proposition 31 *The Leibniz operator $\Omega^{\mathcal{A}}$ is order reflecting on* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *iff every $\mathcal{I}$-filter family T of $\mathcal{A}$ is a Leibniz filter family.*

We are now ready to provide the promised characterization of truth-equational π-institutions, an analog of Theorem 6.10 of [1]:

Theorem 32 *For a π-institution $\mathcal{I}$, the following are equivalent:*

(i) *$\mathcal{I}$ is truth-equational.*

(ii) *For all $\mathcal{A}$, every $\mathcal{I}$-filter family on $\mathcal{A}$ is a Suszko filter family.*

(iii) *For all* $\mathcal{A} \in \mathsf{AlgSys}(\mathcal{I})$*, every* $\mathcal{I}$*-filter family on* $\mathcal{A}$ *is a Suszko filter family.*

Proof:

(i)⇒(ii) A consequence of Proposition 30, taking into account Definition 5.

(ii)⇒(iii) Obvious.

(iii)⇒(i) Let $\mathcal{A}$ be an N-algebraic system, $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and consider the projection morphism $\langle I, \pi\rangle := \langle I_{\mathbf{Sign}'}, \pi^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)}\rangle : \mathcal{A} \to \mathcal{A}/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)$. By Lemma 1, $\pi(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T))$. On the quotient, define $T' = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T))$. Since $T' \leq \pi(T)$, by monotonicity of $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}$ and Lemma P.45,

$$\widetilde{\Omega}^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T),\mathcal{I}}(T') \leq \widetilde{\Omega}^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T),\mathcal{I}}(\pi(T)) = \Delta^{\mathrm{SEN}'/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)}.$$

Thus, $\widetilde{\Omega}^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T),\mathcal{I}}(T') = \widetilde{\Omega}^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T),\mathcal{I}}(\pi(T))$. By hypothesis, T' and $\pi(T)$ are Suszko filter families on $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)$, since $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T) \in \mathsf{AlgSys}^{\widetilde{\Omega}^{\mathcal{I}}}(\mathcal{I}) = \mathsf{AlgSys}(\mathcal{I})$. By Proposition P.23, the Suszko operator is injective on Suszko filter families, whence

$$T/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T) = T' = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)).$$

By Proposition 19, T is a Suszko filter family. By Proposition 30 and Definition 5, $\mathcal{I}$ is truth equational.

■

The following results assert that, under truth equationality, the converse implications of those in Corollary 26 hold:

Proposition 33 *If* $\mathcal{I}$ *is truth-equational and* $\mathsf{AlgSys}^*(\mathcal{I}) = \mathsf{AlgSys}(\mathcal{I})$*, then* $\mathcal{I}$ *is protoalgebraic.*

Proof: By Proposition 24, every full class of $\mathcal{I}$-filter families is of the form $[\![T']\!]^{\Omega^{\mathcal{A}}}$, for some $\mathcal{A}$ and some $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Thus, so are Suszko classes. If $\mathcal{A}$ is an arbitrary N-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, then $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}} = [\![T']\!]^{\Omega^{\mathcal{A}}}$, for some $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Hence $T^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}} = T'^{\Omega^{\mathcal{A}}}$. By Theorem 32, every filter family is Suszko and, in general, every Suszko filter family is Leibniz, whence $T = T^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}} = T'^{\Omega^{\mathcal{A}}} = T'$. Now we conclude that $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}} = [\![T]\!]^{\Omega^{\mathcal{A}}}$, whence, by Proposition 28, $\mathcal{I}$ is protoalgebraic. ■

Corollary 34 *A π-institution $\mathcal{I}$ is weakly algebraizable iff it is truth equational and* $\mathsf{AlgSys}^*(\mathcal{I}) = \mathsf{AlgSys}(\mathcal{I})$.

Proof: If $\mathcal{I}$ is weakly algebraizable, by Definition 5, it is protoalgebraic and truth-equational. Thus, by Corollary 26, it is truth equational and $\mathsf{AlgSys}^*(\mathcal{I}) = \mathsf{AlgSys}(\mathcal{I})$. The converse follows by Proposition 33. ∎

Finally, for a characterization of truth equationality in terms of the form of full $\mathcal{I}$-gmatrix systems, that parallels the characterization of protoalgebraicity proven in Theorem 27, we obtain the following analog of Theorem 6.13 of [1]:

Theorem 35 *For a π-institution $\mathcal{I}$ the following are equivalent:*

(i) *$\mathcal{I}$ is truth equational.*

(ii) $\langle\mathcal{A}, (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T\rangle$ *is a full $\mathcal{I}$-gmatrix system, for all $\mathcal{A}$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.*

(iii) $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}} = (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$, *for all $\mathcal{A}$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.*

Proof: The equivalences follow from Theorem 32 through the application of the characterizations of Suszko filter families obtained in Lemma 17 and Theorem 18. ∎

Finally, by Theorems 27 and 35, we get the following characterization of weakly algebraizable π-institutions in terms of the form of their full classes of filter families.

Corollary 36 *A π-institution $\mathcal{I}$ is weakly algebraizable iff the full classes of $\mathcal{I}$-filter families are exactly the ones of the form* $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}} = (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$, *for all N-algebraic systems $\mathcal{A}$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.*

In terms of the Suszko property, we obtain the following characterization of weakly algebraizable π-institutions:

Proposition 37 *A π-institution $\mathcal{I}$ is weakly algebraizable iff all full classes of $\mathcal{I}$-filter families are Suszko classes and all $\mathcal{I}$-filter families are Suszko filter families.*

Proof: Suppose $\mathcal{I}$ is weakly algebraizable. Then, it is protoalgebraic. By Theorem 27, every full class of $\mathcal{I}$-filter families is of the form $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$, for some $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. On the other hand, $\mathcal{I}$ is also truth equational.

Thus, by Theorem 35, $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T = [\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}$, whence every full class is a Suszko class. Finally, by Theorem 32, every filter family is a Suszko filter family.

Suppose, conversely, that all full classes of $\mathcal{I}$-filter families are Suszko classes and all $\mathcal{I}$-filter families are Suszko filter families. By the second property and Theorem 32, $\mathcal{I}$ is truth equational. By Theorem 35, every Suszko class has the form $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$, for some $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Thus, every full class of $\mathcal{I}$-filter families has the form $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$, for some $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. By Theorem 27, $\mathcal{S}$ is protoalgebraic, whence $\mathcal{I}$ is weakly algebraizable. ∎

Yet another characterization of truth equationality involves the class of filter families on quotient algebraic systems by the Suszko congruence system of a filter family:

Corollary 38 *A π-institution $\mathcal{I}$ is truth equational iff*

$$\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)) = (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T),$$

for all $\mathcal{A}$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$.

Proof: Let $\mathcal{A}$ be an N-algebraic system and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Consider $\langle I, \pi\rangle := \langle I_{\mathbf{Sign}'}, \pi^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)}\rangle : \mathcal{A} \to \mathcal{A}/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)$. By Proposition P.14 and the definition of the Suszko class, $\pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T))) = \Omega^{\mathcal{A}^{-1}}(\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)) = [\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}$. Now, under truth equationality, we get

$$\begin{aligned} &\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)) \\ &\qquad = \pi((\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T) \quad \text{(Theorem 35 and surjectivity)} \\ &\qquad = (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T). \end{aligned}$$

If, conversely, $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)) = \pi((\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T)$, then, again by Proposition P.14, we have $\pi([\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}) = \pi((\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T)$. Since $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)$ is compatible with all filter families in $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}$ and $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T \subseteq [\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}$, it follows that $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}} = (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$. Then, by Theorem 35, $\mathcal{I}$ is truth equational. ∎

The following theorem is an abstraction of a well-known result of Blok and Pigozzi [2] (see, also Theorem 2.7 of [16] and Theorem 1.1.8 of [8]). It has been abstracted in the CAAL context in [28]. We revisit it here, equipped with the more general tools developed in [31], taking after [1].

Theorem 39 (Protoalgebraic Correspondence Thm) *A π-institution $\mathcal{I}$ is protoalgebraic iff every strict surjective N-matrix system morphism $\langle H,\gamma\rangle : \langle\mathcal{A},T\rangle \to \langle\mathcal{B},T'\rangle$ between $\mathcal{I}$-matrix systems, with H an isomorphism, induces an order isomorphism between the posets* $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$ *and* $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}))^{T'}$*, whose inverse is γ^{-1}.*

Proof: First, suppose that $\mathcal{I}$ is protoalgebraic. By strictness, $T = \gamma^{-1}(T')$, whence, by surjectivity of $\langle H,\gamma\rangle$, $T' = \gamma(\gamma^{-1}(T')) = \gamma(T)$. Thus, $T = \gamma^{-1}(\gamma(T))$ and, therefore, $\langle H,\gamma\rangle$ is compatible with T. By Theorem 14, $\langle H,\gamma\rangle$ induces an order isomorphism between $[\![T]\!]^{\Omega^{\mathcal{A}}}$ and $[\![T']\!]^{\Omega^{\mathcal{B}}}$, whose inverse is γ^{-1}. Taking into account protoalgebraicity and Lemma 17, the upsets $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$ and $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}))^{T'}$ are contained in these two posets, respectively, and T, T' are corresponding filter families, whence the order isomorphism restricts to one between $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$ and $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}))^{T'}$.

For the converse, we follow the proof of (v)⇒(i) of Theorem 1.1.8 of [8]. Let $\mathcal{A}$ be an N-algebraic system and $T,T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, such that $T \leq T'$. Then for $\langle I,\pi\rangle := \langle I_{\mathbf{Sign}'},\pi^{\Omega^{\mathcal{A}}(T)}\rangle : \mathcal{A} \to \mathcal{A}/\Omega^{\mathcal{A}}(T)$, we get that $\langle I,\pi\rangle : \langle\mathcal{A},T\rangle \to \langle\mathcal{A}/\Omega^{\mathcal{A}}(T),T/\Omega^{\mathcal{A}}(T)\rangle$ is a strict surjective N-morphism between two $\mathcal{I}$-matrix systems, with I an isomorphism. By hypothesis, since $T' \in (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$, we get that $\pi(T') \in (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\Omega^{\mathcal{A}}(T)))^{T/\Omega^{\mathcal{A}}(T)}$. Therefore, $\pi^{-1}(\pi(T')) \in (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$. By the surjectivity of π, $\pi(T') = \pi(\pi^{-1}(\pi(T')))$ and, by the injectivity of π, $T' = \pi^{-1}(\pi(T'))$, whence $\Omega^{\mathcal{A}}(T)$ is compatible with T' and, hence $\Omega^{\mathcal{A}}(T) \leq \Omega^{\mathcal{A}}(T')$, showing that $\mathcal{I}$ is protoalgebraic. ∎

By altering the classes of matrix system morphisms and the filter families, we can also obtain a similar correspondence theorem for truth equational π-institutions. We first establish a property of strict surjective matrix system morphisms implied by truth equationality.

Theorem 40 *If a π-institution $\mathcal{I}$ is truth-equational, then every strict surjective N-matrix system morphism $\langle H,\gamma\rangle : \langle\mathcal{A},T\rangle \to \langle\mathcal{B},T'\rangle$ between $\mathcal{I}$-matrix systems, with H an isomorphism, which is $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}$-compatible with T, induces an order isomorphism between* $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$ *and* $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}))^{T'}$*, whose inverse is γ^{-1}.*

Proof: If $\mathcal{I}$ is truth equational, and $\langle H,\gamma\rangle : \langle\mathcal{A},T\rangle \to \langle\mathcal{B},T'\rangle$ is strict and surjective, with H is an isomorphism, which is $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}$-compatible with T, then $T' = \gamma(T)$ and, by Theorem 21, $\langle H,\gamma\rangle$ induces an order isomorphism between $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}$ and $[\![T']\!]^{\widetilde{\Omega}^{\mathcal{B},\mathcal{I}}}$, whose inverse is γ^{-1}. By Theorem 32, every filter family is Suszko, whence $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}} = (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$ and $[\![T']\!]^{\widetilde{\Omega}^{\mathcal{B},\mathcal{I}}} = (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}))^{T'}$, whence the conclusion follows. ∎

Theorem 41 (Truth-Equational Correspondence Theorem) *The π-institution $\mathcal{I}$ is truth-equational iff every strict surjective N-matrix system morphism $\langle H,\gamma\rangle : \langle\mathcal{A},T\rangle \to \langle\mathcal{B},T'\rangle$ between $\mathcal{I}$-matrix systems, with H an isomorphism, that is $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}$-compatible with T, induces an order isomorphism between $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$ and $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}))^{T'^{\widetilde{\Omega}^{\mathcal{B},\mathcal{I}}}}$, whose inverse is γ^{-1}.*

Proof: Suppose, first, that $\mathcal{I}$ is truth equational. By Theorem 32, every filter family T' is Suszko, whence $T'^{\widetilde{\Omega}^{\mathcal{B},\mathcal{I}}} = T'$. Therefore, the postulated property coincides with the one proved in Theorem 40.

For the converse, assuming the postulated property, it is enough to show, by Theorem 32, that every filter family is Suszko. Let $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and $\mathcal{B} = \mathcal{A}/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)$. Let $\langle I,\pi\rangle = \langle I_{\mathbf{Sign}'},\pi^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)}\rangle : \mathcal{A} \to \mathcal{B}$ be the corresponding projection N-morphism. $\langle I,\pi\rangle$ is a strict and surjective N-matrix system morphism from $\langle\mathcal{A},T\rangle$ to $\langle\mathcal{B},T/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)\rangle$ and clearly $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)$-compatible with T. By hypothesis, we get an order isomorphism between the posets $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$ and $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}))^{\pi(T)^{\widetilde{\Omega}^{\mathcal{B},\mathcal{I}}}}$, with inverse π^{-1}. The Suszko operator is a weakly coherent family of $\mathcal{I}$-compatibility operators, whence, by Lemma P.33, $\widetilde{\Omega}^{\mathcal{B},\mathcal{I}}(\pi(T)) = \pi(\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)) = \Delta^{\mathrm{SEN}'}$. Thus, $[\![\pi(T)]\!]^{\widetilde{\Omega}^{\mathcal{B},\mathcal{I}}} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, whence $\pi(T)^{\widetilde{\Omega}^{\mathcal{B},\mathcal{I}}} = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$ and $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}))^{\pi(T)^{\widetilde{\Omega}^{\mathcal{B},\mathcal{I}}}} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$. Applying Theorem 21 to $\langle I,\pi\rangle$, we get an order isomorphism from $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}$ and $[\![\pi(T)]\!]^{\widetilde{\Omega}^{\mathcal{B},\mathcal{I}}} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{B})$, with inverse π^{-1}. Therefore, necessarily, $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}} = (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$, which shows, by Theorem 18, that T is a Suszko filter family. ∎

Dropping the requirement of compatibility results in a characterization of weakly algebraizable π-institutions:

Theorem 42 *A π-institution $\mathcal{I}$ is weakly algebraizable iff every strict surjective N-matrix system morphism $\langle H,\gamma\rangle : \langle\mathcal{A},T\rangle \to \langle\mathcal{B},T'\rangle$ between $\mathcal{I}$-matrix systems, with H an isomorphism, induces an order isomorphism between $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$ and $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}))^{T'^{\widetilde{\Omega}^{\mathcal{B},\mathcal{I}}}}$, whose inverse is γ^{-1}.*

Proof: If $\mathcal{I}$ is weakly algebraizable, then, it is protoalgebraic. By Theorem 39, we have an order isomorphism from $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$ to $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}))^{T'}$. Since $\mathcal{I}$ is truth equational, by Theorem 32, every filter family is Suszko, whence $T'^{\widetilde{\Omega}^{\mathcal{B},\mathcal{I}}} = T'$, yielding the required isomorphism.

Conversely, assume that the property of the statement holds. Thus, it holds also, in particular, for all $\langle H, \gamma\rangle$, with H an isomorphism, that are $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}$-compatible with T. By Theorem 40, $\mathcal{I}$ is truth equational. This implies, again by Theorem 32, that all filter families are Suszko, whence $T'^{\widetilde{\Omega}^{\mathcal{B},\mathcal{I}}} = T'$ and the hypothesis establishes an order isomorphism between $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$ and $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{B}))^{T'}$, whose inverse is γ^{-1}. By Theorem 39, $\mathcal{I}$ is protoalgebraic and, therefore, weakly algebraizable. ■

Finally, we focus on characterizing the classes of the categorical abstract algebraic hierarchy using the Leibniz operator. The range of the Leibniz operator $\Omega^{\mathcal{A}}$ is the class $\mathrm{ConSys}_{\mathsf{AlgSys}^*(\mathcal{I})}(\mathcal{A})$. By Proposition P.23, the Leibniz operator is order reflecting and, hence, also injective, on the collection of Leibniz filter families of $\mathcal{A}$. If $\mathcal{I}$ is protoalgebraic, then we obtain monotonicity on $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, which is inherited by the collection of Leibniz filters:

Proposition 43 *If $\mathcal{I}$ is protoalgebraic, then, for all $\mathcal{A}$, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\Omega^{\mathcal{A}}}(\mathcal{A}) \to \mathrm{ConSys}_{\mathsf{AlgSys}^*(\mathcal{I})}(\mathcal{A})$ is an order isomorphism.*

To work with Suszko filter families instead of Leibniz filter families, we prove the following lemmas, showing that, if the preceding isomorphism holds, then the classes $\mathsf{AlgSys}(\mathcal{I})$ and $\mathsf{AlgSys}^*(\mathcal{I})$ are identical.

Lemma 44 *If $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\Omega^{\mathcal{A}}}(\mathcal{A}) \to \mathrm{ConSys}_{\mathsf{AlgSys}^*(\mathcal{I})}(\mathcal{A})$ is an order isomorphism, for all $\mathcal{A}$, then $\mathsf{AlgSys}(\mathcal{I}) = \mathsf{AlgSys}^*(\mathcal{I})$.*

Proof: Suppose $\mathcal{A} \in \mathsf{AlgSys}(\mathcal{I})$. Consider $T^0 = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Since it is the smallest Leibniz filter family on $\mathcal{A}$, by hypothesis, $\Omega^{\mathcal{A}}(T^0) \leq \Omega^{\mathcal{A}}(T)$, for all $T \in \mathrm{FiFam}^{\Omega^{\mathcal{A}}}(\mathcal{A})$. Thus, $[\![T]\!]^{\Omega^{\mathcal{A}}} \subseteq [\![T^0]\!]^{\Omega^{\mathcal{A}}}$, for all $T \in \mathrm{FiFam}^{\Omega^{\mathcal{A}}}(\mathcal{A})$. Let $T' \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Since $\Omega^{\mathcal{A}}(T') \in \mathrm{ConSys}_{\mathsf{AlgSys}^*(\mathcal{I})}(\mathcal{A})$, there exists, by hypothesis, $T \in \mathrm{FiFam}^{\Omega^{\mathcal{A}}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T') = \Omega^{\mathcal{A}}(T)$. Thus, $T' \in [\![T']\!]^{\Omega^{\mathcal{A}}} = [\![T]\!]^{\Omega^{\mathcal{A}}} \subseteq [\![T^0]\!]^{\Omega^{\mathcal{A}}}$, showing that $[\![T^0]\!]^{\Omega^{\mathcal{A}}} = \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. By Proposition 23, $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) \in \mathsf{AlgSys}^*(\mathcal{I})$. The assumption that $\mathcal{A} \in \mathsf{AlgSys}(\mathcal{I})$ implies that $\widetilde{\Omega}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})) = \Delta^{\mathrm{SEN}'}$ and this, in turn, yields $\mathcal{A} \cong \mathcal{A}/\widetilde{\Omega}^{\mathcal{A}}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))$. Therefore, $\mathcal{A} \in \mathsf{AlgSys}^*(\mathcal{I})$, showing that $\mathsf{AlgSys}(\mathcal{I}) \subseteq \mathsf{AlgSys}^*(\mathcal{I})$. This is all that was needed since the reverse inclusion is universally true. ■

Moreover, identity of the classes $\mathsf{AlgSys}(\mathcal{I})$ and $\mathsf{AlgSys}^*(\mathcal{I})$ may also be drawn under the hypothesis that $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}(\mathcal{A}) \to \mathrm{ConSys}_{\mathsf{AlgSys}^*(\mathcal{I})}(\mathcal{A})$ is an order isomorphism.

Lemma 45 *If* $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}(\mathcal{A}) \to \mathrm{ConSys}_{\mathsf{AlgSys}^*(\mathcal{I})}(\mathcal{A})$ *is an order isomorphism, for all* $\mathcal{A}$*, then* $\mathsf{AlgSys}(\mathcal{I}) = \mathsf{AlgSys}^*(\mathcal{I})$.

Proof: Follow the same steps as in the proof of Lemma 44, taking into account that $T^0 = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) \in \mathrm{FiFam}^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}(\mathcal{A})$. ∎

The following theorem abstracts to the categorical context Theorem 6.25 of [1], which itself generalizes preceding theorems applicable to classes of logics narrower than protoalgebraic, e.g., Theorem 4.8 of [10] (see, also, [18]).

Theorem 46 *A* π*-institution* $\mathcal{I}$ *is protoalgebraic iff the Leibniz operator restricted to the Suszko filter families*

$$\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}(\mathcal{A}) \to \mathrm{ConSys}_{\mathsf{AlgSys}^*(\mathcal{I})}(\mathcal{A})$$

is an order isomorphism, for all N*-algebraic systems* $\mathcal{A}$.

Proof: The left-to-right implication follows from Proposition 43, since, by Proposition 28, for a protoalgebraic π-institution, the class of all Leibniz filter families and the class of all Suszko filter families are identical.

Suppose, conversely, that $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}(\mathcal{A}) \to \mathrm{ConSys}_{\mathsf{AlgSys}^*(\mathcal{I})}(\mathcal{A})$ is an order isomorphism, for all N-algebraic systems $\mathcal{A}$. We prove $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T) = \Omega^{\mathcal{A}}(T^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}})$ and, also, that $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}})$, for all $\mathcal{A}$ and all $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. These equalities show that the Leibniz and Suszko operators coincide, which, by Proposition 25, characterizes protoalgebraicity.

- If $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T) \in \mathrm{ConSys}_{\mathsf{AlgSys}(\mathcal{I})}(\mathcal{A})$. By Lemma 45 and the surjectivity of $\Omega^{\mathcal{A}}$, there exists $T' \in \mathrm{FiFam}^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T) = \Omega^{\mathcal{A}}(T')$. Thus, $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}} = [\![T']\!]^{\Omega^{\mathcal{A}}}$. Hence, since every Suszko filter family is Leibniz, $T^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}} = T'^{\Omega^{\mathcal{A}}} = T'$. Now, we get $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T) = \Omega^{\mathcal{A}}(T^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}})$, as claimed.

- Since $\Omega^{\mathcal{A}}(T) \in \mathrm{ConSys}_{\mathsf{AlgSys}^*(\mathcal{I})}(\mathcal{A})$, by hypothesis, there exists $T' \in \mathrm{FiFam}^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}(\mathcal{A})$, such that $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T')$. Then $[\![T]\!]^{\Omega^{\mathcal{A}}} = [\![T']\!]^{\Omega^{\mathcal{A}}}$, whence, taking into account that all Suszko filter families are Leibniz, $T^{\Omega^{\mathcal{A}}} = T'^{\Omega^{\mathcal{A}}} = T'$. Now we get $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T^{\Omega^{\mathcal{A}}})$.

  Since $T^{\Omega^{\mathcal{A}}} = T'$ is a Suszko filter family, we have $(T^{\Omega^{\mathcal{A}}})^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}} = T^{\Omega^{\mathcal{A}}}$.

  - Since $T^{\Omega^{\mathcal{A}}} \le T$, $[\![T]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}} \subseteq [\![T^{\Omega^{\mathcal{A}}}]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}$, whence $(T^{\Omega^{\mathcal{A}}})^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}} \le T^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}$ and, therefore, $T^{\Omega^{\mathcal{A}}} \le T^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}$.

– By Lemma 17, the converse inclusion always holds.

We see that $T^{\Omega^{\mathcal{A}}} = T^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}$, implying that $\Omega^{\mathcal{A}}(T) = \Omega^{\mathcal{A}}(T^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}})$. ∎

One obtains now a characterization of equivalential π-institutions as well:

Corollary 47 *A π-institution $\mathcal{I}$ is equivalential iff the Leibniz operator commutes with inverse N-morphisms and, restricted to the Suszko filter families, $\Omega^{\mathcal{A}} : \mathrm{FiFam}^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}(\mathcal{A}) \to \mathrm{ConSys}_{\mathsf{AlgSys}^*(\mathcal{I})}(\mathcal{A})$ is an order isomorphism, for all $\mathcal{A}$.*

If one adds to the hypotheses of Theorem 46 the condition that the Leibniz operator commute with inverse N-morphisms, then, by Corollary 47, we obtain a characterization of algebraizable π-institutions as those in which the Leibniz operator commutes with inverse N-morphisms and, for all $\mathcal{A}$, $\Omega^{\mathcal{A}}$ is an order isomorphism between $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ and $\mathrm{ConSys}_{\mathsf{AlgSys}^*(\mathcal{I})}(\mathcal{A})$. This is a categorical analog of Corollary 3.14 of [15].

Finally, we close this work with some results characterizing the various levels of the CAAL hierarchy by means of properties of the categorical Suszko operator. We start with truth equational and protoalgebraic π-institutions and conclude with an all-encompassing theorem that combines these two characterizations with the relevant definitions to extend them to the remaining classes of the hierarchy.

Theorem 48 *For a π-institution $\mathcal{I}$ the following are equivalent:*

(i) *$\mathcal{I}$ is truth equational.*

(ii) *The Suszko operator $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}$ is injective on $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, for all $\mathcal{A}$.*

(iii) *$\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}$ is injective on $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$, for all $\mathcal{A} \in \mathsf{AlgSys}(\mathcal{I})$.*

Proof:

(i)⇒(ii) By Proposition P.23, the Suszko operator is injective on the Suszko filter families. If $\mathcal{I}$ is truth equational, by Theorem 32, every filter family is Suszko, whence the Suszko operator is globally injective.

(ii)⇒(iii) (iii) is a special case of (ii).

(iii)⇒(i) Consider an $\mathcal{A}$ and $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Let $T' \in \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T))$. We have $\mathcal{A}/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T) \in \mathsf{AlgSys}^{\mathrm{Su}}(\mathcal{I}) = \mathsf{AlgSys}(\mathcal{I})$. Lemma 1 yields

$T/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T) \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T))$. Thus, $T' \leq T/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)$, whence by order preservation and Lemma P.45,

$$\widetilde{\Omega}^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T),\mathcal{I}}(T') \leq \widetilde{\Omega}^{\mathcal{A}/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T),\mathcal{I}}(T/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)) = \Delta^{\mathrm{SEN}'/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T)}.$$

By the hypothesis, $T/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T) = T' = \bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T))$. By Proposition 19, T is a Suszko filter family. Since every filter family is Suszko, by Theorem 32, $\mathcal{I}$ is truth equational. ■

Proposition 49 *The Suszko operator restricted to Suszko filter families* $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}} : \mathrm{FiFam}^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}(\mathcal{A}) \to \mathsf{ConSys}_{\mathsf{AlgSys}(\mathcal{I})}(\mathcal{A})$ *is an order embedding.*

Proof: The Suszko operator $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}$ is order-preserving on $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. In particular, also on $\mathrm{FiFam}^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}(\mathcal{A})$. By Proposition P.23, it is also order reflecting on $\mathrm{FiFam}^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}(\mathcal{A})$. Since it is into $\mathrm{ConSys}_{\mathsf{AlgSys}^{\mathsf{Su}}(\mathcal{I})}(\mathcal{A})$ and, by Lemma 4, $\mathsf{AlgSys}^{\mathsf{Su}}(\mathcal{I}) = \mathsf{AlgSys}(\mathcal{I})$, we obtain the result. ■

Requiring that $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}} : \mathrm{FiFam}^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}(\mathcal{A}) \to \mathsf{ConSys}_{\mathsf{AlgSys}(\mathcal{I})}(\mathcal{A})$ be surjective turns out to characterize protoalgebraicity of $\mathcal{I}$. Moreover, it amounts to commutativity of the Suszko operator with inverse surjective N-morphisms.

Theorem 50 *For a π-institution $\mathcal{I}$ the following are equivalent:*

(i) *$\mathcal{I}$ is protoalgebraic.*

(ii) *$\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}$ commutes with inverse surjective N-morphisms.*

(iii) *$\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}$ restricted to the Suszko filter families $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}} : \mathrm{FiFam}^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}(\mathcal{A}) \to \mathsf{ConSys}_{\mathsf{AlgSys}(\mathcal{I})}(\mathcal{A})$ is surjective, for all $\mathcal{A}$.*

Proof:

(i)⇒(ii) By Proposition 25, if $\mathcal{I}$ is protoalgebraic, the Suszko and Leibniz operators coincide. Moreover, by Proposition 2, the Leibniz operator commutes with inverse surjective N-morphisms.

(ii)⇒(iii) Consider $\mathcal{A}$ and $\theta \in \mathrm{ConSys}_{\mathsf{AlgSys}(\mathcal{I})}(\mathcal{A})$. Since $\mathcal{A}/\theta \in \mathsf{AlgSys}(\mathcal{I})$, we obtain $\widetilde{\Omega}^{\mathcal{A}/\theta,\mathcal{I}}(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)) = \Delta^{\mathrm{SEN}'/\theta}$. Let $\langle I,\pi\rangle := \langle I_{\mathbf{Sign}'},\pi^{\theta}\rangle : \mathcal{A} \to \mathcal{A}/\theta$ be the projection N-morphism. Using the postulated commutativity,

$$\begin{array}{rcl}\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(\bigcap \pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta))) & = & \widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(\pi^{-1}(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta))) \\ & = & \pi^{-1}(\widetilde{\Omega}^{\mathcal{A}/\theta,\mathcal{I}}(\bigcap \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta))) \\ & = & \pi^{-1}(\Delta^{\mathrm{SEN}'/\theta}) = \mathrm{Ker}(\langle I,\pi\rangle) = \theta.\end{array}$$

We also have

$$\begin{array}{rcl}\pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta)) & = & \Omega^{\mathcal{A}^{-1}}(\theta) \quad \text{(by Proposition P.14)}\\ & = & \Omega^{\mathcal{A}^{-1}}(\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(\bigcap\pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta))))\\ & & \text{(previous equality)}\\ & = & [\![\bigcap\pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta))]\!]^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}.\\ & & \text{(Suszko Class)}\end{array}$$

Therefore, $\bigcap\pi^{-1}(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}/\theta))$ is a Suszko filter family, with Suszko congruence θ. Thus, $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}$ is surjective when restricted to Suszko filter families.

(iii)⇒(i) By Theorem 27, it suffices to show that, for all $\mathcal{A}$, every full $\mathcal{I}$-gmatrix system is of the form $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$, for some $T \in \mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$. Let $\mathcal{T}$ be a full $\mathcal{I}$-gmatrix system on some $\mathcal{A}$. Then, by definition of $\mathsf{AlgSys}(\mathcal{I})$, $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) \in \mathrm{ConSys}_{\mathsf{AlgSys}(\mathcal{I})}(\mathcal{A})$. By hypothesis, there exists a Suszko filter family $T \in \mathrm{FiFam}^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}(\mathcal{A})$, such that $\widetilde{\Omega}^{\mathcal{A}}(\mathcal{T}) = \widetilde{\Omega}^{\mathcal{A},\mathcal{I}}(T) = \widetilde{\Omega}^{\mathcal{A}}((\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T)$. But, T is a Suszko filter family, whence, by Theorem 18, $(\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$ is full. Hence, by the Isomorphism Theorem, Corollary P.18, $\mathcal{T} = (\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}))^T$.

■

The following analog of Theorem 6.30 of [1] provides characterizations of the classes of the CAAL hierarchy of π-institutions in terms of properties of the categorical Suszko operator.

Theorem 51 *Let $\mathcal{I}$ be a π-institution.*

(1) *$\mathcal{I}$ is protoalgebraic iff the Suszko operator commutes with inverse surjective N-morphisms.*

(2) *$\mathcal{I}$ is equivalential iff the Suszko operator commutes with inverse N-morphisms.*

(3) *$\mathcal{I}$ is truth equational iff the Suszko operator is globally injective.*

(4) *$\mathcal{I}$ is weakly algebraizable iff the Suszko operator is globally injective and commutes with inverse surjective N-morphisms.*

(5) *$\mathcal{I}$ is algebraizable iff the Suszko operator is globally injective and commutes with inverse N-morphisms.*

Proof:

(1) By Theorem 50.

(2) Assume $\mathcal{I}$ is equivalential. By definition, $\mathcal{I}$ is protoalgebraic and the Leibniz operator commutes with inverse N-morphisms. Since, under protoalgebraicity, the Leibniz and the Suszko operators coincide, the Suszko operator commutes with inverse N-morphisms.

Conversely, assume that the Suszko operator commutes with inverse N-morphisms. A fortiori, it commutes with inverse surjective N-morphisms. By Part (1), $\mathcal{I}$ is protoalgebraic. Therefore, the Leibniz and Suszko operators coincide. By hypothesis, the Leibniz operator commutes with inverse N-morphisms. Hence $\mathcal{I}$ is equivalential.

(3) By Theorem 48.

(4) Since weak algebraizability is equivalent to protoalgebraicity and truth equationality, this follows from Parts (1) and (3).

(5) Similarly, since algebraizability is equivalent to equivalentiality and truth equationality, this follows from Parts (2) and (3). ■

We finally conclude with an analog of Theorem 6.31 of [1], providing alternative characterizations in terms of order embeddings/isomorphisms defined by the Suszko operator on the collections of filter families on arbitrary N-algebraic systems of a given π-institution.

Theorem 52 *Let $\mathcal{I}$ be a π-institution.*

(1) *$\mathcal{I}$ is protoalgebraic iff, for all $\mathcal{A}$, $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}$ restricts to an order isomorphism between* $\mathrm{FiFam}^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}(\mathcal{A})$ *and* $\mathrm{ConSys}_{\mathsf{AlgSys}(\mathcal{I})}(\mathcal{A})$.

(2) *$\mathcal{I}$ is truth equational iff, for all $\mathcal{A}$, $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}$ is an order embedding of* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *into* $\mathrm{ConSys}_{\mathsf{AlgSys}(\mathcal{I})}(\mathcal{A})$.

(3) *$\mathcal{I}$ is weakly algebraizable iff, for all $\mathcal{A}$, $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}$ is an order isomorphism between* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and* $\mathrm{ConSys}_{\mathsf{AlgSys}(\mathcal{I})}(\mathcal{A})$.

(4) *$\mathcal{I}$ is algebraizable iff the Suszko operator commutes with inverse N-morphisms and, for all $\mathcal{A}$, $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}$ is an order isomorphism between* $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ *and* $\mathrm{ConSys}_{\mathsf{AlgSys}(\mathcal{I})}(\mathcal{A})$.

Proof:

(1) Suppose $\mathcal{I}$ is protoalgebraic. By Theorem 46, for all $\mathcal{A}$, $\Omega^{\mathcal{A}}$ is an order isomorphism from $\mathrm{FiFam}^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}(\mathcal{A})$ onto $\mathrm{ConSys}_{\mathsf{AlgSys}^*(\mathcal{I})}(\mathcal{A})$. Since, by protoalgebraicity, the Leibniz and Suszko operators coincide, we have $\mathsf{AlgSys}^*(\mathcal{I}) = \mathsf{AlgSys}(\mathcal{I})$, whence the desired isomorphism follows.

If, conversely, $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}$ is an order isomorphism from $\mathrm{FiFam}^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}(\mathcal{A})$ onto $\mathrm{ConSys}_{\mathsf{AlgSys}(\mathcal{I})}(\mathcal{A})$, it must be surjective. Therefore, by Theorem 50, we get that $\mathcal{I}$ is protoalgebraic.

(2) By Proposition 49, $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}$ is an order embedding from $\mathrm{FiFam}^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}(\mathcal{A})$ into $\mathrm{ConSys}_{\mathsf{AlgSys}(\mathcal{I})}(\mathcal{A})$. By Proposition 32, truth equationality implies that $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A}) = \mathrm{FiFam}^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}(\mathcal{A})$. Thus, $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}$ is an order embedding from $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ into $\mathrm{ConSys}_{\mathsf{AlgSys}(\mathcal{I})}(\mathcal{A})$.

If, conversely, $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}$ is an order embedding, it is injective. Thus, by Theorem 48, $\mathcal{I}$ is truth equational.

(3) If $\mathcal{I}$ is weakly algebraizable, by Theorem 46 and Proposition 32, $\Omega^{\mathcal{A}}$ is an isomorphism from $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ onto $\mathrm{ConSys}_{\mathsf{AlgSys}^*(\mathcal{I})}(\mathcal{A})$. Since $\mathcal{I}$ is also protoalgebraic, the coincidence of the Suszko and Leibniz operators implies that $\mathsf{AlgSys}^*(\mathcal{I}) = \mathsf{AlgSys}(\mathcal{I})$, yielding the claimed isomorphism.

If, conversely, $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}$ is an order isomorphism from $\mathrm{FiFam}^{\mathcal{I}}(\mathcal{A})$ onto $\mathrm{ConSys}_{\mathsf{AlgSys}(\mathcal{I})}(\mathcal{A})$, then it is injective on filter families, whence, by Theorem 48, $\mathcal{I}$ is truth equational. Since, by Proposition 32, every filter family is Suszko, $\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}$ is surjective when restricted to the collection $\mathrm{FiFam}^{\widetilde{\Omega}^{\mathcal{A},\mathcal{I}}}(\mathcal{A})$. Thus, by Theorem 50, $\mathcal{I}$ is also protoalgebraic. Therefore, $\mathcal{I}$ is weakly algebraizable.

(4) This follows from Part (3) and Theorem 51. ∎

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