Categorical Abstract Algebraic Logic:
Fibring of π-Institutions

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Abstract

Inspired by work of Fernández and Coniglio, we study a framework for combining, by fibring, logics formalized as π-institutions. Fibring of deductive systems, i.e., finitary and structural consequence relations over sets of formulas, as presented by Fernández and Coniglio, becomes a special case of fibring of π-institutions. Moreover, we show how their study of preservation of algebraic properties, such as protoalgebraicity, may be lifted to this more general context. Finally, with an eye towards more applied logics, we illustrate how, using our framework, one may obtain richer extensions of basic description logics by fibring simpler extensions with various features that have been studied independently.

1 Introduction

In [14] Gabbay introduced the method of fibring as a way of combining modal logics by associating with possible worlds of one logic Kripke models of the other (see, also, [15]). Subsequent work by Sernadas, Sernadas and Caleiro [23], which was also motivated by their own previous work [21] (see also [22]), generalized the method of fibring, using category theoretic tools, and applied it to a variety of logical systems without terms. This work culminated in Caleiro's Ph.D. Thesis [6], where an extensive study of the method and an analysis of various properties that are preserved when combining logics by fibring are presented (see, also, [7]).

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It is self-evident that, when logics are combined to obtain richer logical systems, it is of great interest to study preservation of properties. These, so-called, transfer theorems assert that properties that hold for the logical systems to be combined using general constructions, such as fibring, also hold in the resulting logical system. Various properties have been studied in this context, most notably, soundness [23], completeness [30] and interpolation [8].

Taking after this line of work, Fernández and Coniglio [10] specialize the categorical framework of [23] to provide a platform over which one may study fibring of sentential logics or deductive systems. In the abstract algebraic logic (AAL) sense [4, 12, 9, 13], these are finitary and structural consequence relations on sets of propositional formulas. In AAL, sentential logics are classified in a hierarchy, called the Leibniz hierarchy, which reflects the extent to which a logic is amenable to study via algebraic methods and tools. More precisely, in AAL a canonical way exists for associating with a sentential logic a class of universal algebras, perceived as its algebraic counterpart. The level in the Leibniz hierarchy into which the logic is classified, corresponds in a precise technical sense to the strength of the ties between the consequence relation of the logic and the equational consequence of its algebraic counterpart. When the tie is strongest, i.e., in the case when the logic is algebraizable, the study of various metalogical properties may be replaced by the study of corresponding algebraic properties of the algebraic counterpart.

As this analysis indicates, it is very beneficial, when one combines logics by fibring, to be able to ensure that, whenever the component logical systems belong to a certain level of the Leibniz hierarchy, so does their combination. In fact, the main interest of [10] lies in being able to perform fibring inside the various classes of the Leibniz hierarchy. Work along these lines was pioneered by Jánossy, Kurucz, and Eiben [18], who considered fibring inside the class of algebraizable logics. The work of Fernández and Coniglio [10] extends some features of their work by considering fibring in several of the levels of the Leibniz hierarchy, namely in the classes of protoalgebraic, equivalential and algebraizable logics. They define a category of consequence relations **Cons** over a category **Sig** of algebraic or logical signatures. Morphisms in the category **Cons** are, roughly speaking, signature morphisms which preserve consequence. They are able to show that the subcategories of this category consisting of the protoalgebraic, the equivalential and the algebraizable logics are all full subcategories. As a consequence, they are able to conclude that fibring can be carried out inside the realm of each subcategory, i.e., that fibring of protoalgebraic, equivalential or algebraizable logics results, respectively, in a combined logic within the same class.

One of the limitations of the framework of [23], as specialized in [10], is that it can cope only with sentential logics. The framework of π-institutions, introduced by Fiadeiro and Sernadas [11], on the other hand, as an abstract version of the model theoretic notion of institution [16, 17], has been shown by the author (e.g., [24, 25, 26, 27]) to provide within AAL a platform over which the relation between metalogical and algebraic properties of more general logical systems may be studied. Thus, in [10], a call is made for an extension of the sentential framework to accommodate fibring of π-institutions and, therefore, to open the way for studying some transfer results in a more general context. This would make them available in case fibring of more complex logical systems is called for. We view the present work as a first attempt towards fulfilling this program. More precisely, using the basic categorical framework of Sernadas et al. [23], we provide a platform over which one may apply fibring to π-institutions. We then show that this framework properly includes that of [10] by explicitly providing the details of how this inclusion may be accomplished. To initiate the study of preservation of algebraic properties, we lift some of the results on protoalgebraic logics of [10] to logics formalized as π-institutions. We only accomplish this for what we call poly-term π-institutions, a restricted class of π-institutions, that contains, however, all sentential logics. Finally, we illustrate the theory with an example drawn from the realm of description logics [1]. Informally speaking, we set up the special framework needed for fibring two description logics that are extensions of the basic description logic $\mathcal{ALC}$, each potentially consisting of additional logical connectives.

A short overview of the structure of the paper is now provided. In Section 2 the category **Ins** of π-institutions is introduced. Because we are interested in fibring logical systems that refer to the same semantic entities, we require that all π-institutions consist of signature categories with identical collections of objects. For example, when propositional logics are to be fibred, they will all be assumed to have the same propositional variables and when description logics are to be fibred, they will all be assumed to refer to the same collections of concept and role names. This seems consistent with the idea that fibring is supposed to produce a richer logic for reasoning about the same semantical entities, i.e., increase our reasoning capability about a fixed collection of objects and relations between them. In Section 3, the unconstrained fibring of π-institutions is discussed. Since unconstrained fibring corresponds to coproducts, the existence of the unconstrained fibring of π-institutions follows from the existence of colimits in **Ins**. In section

4, constrained fibring is introduced. It is defined in the category **Ins** in the same way as constrained fibring was defined in [10] in the category **Cons** of consequence relations, representing deductive systems. Apart from the existence of coproducts, the existence of coequalizers, as well as the fact that the forgetful functor from the category of π-institutions to the underlying category of logical connectives is a cofibration, play a crucial role in this construction. In Section 5, we revisit the framework of [10] and show how their general constructions can be obtained as special cases of the ones introduced here. Moreover, in Section 6, we extend their results on the class of protoalgebraic deductive systems to the class of protoalgebraic poly-term π-institutions. These form a subcategory of the category **Ins** of π-institutions, which is rich enough to include all protoalgebraic deductive systems in the sense of AAL [3, 9] and, also, many of the protoalgebraic term π-institutions in the sense of [25]. It is shown that it is a full subcategory of the category of all π-institutions (on the same poly-term system of sentence functors) and this fact is used to show that it admits both unconstrained and constrained fibring. Finally, in Section 7, an additional example is provided of constrained fibring. Namely, we adjust the general framework so as to be able to fibre two π-institutions $\mathcal{I}'$ and $\mathcal{I}''$, corresponding to extensions of the well-known basic description logic $\mathcal{ALC}$ over a π-institution $\mathcal{I}^{\mathcal{ALC}}$, corresponding to $\mathcal{ALC}$ itself. The resulting logical system corresponds to a description logic that extends $\mathcal{ALC}$ with features of both extensions $\mathcal{I}'$ and $\mathcal{I}''$. This example illustrates the potential for the applicability of fibring to the realm of description logics to obtain richer description logics by combining various logics in which different features have already been introduced and studied. Of course, a very interesting problem in this domain would be to study under which conditions fibring preserves decidability or membership of the satisfiability problem of the corresponding description logics inside specific complexity classes. These studies are beyond the scope of this work.

For standard categorical notation that will mostly remain unexplained, the reader is advised to consult any of the standard references [2, 5, 19].

2 The Category of π-Institutions

The basic building blocks for our treatment of logical systems will be π-institutions. Our goal, therefore, in this section is to develop the category of π-institutions **Ins**. Since focus will be on constructing unconstrained and constrained fibring of π-institutions, the category will be specifically tailored to this task. The objects of this category will be built over objects of more basic categories. The first will be the category of "logical connectives", denoted by **Con**. Its objects will be perceived as the logical types or sets of logical connectives and morphisms between them will be perceived as substitutions of basic connectives for basic connectives preserving the corresponding arities. Given an object $\mathcal{L}$ in the category **Con**, there will be a category $\mathbf{Sign}_{\mathcal{L}}$ of "variables". These may be either propositional variables, in case we are dealing with a propositional language, or relations and function symbols, in case we are dealing with a first-order language. Since, in the context of fibring, we will be making the basic hypothesis that all our logical systems refer to the same semantic entities, we assume that, for every $\mathcal{L}, \mathcal{L}'$ in the category **Con**, $|\mathbf{Sign}_{\mathcal{L}}| = |\mathbf{Sign}_{\mathcal{L}'}|$. Of course, given objects Σ and Σ' that are contained in $|\mathbf{Sign}_{\mathcal{L}}|$, we allow that $\mathbf{Sign}_{\mathcal{L}}(\Sigma, \Sigma') \neq \mathbf{Sign}_{\mathcal{L}'}(\Sigma, \Sigma')$ in accordance with the fact that morphisms are perceived as substitutions of $\mathcal{L}$-terms or $\mathcal{L}'$-terms, respectively, for variables and the sets of these substitutions are different when the two logical languages are different.

In the sequel, and throughout the paper, we fix the following framework, which makes the intuitions above precise:

- A category **Con**, which we perceive as the category of **logical connectives** and **arity preserving mappings** between them;
- For every $\mathcal{L} \in |\mathbf{Con}|$, a category $\mathbf{Sign}_{\mathcal{L}}$, such that for every $\mathcal{L}, \mathcal{L}' \in |\mathbf{Con}|$, $|\mathbf{Sign}_{\mathcal{L}}| = |\mathbf{Sign}_{\mathcal{L}'}|$. We call the objects of these categories **signatures** and let **Sign** denote the discrete category with objects all signatures;
- For every $\mathcal{L} \in |\mathbf{Con}|$, a functor $\mathrm{SEN}_{\mathcal{L}} : \mathbf{Sign}_{\mathcal{L}} \to \mathbf{Set}$, giving, for all $\mathcal{L} \in |\mathbf{Con}|$ and all $\Sigma \in |\mathbf{Sign}_{\mathcal{L}}|$, the set $\mathrm{SEN}_{\mathcal{L}}(\Sigma)$ of **sentences** with connectives in $\mathcal{L}$ and variables in Σ;
- For every $\alpha \in \mathbf{Con}(\mathcal{L}, \mathcal{L}')$, a functor $F^{\alpha} : \mathbf{Sign}_{\mathcal{L}} \to \mathbf{Sign}_{\mathcal{L}'}$, which is the identity on objects, and a natural transformation (also denoted by) $\alpha : \mathrm{SEN}_{\mathcal{L}} \to \mathrm{SEN}_{\mathcal{L}'} \circ F^{\alpha}$.

$$\begin{tikzcd} \mathrm{SEN}_{\mathcal{L}}(\Sigma) \arrow[r, "\alpha_{\Sigma}"] \arrow[d, "\mathrm{SEN}_{\mathcal{L}}(f)"'] & \mathrm{SEN}_{\mathcal{L}'}(\Sigma) \arrow[d, "\mathrm{SEN}_{\mathcal{L}'}(F^{\alpha}(f))"] \\ \mathrm{SEN}_{\mathcal{L}}(\Sigma') \arrow[r, "\alpha_{\Sigma'}"'] & \mathrm{SEN}_{\mathcal{L}'}(\Sigma') \end{tikzcd}$$

Given a substitution $f \in \mathbf{Sign}_{\mathcal{L}}(\Sigma, \Sigma')$ of $\mathcal{L}$-terms over Σ' for variables in Σ, and a mapping $\alpha : \mathcal{L} \to \mathcal{L}'$ of connectives in $\mathcal{L}$ to connectives in $\mathcal{L}'$, the morphism $F^\alpha(f) \in \mathbf{Sign}_{\mathcal{L}'}(\Sigma, \Sigma')$ is supposed to represent the substitution of $\mathcal{L}'$-terms over Σ' for variables in Σ that results by composing f with the appropriate extension of α on $\mathcal{L}$-terms over Σ'.

Moreover, for all $\alpha \in \mathbf{Con}(\mathcal{L}, \mathcal{L}')$ and all $\beta \in \mathbf{Con}(\mathcal{L}', \mathcal{L}'')$, $F^{\beta\alpha} = F^\beta \circ F^\alpha$ and the natural transformation corresponding to $\beta \circ \alpha$ is the composite of the one corresponding to α and the one corresponding to β.

$$\begin{tikzcd}
\mathrm{SEN}_{\mathcal{L}}(\Sigma) \arrow[r, "\alpha_\Sigma"] \arrow[d, "\mathrm{SEN}_{\mathcal{L}}(f)"'] & \mathrm{SEN}_{\mathcal{L}'}(\Sigma) \arrow[r, "\beta_\Sigma"] \arrow[d, "\mathrm{SEN}_{\mathcal{L}'}(F^\alpha(f))"] & \mathrm{SEN}_{\mathcal{L}''}(\Sigma) \arrow[d, "\mathrm{SEN}_{\mathcal{L}''}(F^\beta(F^\alpha(f)))"] \\
\mathrm{SEN}_{\mathcal{L}}(\Sigma') \arrow[r, "\alpha_{\Sigma'}"'] & \mathrm{SEN}_{\mathcal{L}'}(\Sigma') \arrow[r, "\beta_{\Sigma'}"'] & \mathrm{SEN}_{\mathcal{L}''}(\Sigma')
\end{tikzcd}$$

Definition 1 *Given $\mathcal{L} \in |\mathbf{Con}|$, a* **consequence system over** $\mathcal{L}$ *is a collection $\vdash = \{\vdash_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ of relations $\vdash_\Sigma \subseteq \mathcal{P}(\mathrm{SEN}_{\mathcal{L}}(\Sigma)) \times \mathrm{SEN}_{\mathcal{L}}(\Sigma)$, such that, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}_{\mathcal{L}}(\Sigma, \Sigma')$ and $\Gamma \cup \Delta \cup \{\phi\} \subseteq \mathrm{SEN}_{\mathcal{L}}(\Sigma)$,*

- $\Gamma \vdash_\Sigma \phi$, *for all* $\phi \in \Gamma$; [Reflexivity]
- $\Gamma \vdash_\Sigma \phi$ *and* $\Delta \vdash_\Sigma \Gamma$ *imply* $\Delta \vdash_\Sigma \phi$; [Transitivity]
- $\Gamma \vdash_\Sigma \phi$ *implies* $\mathrm{SEN}_{\mathcal{L}}(f)(\Gamma) \vdash_{\Sigma'} \mathrm{SEN}_{\mathcal{L}}(f)(\phi)$. [Structurality]

If $\vdash$ is a consequence system over $\mathcal{L}$, then we also have

$$\Gamma \vdash_\Sigma \phi \quad \text{implies} \quad \Delta \vdash_\Sigma \phi, \quad \text{if} \quad \Gamma \subseteq \Delta, \quad \text{[Monotonicity]}$$

for all $\Sigma \in |\mathbf{Sign}|$ and all $\Gamma \cup \Delta \cup \{\phi\} \subseteq \mathrm{SEN}_{\mathcal{L}}(\Sigma)$.

Definition 2 *A* π-**institution** *is a pair $\mathcal{I} = \langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L} \in |\mathbf{Con}|$ and $\vdash$ is a consequence system over $\mathcal{L}$.*

Given $\mathcal{L} \in |\mathbf{Con}|$, by $\mathrm{Cons}_{\mathcal{L}}$ will be denoted the collection of all consequence systems over $\mathcal{L}$. Given two closure systems $\vdash^1 = \{\vdash^1_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ and $\vdash^2 = \{\vdash^2_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ over $\mathcal{L}$, we define $\vdash^1 \leq_{\mathcal{L}} \vdash^2$ to mean that, for all $\Sigma \in |\mathbf{Sign}|$, $\vdash^1_\Sigma \subseteq \vdash^2_\Sigma$. We call $\leq_{\mathcal{L}}$ signature-wise inclusion. The following proposition is well-known in the context of categorical abstract algebraic logic.

Proposition 3 *The pair $\mathbf{Cons}_{\mathcal{L}} = \langle \mathrm{Cons}_{\mathcal{L}}, \leq_{\mathcal{L}} \rangle$ forms a complete lattice.*

Definition 4 introduces the notion of morphism of π-institutions that we will adopt in the category of π-institutions.

Definition 4 *Let $\mathcal{I} = \langle \mathcal{L}, \vdash \rangle$ and $\mathcal{I}' = \langle \mathcal{L}', \vdash' \rangle$ be two π-institutions. A* **semi-interpretation from $\mathcal{I}$ to $\mathcal{I}'$** *is a morphism $\alpha \in \mathbf{Con}(\mathcal{L}, \mathcal{L}')$, such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Gamma \cup \{\phi\} \subseteq \mathrm{SEN}_{\mathcal{L}}(\Sigma)$,*

$$\Gamma \vdash_\Sigma \phi \quad \textit{implies} \quad \alpha_\Sigma(\Gamma) \vdash'_\Sigma \alpha_\Sigma(\phi).$$

Theory families of π-institutions enable us to provide an alternative characterization of semi-interpretations. Since these results are well-known in the theory of categorical abstract algebraic logic, we omit the proofs.

Definition 5 *Let $\mathcal{I} = \langle \mathcal{L}, \vdash \rangle$ be a π-institution. A* **theory family** *of $\mathcal{I}$ is a collection $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}_{\mathcal{L}}(\Sigma)$,*

$$T_\Sigma \vdash_\Sigma \phi \quad \textit{implies} \quad \phi \in T_\Sigma.$$

Let $\mathrm{ThFam}(\mathcal{I})$ denote the collection of all theory families of a π-institution $\mathcal{I} = \langle \mathcal{L}, \vdash \rangle$. Given $T^1, T^2 \in \mathrm{ThFam}(\mathcal{I})$, we define $T^1 \leq T^2$ iff, for all $\Sigma \in |\mathbf{Sign}_{\mathcal{L}}|$, $T^1_\Sigma \subseteq T^2_\Sigma$. We call $\leq$ **signature-wise inclusion**.

Proposition 6 (1) *The pair $\mathbf{ThFam}(\mathcal{I}) = \langle \mathrm{ThFam}(\mathcal{I}), \leq \rangle$ is a complete lattice, for every π-institution $\mathcal{I}$.*

(2) *Let $\mathcal{I} = \langle \mathcal{L}, \vdash \rangle$, $\mathcal{I}' = \langle \mathcal{L}', \vdash' \rangle$ be two π-institutions and $\alpha \in \mathbf{Con}(\mathcal{L}, \mathcal{L}')$ a morphism. Then $\alpha : \mathcal{I} \to \mathcal{I}'$ is a semi-interpretation iff, for every $T' \in \mathrm{ThFam}(\mathcal{I}')$, $\alpha^{-1}(T') := \{\alpha_\Sigma^{-1}(T'_\Sigma)\}_{\Sigma \in |\mathbf{Sign}|} \in \mathrm{ThFam}(\mathcal{I})$.*

Definition 7 *Let us denote by* **Ins** *the category that is defined as follows:*

(a) *Objects: π-institutions;*

(b) *Morphisms: An* **Ins**-*morphism $\alpha : \langle \mathcal{L}, \vdash \rangle \to \langle \mathcal{L}', \vdash' \rangle$ is a* **Con**-*morphism $\alpha \in \mathbf{Con}(\mathcal{L}, \mathcal{L}')$, that is also a semi-interpretation;*

(c) *Composition and Identities: As in* **Con**.

3 Unconstrained Fibring in Ins

The goal of this section is to define the unconstrained fibring of two π-institutions and to show that it always exists in the category **Ins**, provided that the category of connectives **Con** has all small colimits. We start with a technical lemma, showing that, given a morphism α in **Con** between a language $\mathcal{L}$ and a language $\mathcal{L}'$, a consequence system may be constructed over $\mathcal{L}$, whenever one is given over $\mathcal{L}'$, by pulling back along α.

Definition 8 *Let $\mathcal{L}, \mathcal{L}' \in |\mathbf{Con}|$, $\alpha \in \mathbf{Con}(\mathcal{L}, \mathcal{L}')$ and $\vdash'$ a consequence system over $\mathcal{L}'$. Define $\vdash'^{\alpha} = \{\vdash'^{\alpha}_{\Sigma}\}_{\Sigma \in |\mathbf{Sign}|}$ by letting, for all $\Sigma \in |\mathbf{Sign}|$, $\vdash'^{\alpha}_{\Sigma} \subseteq \mathcal{P}(\mathrm{SEN}_{\mathcal{L}}(\Sigma)) \times \mathrm{SEN}_{\mathcal{L}}(\Sigma)$ be defined, for all $\Gamma \cup \{\phi\} \subseteq \mathrm{SEN}_{\mathcal{L}}(\Sigma)$, by*

$$\Gamma \vdash'^{\alpha}_{\Sigma} \phi \quad \textit{iff} \quad \alpha_{\Sigma}(\Gamma) \vdash'_{\Sigma} \alpha_{\Sigma}(\phi).$$

Lemma 9 *Let $\mathcal{L}, \mathcal{L}' \in |\mathbf{Con}|$, $\alpha \in \mathbf{Con}(\mathcal{L}, \mathcal{L}')$ and $\vdash'$ a consequence system over $\mathcal{L}'$. Then $\vdash'^{\alpha} = \{\vdash'^{\alpha}_{\Sigma}\}_{\Sigma \in |\mathbf{Sign}|}$ is a consequence system over $\mathcal{L}$.*

Proof:

Reflexivity is straightforward. We just show transitivity and structurality. Let $\Sigma \in |\mathbf{Sign}|$, $\Gamma \cup \Delta \cup \{\phi\} \subseteq \mathrm{SEN}_{\mathcal{L}}(\Sigma)$, such that $\Gamma \vdash'^{\alpha}_{\Sigma} \phi$ and $\Delta \vdash'^{\alpha}_{\Sigma} \Gamma$. Thus, we have $\alpha_{\Sigma}(\Gamma) \vdash'_{\Sigma} \alpha_{\Sigma}(\phi)$ and $\alpha_{\Sigma}(\Delta) \vdash'_{\Sigma} \alpha_{\Sigma}(\Gamma)$. Hence, since $\vdash'$ is a closure system over $\mathcal{L}'$, we get that $\alpha_{\Sigma}(\Delta) \vdash'_{\Sigma} \alpha_{\Sigma}(\phi)$, which yields that $\Delta \vdash'^{\alpha}_{\Sigma} \phi$. Thus, $\vdash'^{\alpha}$ is transitive. To show structurality, assume that $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}_{\mathcal{L}}(\Sigma, \Sigma')$ and $\Gamma \cup \{\phi\} \subseteq \mathrm{SEN}_{\mathcal{L}}(\Sigma)$, such that $\Gamma \vdash'^{\alpha}_{\Sigma} \phi$. Thus, $\alpha_{\Sigma}(\Gamma) \vdash'_{\Sigma} \alpha_{\Sigma}(\phi)$. But, then, since $\vdash'$ is structural, we obtain that $\mathrm{SEN}_{\mathcal{L}'}(F^{\alpha}(f))(\alpha_{\Sigma}(\Gamma)) \vdash'_{\Sigma'} \mathrm{SEN}_{\mathcal{L}'}(F^{\alpha}(f))(\alpha_{\Sigma}(\phi))$. This is equivalent to $\alpha_{\Sigma'}(\mathrm{SEN}_{\mathcal{L}}(f)(\Gamma)) \vdash'_{\Sigma'} \alpha_{\Sigma'}(\mathrm{SEN}_{\mathcal{L}}(f)(\phi))$, which yields that $\mathrm{SEN}_{\mathcal{L}}(f)(\Gamma) \vdash'^{\alpha}_{\Sigma'} \mathrm{SEN}_{\mathcal{L}}(f)(\phi)$. Therefore, $\vdash'^{\alpha}$ is also structural. □

A key component in being able to fibre π-institutions is the existence of coproducts in the category **Ins**. We show that **Ins** is small cocomplete provided the underlying category **Con** of our languages is small cocomplete.

Proposition 10 *If* **Con** *is small cocmplete, then the category* **Ins** *is small cocomplete.*

Proof:

Let $D : I \to \mathbf{Ins}$ be a small diagram in **Ins** with index graph I. We use the notation $\langle \mathcal{L}^i, \vdash^i \rangle = D(i)$ and $\alpha^e = D(e)$, for $i \xrightarrow{e} j$ in I. Thus, we have $\langle \mathcal{L}^i, \vdash^i\rangle \stackrel{\alpha^e}{\to} \langle \mathcal{L}^j, \vdash^j\rangle$ is a morphism in **Ins**, if $i \stackrel{e}{\to} j$ is an edge in I. If $U : \mathbf{Ins} \to \mathbf{Con}$ is the forgetful functor, forgetting the consequence system, then $U \circ D : I \to \mathbf{Con}$ is a small diagram in **Con**. Since **Con** is small cocomplete, $U \circ D$ has a colimit. Let $\mathcal{L}$ be its colimit in **Con**, with colimit morphisms $\mathcal{L}^i \stackrel{\lambda^i}{\to} \mathcal{L}$. Let also $\vdash = \{\vdash_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ be the least closure system over $\mathcal{L}$, such that, for all $i \in I$, all $\Sigma \in |\mathbf{Sign}|$ and all $\Gamma \cup \{\phi\} \subseteq \mathrm{SEN}_{\mathcal{L}^i}(\Sigma)$,

$$\Gamma \vdash^i_\Sigma \phi \quad \text{implies} \quad \lambda^i_\Sigma(\Gamma) \vdash_\Sigma \lambda^i_\Sigma(\phi).$$

By definition of $\vdash$, $\langle \langle \mathcal{L}, \vdash\rangle, \{\lambda^i\}_{i \in I}\rangle$ forms a coconone over D in **Ins**, i.e., for all $i \stackrel{e}{\to} j$ in I, the following triangle commutes:

$$\begin{tikzcd} & \langle \mathcal{L}, \vdash\rangle & \\ \langle \mathcal{L}^i, \vdash^i\rangle \arrow[ur, "\lambda^i"] \arrow[rr, "\alpha^e"'] & & \langle \mathcal{L}^j, \vdash^j\rangle \arrow[ul, "\lambda^j"'] \end{tikzcd}$$

It suffices, now, to show that this is a colimiting cocone. To this end, consider a cocone $\langle \langle \mathcal{L}', \vdash'\rangle, \kappa^i\rangle$ over D in **Ins**.

$$\begin{tikzcd} & \langle \mathcal{L}', \vdash'\rangle & \\ \langle \mathcal{L}^i, \vdash^i\rangle \arrow[ur, "\kappa^i"] \arrow[rr, "\alpha^e"'] & & \langle \mathcal{L}^j, \vdash^j\rangle \arrow[ul, "\kappa^j"'] \end{tikzcd}$$

By applying U, we obtain a cocone in **Con**. Since $\mathcal{L}$ is the colimit of $U \circ D$ in **Con**, we get a unique morphism $\mu : \mathcal{L} \to \mathcal{L}'$ in **Con**, that makes the following diagram commute, for all $i \in I$,

$$\begin{tikzcd} & \mathcal{L}' & \\ \mathcal{L}^i \arrow[ur, "\kappa^i"] \arrow[rr, "\lambda^i"'] & & \mathcal{L} \arrow[ul, "\mu"'] \end{tikzcd}$$

Since $\kappa^i : \langle \mathcal{L}^i, \vdash^i\rangle \to \langle \mathcal{L}', \vdash'\rangle$ is in **Ins**, we get that, for all $i \in I$, all $\Sigma \in |\mathbf{Sign}|$ and all $\Gamma \cup \{\phi\} \subseteq \mathrm{SEN}_{\mathcal{L}^i}(\Sigma)$,

$$\Gamma \vdash^i_\Sigma \phi \quad \text{implies} \quad \kappa^i_\Sigma(\Gamma) \vdash'_\Sigma \kappa^i_\Sigma(\phi).$$

Since $\mu \circ \lambda^i = \kappa^i$, we get that, for all $i \in I$, all $\Sigma \in |\mathbf{Sign}|$ and all $\Gamma \cup \{\phi\} \subseteq \mathrm{SEN}_{\mathcal{L}^i}(\Sigma)$,

$$\Gamma \vdash^i_\Sigma \phi \quad \text{implies} \quad \mu_\Sigma(\lambda^i_\Sigma(\Gamma)) \vdash'_\Sigma \mu_\Sigma(\lambda^i_\Sigma(\phi)).$$

Thus, by Lemma 9, for all $i \in I$, all $\Sigma \in |\mathbf{Sign}|$ and all $\Gamma \cup \{\phi\} \subseteq \mathrm{SEN}_{\mathcal{L}^i}(\Sigma)$,

$$\Gamma \vdash^i_\Sigma \phi \quad \text{implies} \quad \lambda^i_\Sigma(\Gamma) \vdash'^\mu_\Sigma \lambda^i_\Sigma(\phi).$$

But, by definition, $\vdash$ is the least closure system over $\mathcal{L}$ having this property, whence $\vdash \leq_{\mathcal{L}} \vdash'^\mu$. This shows that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Gamma \cup \{\phi\} \subseteq \mathrm{SEN}_{\mathcal{L}}(\Sigma)$,

$$\Gamma \vdash_\Sigma \phi \quad \text{implies} \quad \mu_\Sigma(\Gamma) \vdash'_\Sigma \mu_\Sigma(\phi).$$

Since, uniqueness of μ is inherited to **Ins** from **Con**, this concludes the proof that $\langle\langle\mathcal{L},\vdash\rangle, \{\lambda^i\}_{i\in I}\rangle$ is a colimit of D in **Ins**. $\square$

The notion of unconstrained fibring of two π-institutions is defined next. It is one of the key definitions in our work.

Definition 11 *The* **(unconstrained) fibring** *in* **Ins** *of two π-institutions $\mathcal{I} = \langle\mathcal{L},\vdash\rangle$ and $\mathcal{I}' = \langle\mathcal{L}',\vdash'\rangle$ is given by their coproduct $\mathcal{I} \amalg \mathcal{I}'$ computed in* **Ins**.

As a consequence of Definition 11 and Proposition 10, we immediately obtain the main theorem of this section.

Theorem 12 *If* **Con** *is small cocomplete, then the category* **Ins** *has unconstrained fibring.*

4 Constrained Fibring in Ins

In this section we study constraint fibring in the category **Ins** of π-institutions. We adopt the definition introduced in [23] and used in the case of deductive systems also in [10]. First, we need to remind the reader of the definition of a co-structured morphism over a functor and of a cocartesian lifting, which will form the categorical basis for the definition of constrained fibring (see also [2] and [10]).

Definition 13 *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $F : \mathcal{C} \to \mathcal{D}$ be a functor.*

*An F-***co-structured morphism with codomain** *$d \in \mathcal{D}$ is a pair $\langle c, f\rangle$, such that $c \in |\mathcal{C}|$ and $f : F(c) \to d \in \mathrm{Mor}(\mathcal{D})$.*

A **cocartesian lifting** *of an F-co-structured morphism $\langle c, f\rangle$ is a $\mathcal{C}$-morphism $f^* : c \to c'$, such that $F(f^*) = f$ and it satisfies the following universal property: For all $\mathcal{C}$-morphisms $g : c \to c''$ and all $\mathcal{D}$-morphisms*

$h : d \to F(c'')$, such that $h \circ f = F(g)$, there exists a unique $\mathcal{C}$-morphism $h^* : c' \to c''$, with $F(h^*) = h$ and $h^* \circ f^* = g$.

$$\begin{tikzcd} c \arrow[r, "f^*"] \arrow[d, "g"'] & c' \arrow[dl, "h^*"] \\ c'' & \end{tikzcd} \qquad \begin{tikzcd} F(c) \arrow[r, "F(f^*)"] \arrow[d, "F(g)"'] & F(c') = d \arrow[dl, "h"] \\ F(c'') & \end{tikzcd}$$

The functor F is called a **cofibration** *if every F-co-structured morphism admits a cocartesian lifting.*

We show next that the forgetful functor U : **Ins** $\to$ **Con**, given by $\langle \mathcal{L}, \vdash \rangle \overset{U}{\mapsto} \mathcal{L}$ and $\langle \mathcal{L}, \vdash \rangle \overset{\alpha}{\to} \langle \mathcal{L}', \vdash' \rangle \overset{U}{\mapsto} \mathcal{L} \overset{\alpha}{\to} \mathcal{L}'$, is a cofibration.

Proposition 14 *The forgetful functor U :* **Ins** $\to$ **Con** *is a cofibration.*

Proof:

Consider the U-co-structured morphism $\langle \langle \mathcal{L}, \vdash \rangle, \alpha \rangle$, with codomain $\mathcal{L}'$. Let $\vdash' = \{\vdash'_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ be the least closure system over $\mathcal{L}'$, such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Gamma \cup \{\phi\} \subseteq \mathrm{SEN}_{\mathcal{L}}(\Sigma)$,

$$\Gamma \vdash_\Sigma \phi \quad \text{implies} \quad \alpha_\Sigma(\Gamma) \vdash'_\Sigma \alpha_\Sigma(\phi).$$

Clearly, $\alpha : \langle \mathcal{L}, \vdash \rangle \to \langle \mathcal{L}', \vdash' \rangle$ is in **Ins**. We claim that it is a cocartesian lifting of the U-co-structured morphism $\langle \langle \mathcal{L}, \vdash \rangle, \alpha \rangle$. To see this, it suffices to prove the associated universal property. Let $\beta : \langle \mathcal{L}, \vdash \rangle \to \langle \mathcal{L}'', \vdash'' \rangle$ be in **Ins**, $\gamma : \mathcal{L}' \to \mathcal{L}''$ in **Con**, such that $\gamma \circ \alpha = \beta$.

$$\begin{tikzcd} \langle \mathcal{L}, \vdash \rangle \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \langle \mathcal{L}', \vdash' \rangle \arrow[dl, "\gamma"] \\ \langle \mathcal{L}'', \vdash'' \rangle & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{L} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \mathcal{L}' \arrow[dl, "\gamma"] \\ \mathcal{L}'' & \end{tikzcd}$$

It suffices to show that $\gamma : \langle \mathcal{L}', \vdash' \rangle \to \langle \mathcal{L}'', \vdash'' \rangle$ is a morphism in **Ins** or, equivalently, that $\vdash' \leq_{\mathcal{L}'} \vdash''^{\gamma}$. This is true because, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Gamma \cup \{\phi\} \subseteq \mathrm{SEN}_{\mathcal{L}}(\Sigma)$,

$$\begin{array}{rcl} \Gamma \vdash_\Sigma \phi & \text{implies} & \beta_\Sigma(\Gamma) \vdash''_\Sigma \beta_\Sigma(\phi) \\ & \text{iff} & \gamma_\Sigma(\alpha_\Sigma(\Gamma)) \vdash''_\Sigma \gamma_\Sigma(\alpha_\Sigma(\phi)) \\ & \text{implies} & \alpha_\Sigma(\Gamma) \vdash''^{\gamma}_\Sigma \alpha_\Sigma(\phi) \end{array}$$

and $\vdash'$ is the least closure system on $\mathcal{L}'$ satisfying this property. $\square$

Following [10], let us illustrate now the idea behind constrained fibring. Suppose $\mathcal{I}' = \langle \mathcal{L}', \vdash' \rangle$ and $\mathcal{I}'' = \langle \mathcal{L}'', \vdash'' \rangle$ are two π-institutions and $\eta' : \mathcal{L}^- \to \mathcal{L}'$ and $\eta'' : \mathcal{L}^- \to \mathcal{L}''$ two injections in **Con**. Intuitively, these represent the syntax that is to be shared through fibring. Let $\mathcal{I}' \amalg \mathcal{I}'' = \langle \mathcal{L}, \vdash \rangle$ be the coproduct in **Ins** of $\mathcal{I}'$ and $\mathcal{I}''$ with canonical injections $\iota' : \mathcal{I}' \to \mathcal{I}' \amalg \mathcal{I}''$ and $\iota'' : \mathcal{I}'' \to \mathcal{I}' \amalg \mathcal{I}''$.

$$\begin{tikzcd} & \mathcal{L}' & & \mathcal{I}' = \langle \mathcal{L}', \vdash' \rangle \arrow[dr, "\iota'"] & \\ \mathcal{L}^- \arrow[ur, "\eta'"] \arrow[dr, "\eta''"'] & & & & \mathcal{I}' \amalg \mathcal{I}'' = \langle \mathcal{L}, \vdash \rangle \\ & \mathcal{L}'' & & \mathcal{I}'' = \langle \mathcal{L}'', \vdash'' \rangle \arrow[ur, "\iota''"'] & \end{tikzcd}$$

Consider now the coequalizer $\nu : \mathcal{L} \to \mathcal{L}^{\triangleright}$ of the following diagram in **Con**:

$$\begin{tikzcd} & \mathcal{L}' \arrow[dr, "\iota'"] & & \\ \mathcal{L}^- \arrow[ur, "\eta'"] \arrow[dr, "\eta''"'] & & \mathcal{L} \arrow[r, "\nu"] & \mathcal{L}^{\triangleright} \\ & \mathcal{L}'' \arrow[ur, "\iota''"'] & & \end{tikzcd}$$

Since $\mathcal{L} = U(\mathcal{I}' \amalg \mathcal{I}'')$, there exists a cocartesian lifting $\nu : \mathcal{I}' \amalg \mathcal{I}'' \to \mathcal{I}^{\triangleright}$ of the U-co-structured morphism $\langle \mathcal{I}' \amalg \mathcal{I}'', \nu \rangle$. The resulting π-institution $\mathcal{I}^{\triangleright}$ is defined to be the fibring of $\mathcal{I}'$ and $\mathcal{I}''$ constrained by the sharing diagram

$$\mathcal{L}' \xleftarrow{\ \eta'\ } \mathcal{L}^- \xrightarrow{\ \eta''\ } \mathcal{L}''$$

This process is formalized in the following definition of the constrained fibring of two π-institutions in **Ins** over a given sharing diagram.

Definition 15 *Let $\mathcal{I}' = \langle \mathcal{L}', \vdash' \rangle$ and $\mathcal{I}'' = \langle \mathcal{L}'', \vdash'' \rangle$ be two π-institutions and D a sharing diagram formed by two injective translations $\eta' : \mathcal{L}^- \to U(\mathcal{I}')$ and $\eta'' : \mathcal{L}^- \to U(\mathcal{I}'')$ in* **Con**. *The* **fibring of $\mathcal{I}'$ and $\mathcal{I}''$ constrained by the sharing** *D is the codomain $\mathcal{I}' \amalg_D \mathcal{I}''$ of the cocartesian lifting of the coequalizer $\nu : U(\mathcal{I}' \amalg \mathcal{I}'') \to \mathcal{L}^{\triangleright}$ in* **Con** *of*

$$\begin{tikzcd} \mathcal{L}^- \arrow[r, shift left, "\iota' \circ \eta'"] \arrow[r, shift right, "\iota'' \circ \eta''"'] & U(\mathcal{I}' \amalg \mathcal{I}'') \end{tikzcd} \qquad (1)$$

where $\iota' : \mathcal{I}' \to \mathcal{I}' \amalg \mathcal{I}''$ and $\iota'' : \mathcal{I}'' \to \mathcal{I}' \amalg \mathcal{I}''$ are the canonical injections of the coproduct $\mathcal{I}' \amalg \mathcal{I}''$ in **Ins** *of $\mathcal{I}'$ and $\mathcal{I}''$.*

Theorem 16 *If* **Con** *is small cocomplete, then the category* **Ins** *has constrained fibring.*

Proof:
By Proposition 10, the coproduct $\mathcal{I}' \coprod \mathcal{I}''$ exists. By hypothesis, the coequalizer $\nu : U(\mathcal{I}' \amalg \mathcal{I}'') \to \mathcal{L}^{\triangleright}$ of Diagram (1) exists. Finally, by Proposition 14, the cocartesian lifting of the U-co-structured morphism ν exists. By definition, its codomain $\mathcal{I}' \amalg_D \mathcal{I}''$ is the constrained fibring of $\mathcal{I}'$ and $\mathcal{I}''$ by the sharing D. □

5 Fibring Propositional Deductive Systems

In this section we show how the framework of Fernández and Coniglio [10] can be accommodated inside the general framework that was developed in the previous sections. To this end, a denumerable set $\mathcal{V}$ of propositional variables $p_0, p_1, \ldots$ is fixed. A **signature** is a family $C = \{C_k\}_{k \in \mathbb{N}}$ of sets, such that $C_k \cap C_n = \emptyset = C_k \cap \mathcal{V}$, for all $k \neq n$. C_k consists of the **connectives of arity** k. The **propositional language** $L(C)$ is the universe of the free algebra generated by applying the operations in C (respecting arities) to the propositional variables in $\mathcal{V}$ in the ordinary recursive way.

Given a signature C, a function $\sigma : \mathcal{V} \to L(C)$ is called a **substitution in** C. It can be extended to an endomorphism $\hat{\sigma} : L(C) \to L(C)$ due to the freeness of $L(C)$ over $\mathcal{V}$.

Let **Con** be the category with objetcs all signatures and morphisms $f : C \to C'$ families $f = \{f_k\}_{k \in \mathbb{N}}$, such that $f_k : C_k \to C'_k$ a function, for all $k \in \mathbb{N}$. Composition is defined pointwise, $g \circ f = \{g_k f_k\}_{k \in \mathbb{N}}$, and identities are the pointwise identities.

Note that, given $C, C' \in |\mathbf{Con}|$ and $f : C \to C'$, f defines a unique map $\hat{f} : L(C) \to L(C')$, which is the identity on the variables. More precisely, we have

- $\hat{f}(p) = p$, for all $p \in \mathcal{V}$;
- $\hat{f}(c_k(t_0, \ldots, t_{k-1})) = f_k(c_k)(\hat{f}(t_0), \ldots, \hat{f}(t_{k-1}))$, for all $c_k \in C_k$ and all $t_0, \ldots, t_{k-1} \in L(C)$.

For every $C \in |\mathbf{Con}|$, $\mathbf{Sign}_C$ consists of the single object $\mathcal{V}$ with morphisms all substitutions $\sigma : \mathcal{V} \to L(C)$ in C. Composition is defined by

$\tau \circ \sigma = \hat{\tau}\sigma$ (the latter being ordinary composition of functions) and the identity is the insertion-of-variables map.

Let $\mathrm{SEN}_C(\mathcal{V}) = L(C)$ and, for all $\sigma \in \mathbf{Sign}_C(\mathcal{V},\mathcal{V})$, $\mathrm{SEN}_C(\sigma) = \hat{\sigma}$. Furthermore, associate with $f : C \to C'$ the functor $F^f : \mathbf{Sign}_C \to \mathbf{Sign}_{C'}$, sending $\sigma : \mathcal{V} \to L(C)$ to $F^f(\sigma) = \hat{f}\sigma : \mathcal{V} \to L(C')$ and the natural transformation $\alpha^f = \{\alpha^f_{\mathcal{V}}\}$, with $\alpha^f_{\mathcal{V}} = \hat{f}$. We check that we have, in fact, a functor and a natural transformation, respectively. We have, for all $\sigma : \mathcal{V} \to L(C)$ and all $\tau : \mathcal{V} \to L(C)$,

$$\begin{tikzcd}
\mathcal{V} \arrow[r, "\sigma"] & \mathcal{V} \arrow[r, "\tau"] & \mathcal{V} \\
\mathcal{V} \arrow[r, "F^f(\sigma)"'] & \mathcal{V} \arrow[r, "F^f(\tau)"'] & \mathcal{V}
\end{tikzcd}$$

$$\begin{aligned}
F^f(\tau) \circ F^f(\sigma) &= (\hat{f}\tau) \circ (\hat{f}\sigma) \\
&= \widehat{\hat{f}\tau}(\hat{f}\sigma) \\
&= (\widehat{\hat{f}\tau}\hat{f})\sigma \\
&= \hat{f}\hat{\tau}\sigma \\
&= \hat{f}(\tau \circ \sigma) \\
&= F^f(\tau \circ \sigma).
\end{aligned}$$

Moreover, for all $\sigma \in \mathcal{V} \to L(C)$,

$$\begin{tikzcd}
\mathrm{SEN}_{\mathcal{L}}(\mathcal{V}) \arrow[r, "\alpha^f_{\mathcal{V}}"] \arrow[d, "\hat{\sigma}"'] & \mathrm{SEN}_{\mathcal{L}'}(\mathcal{V}) \arrow[d, "\widehat{F^f(\sigma)}"] \\
\mathrm{SEN}_{\mathcal{L}}(\mathcal{V}) \arrow[r, "\alpha^f_{\mathcal{V}}"'] & \mathrm{SEN}_{\mathcal{L}'}(\mathcal{V})
\end{tikzcd}$$

$$\begin{aligned}
\widehat{F^f(\sigma)}\ \alpha^f_{\mathcal{V}} &= \widehat{\hat{f}\sigma}\hat{f} \\
&= \hat{f}\hat{\sigma} \\
&= \alpha^f_{\mathcal{V}}\ \hat{\sigma}.
\end{aligned}$$

Finally, for all $f : C \to C'$, all $g : C' \to C''$ and all $\sigma : \mathcal{V} \to L(C)$, we get that

$$\begin{tikzcd}
 & \mathcal{V} \arrow[dl, "\sigma"'] \arrow[d, "F^f(\sigma)"] \arrow[dr, "F^g(F^f(\sigma))"] & \\
L(C) \arrow[r, "\hat{f}"'] & L(C') \arrow[r, "\hat{g}"'] & L(C'')
\end{tikzcd}$$

$$\begin{array}{rcl} F^{g\circ f}(\sigma) & = & \widehat{g\circ f}\sigma \\ & = & \hat{g}\hat{f}\sigma \\ & = & \hat{g}F^f(\sigma) \\ & = & F^g(F^f(\sigma)). \end{array}$$

The category of π-institutions **Ins** over the category **Con** and the categories $\mathbf{Sign}_{\mathcal{L}}, \mathcal{L} \in |\mathbf{Con}|$, of this section, coincides with the category of structural closure operators of [10]. In Proposition 2.5 of [10], it is asserted that **Con** (which is actually called **Sig** in [10]) is small cocomplete. Thus, combining the general results developed in the previous sections with the special framework of this section, we obtain the following corollary of Theorems 12 and 16:

Corollary 17 (Theorem 3.6 of [10]) *The category* **Cons** *of structural consequence relations, as presented in* [10]*, has both unconstrained and constrained fibring.*

Thus, the general results obtained on deductive systems in [10] can be viewed as corollaries of the general results that were presented in Sections 3 and 4.

6 Fibring N-Protoalgebraic π-Institutions

The goal of this section is to use the general framework of Sections 2, 3 and 4 to obtain some results on unconstrained and constrained fibring of protoalgebraic π-institutions. Although we narrow the general context considered in previous sections, we still aim at producing a framework general enough to capture both fibring of protoalgebraic deductive systems (see [3] and [9] for protoalgebraic logics and [10] for fibring inside the class of protoalgebraic logics) and fibring of protoalgebraic term π-institutions (see [25] for term π-institutions). More precisely, in this section we make some additional assumptions on the general framework of Section 2, that was used to obtain the unconstrained and constrained fibring of π-institutions. Namely, we assume that there exists a $V \in |\mathbf{Sign}|$ and $p, q \in \mathrm{SEN}_{\mathcal{L}}(V)$, for all $\mathcal{L} \in |\mathbf{Con}|$ (not depending on $\mathcal{L}$; existential quantification comes first), such that, for all $\mathcal{L} \in |\mathbf{Con}|$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}_{\mathcal{L}}(\Sigma)$, there exists $f^{\mathcal{L}}_{\langle\Sigma,\phi,\psi\rangle} \in \mathbf{Sign}_{\mathcal{L}}(V, \Sigma)$, such that

- $\mathrm{SEN}_{\mathcal{L}}(f^{\mathcal{L}}_{\langle\Sigma,\phi,\psi\rangle})(p) = \phi$, $\mathrm{SEN}_{\mathcal{L}}(f^{\mathcal{L}}_{\langle\Sigma,\phi,\psi\rangle})(q) = \psi$;
- $g \circ f^{\mathcal{L}}_{\langle\Sigma,\phi,\psi\rangle} = f^{\mathcal{L}}_{\langle\Sigma',\mathrm{SEN}_{\mathcal{L}}(g)(\phi),\mathrm{SEN}_{\mathcal{L}}(g)(\psi)\rangle}$, for all $g \in \mathbf{Sign}_{\mathcal{L}}(\Sigma, \Sigma')$;

- $\alpha_V(p) = p, \alpha_V(q) = q$ and $F^\alpha(f^{\mathcal{L}}_{\langle\Sigma,\phi,\psi\rangle}) = f^{\mathcal{L}'}_{\langle\Sigma,\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\rangle}$, for every $\alpha \in \mathbf{Con}(\mathcal{L},\mathcal{L}')$.

The system $\{\mathrm{SEN}_{\mathcal{L}} : \mathbf{Sign}_{\mathcal{L}} \to \mathbf{Set}\}_{\mathcal{L}\in|\mathbf{Con}|}$ satisfying the additional assumptions listed above will be called a **polyterm system**. A π-institution $\mathcal{I} = \langle\mathcal{L},\vdash_{\mathcal{L}}\rangle$, based on the $\mathcal{L}$-th component $\mathrm{SEN}_{\mathcal{L}}$ of a polyterm system is called a **polyterm π-institution**.

Given a set $\Delta \subseteq \mathrm{SEN}_{\mathcal{L}}(V)$, $\Sigma \in |\mathbf{Sign}|$ and $\phi,\psi \in \mathrm{SEN}_{\mathcal{L}}(\Sigma)$, define

$$\Delta_\Sigma(\phi,\psi) := \mathrm{SEN}_{\mathcal{L}}(f^{\mathcal{L}}_{\langle\Sigma,\phi,\psi\rangle})(\Delta).$$

Definition 18, that follows, makes precise the notion of a protoalgebraic polyterm π-institution, which, intuitively, is intended to capture the notion of a protoalgebraic deductive system in the context of polyterm π-institutions. Note that the polyterm property allows us to use a collection of V-sentences Δ as an "internalization" in the set of sentences of the collection of natural transformations over SEN, that plays a similar role in the treatment of (syntactic) N-protoalgebraic π-institutions, as presented in [29, 28].

Definition 18 *A polyterm π-institution $\mathcal{I} = \langle\mathcal{L},\vdash\rangle$ is said to be* **protoalgebraic** *if there exists a (possibly infinite) set $\Delta \subseteq \mathrm{SEN}_{\mathcal{L}}(V)$, such that*

(R) $\vdash_V \Delta_V(p,p)$;

(MP) $p, \Delta_V(p,q) \vdash_V q$.

Note that, because of structurality and the polyterm property, a polyterm π-institution $\mathcal{I} = \langle\mathcal{L},\vdash\rangle$ is protoalgebraic iff, there exists a set $\Delta \subseteq \mathrm{SEN}_{\mathcal{L}}(V)$, such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi,\psi \in \mathrm{SEN}(\Sigma)$,

(R') $\vdash_\Sigma \Delta_\Sigma(\phi,\psi)$;

(MP') $\phi, \Delta_\Sigma(\phi,\psi) \vdash_\Sigma \psi$.

Following [10], we call the collection $\Delta \subseteq \mathrm{SEN}_{\mathcal{L}}(V)$ a **protoalgebraizator** of $\mathcal{I}$. This definition allows us to define the category of all protoalgebraic polyterm π-institutions inside which fibring is to be considered.

Definition 19 *The category* **Prot** *is defined as follows:*

(a) *Objects: Protoalgebraic π-institutions;*

(b) *Morphisms: A morphism $\alpha : \mathcal{I} \to \mathcal{I}'$ from $\mathcal{I} = \langle \mathcal{L}, \vdash \rangle$ to $\mathcal{I}' = \langle \mathcal{L}', \vdash' \rangle$ is an* **Ins**-*morphism, such that, for every protoalgebraizator $\Delta \subseteq \mathrm{SEN}_{\mathcal{L}}(V)$ of $\mathcal{I}$, the collection $\alpha_V(\Delta) \subseteq \mathrm{SEN}_{\mathcal{L}'}(V)$ is a protoalgebraizator of $\mathcal{I}'$.*

(c) *Composition and Identities: As in* **Ins**.

Proposition 20 abstracts Proposition 4.3 of [10]. It asserts that the category of protoalgebraic π-institutions over a polyterm system $\{\mathrm{SEN}_{\mathcal{L}}\}_{\mathcal{L} \in |\mathbf{Con}|}$ is a full subcategory of the category of all π-institutions over the same system.

Proposition 20 **Prot** *is a full subcategory of* **Ins**.

Proof:
Assume that $\alpha : \langle \mathcal{L}, \vdash \rangle \to \langle \mathcal{L}', \vdash' \rangle$ is a morphism in **Ins** and that $\Delta \subseteq \mathrm{SEN}_{\mathcal{L}}(V)$ is a protoalgebraizator of $\langle \mathcal{L}, \vdash \rangle$. Let $\Delta' = \alpha_V(\Delta)$.

- Since Δ is a protoalgebraizator of $\langle \mathcal{L}, \vdash \rangle$, we have

  $$\vdash_V \mathrm{SEN}_{\mathcal{L}}(f^{\mathcal{L}}_{\langle V,p,p \rangle})(\Delta).$$

  Thus, since α is in **Ins**, we get $\vdash'_V \alpha_\Sigma(\mathrm{SEN}_{\mathcal{L}}(f^{\mathcal{L}}_{\langle V,p,p \rangle})(\Delta))$. This holds iff $\vdash'_V \mathrm{SEN}_{\mathcal{L}'}(F^\alpha(f^{\mathcal{L}}_{\langle V,p,p \rangle}))(\alpha_V(\Delta))$ iff $\vdash'_V \mathrm{SEN}_{\mathcal{L}'}(f^{\mathcal{L}'}_{\langle V,p,p \rangle})(\Delta')$. Therefore Δ' satisfies reflexivity.

- Since Δ is a protoalgebraizator of $\langle \mathcal{L}, \vdash \rangle$, we also have

  $$p, \mathrm{SEN}_{\mathcal{L}}(f^{\mathcal{L}}_{\langle V,p,q \rangle})(\Delta) \vdash_V q$$

  Thus, since α is in **Ins**, we get

  $$\alpha_V(p), \alpha_V(\mathrm{SEN}_{\mathcal{L}}(f^{\mathcal{L}}_{\langle V,p,q \rangle})(\Delta)) \vdash'_V \alpha_V(q).$$

  Therefore, we obtain $p, \mathrm{SEN}_{\mathcal{L}'}(F^\alpha(f^{\mathcal{L}}_{\langle V,p,q \rangle}))(\alpha_V(\Delta)) \vdash'_V q$, i.e.,

  $$p, \mathrm{SEN}_{\mathcal{L}'}(f^{\mathcal{L}'}_{\langle V,p,q \rangle})(\Delta') \vdash'_V q,$$

  showing that Δ' satisfies modus ponens.

Thus Δ' is a protoalgebraizator of $\langle \mathcal{L}', \vdash' \rangle$, showing that α is in **Prot**. $\square$

Since, by the proof of Proposition 20, every morphism in **Ins** with domain in **Prot** is itself a morphism in **Prot** and since, by Theorems 12 and 16, **Ins** has both unconstrained and constrained fibring provided that **Con** is small cocomplete, we obtain the following result to the effect that the same holds for the category **Prot**.

Theorem 21 *If* **Con** *is small cocomplete, then the category* **Prot** *has both unconstrained and constrained fibring.*

It is not difficult to see that the system introduced in Section 5 in order to perform fibring over propositional deductive systems is a polyterm system. Therefore, the definition of a protoalgebraic π-institution applies to that system as well, giving us as corollaries the results obtained in [10] on fibring inside the class of protoalgebraic deductive systems (but without restricting to finitary ones).

7 Fibring Description Logics over $\mathcal{ALC}$

In this final section of the paper we provide an additional example of the applicability of the general framework by illustrating how it can be used to produce the constrained fibring of two description logics [1] over the well-known basic description logic $\mathcal{ALC}$ [20]. For simplicity, we will include role names as part of the constructors for concept expressions rather than including them in the set of signatures and we will not consider role name substitutions. In other words, we will assign to existential role quantifications the role of cylindrifications and delegate then to the set of unary connectives over concept expressions. This will allow us to restrict attention to a single-sorted logic and to avoid complications that would obscure the essence of the example.

To this end, a denumerable set $\mathcal{C}$ of concept names $A_0, A_1, \ldots$ and a disjoint denumerable set $\mathcal{R}$ of role names are fixed. As in Section 5, a **signature** is a family $C = \{C_k\}_{k \in \mathbb{N}}$ of sets, such that, for all $k \neq n$, $C_k \cap C_n = \emptyset$ and, for all k, $C_k \cap \mathcal{C} = \emptyset = C_k \cap \mathcal{R}$. The set C_k consists of the **connectives of arity** k. The **propositional language** $L(C)$ is the universe of the free algebra generated by applying the operations in C to the concept names in $\mathcal{C}$ in the ordinary recursive way. For instance, if $C_0 = \{\top\}$, $C_1 = \{\exists \mathsf{hasChild}\}$ and $C_2 = \{\sqcap\}$ and $\texttt{Female} \in \mathcal{C}$, then the concept expression, whose extension is supposed to captures the elements of the universe corresponding to “mothers” is the concept expression $\texttt{Female} \sqcap \exists \mathsf{hasChild}(\top) \in L(C)$.

Given a signature C, a function $\sigma : \mathcal{C} \to L(C)$ is called a **substitution in** C. It can be extended to an endomorphism $\hat{\sigma} : L(C) \to L(C)$ in the usual way, due to the freeness of $L(C)$ over $\mathcal{C}$.

Let **Con** be the category with objetcs all signatures and morphisms $f : C \to C'$ families $f = \{f_k\}_{k \in \mathbb{N}}$, such that $f_k : C_k \to C'_k$ is a function, for all $k \in \mathbb{N}$. Composition is defined pointwise and identities are the pointwise identities, just as in the case of the category **Con** of Section 5.

For every $C \in |\mathbf{Con}|$, $\mathbf{Sign}_C$ consists of the single object $\mathcal{C}$ with morphisms all substitutions $\sigma : \mathcal{C} \to L(C)$. Composition is defined by $\tau \circ \sigma = \hat{\tau}\sigma$ and the identity is the insertion-of-concept-names map.

Let $\mathrm{SEN}_C(\mathcal{C}) = L(C)^2$, perceived as the set of all subsumptions $E \sqsubseteq F$, with $E, F \in L(C)$, and associate with $f : C \to C'$ the functor $F^f : \mathbf{Sign}_C \to \mathbf{Sign}_{C'}$, sending $\sigma : \mathcal{C} \to L(C)$ to $F^f(\sigma) = \hat{f}\sigma : \mathcal{C} \to L(C')$ and the natural transformation $\alpha^f = \{\alpha^f_{\mathcal{C}}\}$, with $\alpha^f_{\mathcal{C}} = \hat{f}^2$. It can be checked, in a similar way as in Section 5, that F^f is a functor and α^f is a natural transformation and that all necessary conditions of the general framework for fibring are satisfied.

Let us fix the signature $C^{\mathcal{ALC}}$ over which the description logic $\mathcal{ALC}$ [20] may be built. It consists of one binary connective $\sqcap$, one unary connective $\neg$ and an additional denumerable collection $\exists R, R \in \mathcal{R}$, of unary connectives. Consider, now, signatures C' and C'', that include (pointwise) the signature $C^{\mathcal{ALC}}$ and form in **Con** the following sharing diagram:

$$C' \xleftarrow{\ \eta'\ } C^{\mathcal{ALC}} \xrightarrow{\ \eta''\ } C''$$

where η' and η'' are the corresponding injections in **Con**.

Consider, next, the π-institution $\mathcal{I}^{\mathcal{ALC}} = \langle C^{\mathcal{ALC}}, \vdash^{\mathcal{ALC}} \rangle$, that results by taking, for all $\mathcal{S} \cup \{E \sqsubseteq F\} \subseteq L(C^{\mathcal{ALC}})^2$,

$$\begin{array}{lll} \mathcal{S} \vdash^{\mathcal{ALC}} E \sqsubseteq F & \text{iff} & \text{for every model } \mathcal{M} = \langle \Delta^{\mathcal{M}}, \cdot^{\mathcal{M}} \rangle \text{ of } C^{\mathcal{ALC}} \\ & & \mathcal{M} \models \mathcal{S} \text{ implies } \mathcal{M} \models E \sqsubseteq F. \end{array}$$

Moreover, assume that we have also a π-institution $\mathcal{I}' = \langle C', \vdash' \rangle$ over C', whose entailment relation contains that of $\mathcal{I}^{\mathcal{ALC}}$. The relation $\vdash'$ could potentially be the semantical entailment relation induced by a Tarski-style semantics, as long as it respects structurality. This is not always the case, but, in general, a fragment of first-order logic may be transformed to a structural counterpart as in Appendix C of [4]. Let $\mathcal{I}'' = \langle C'', \vdash'' \rangle$ be another π-institution over C'', whose entailment relation also includes $\vdash^{\mathcal{ALC}}$. Let D

denote the sharing diagram formed by the two injective morphisms $\eta' : C^{\mathcal{ALC}} \to U(\mathcal{I}')$ and $\eta'' : C^{\mathcal{ALC}} \to U(\mathcal{I}'')$ in **Con**. The fibring of $\mathcal{I}'$ and $\mathcal{I}''$ constrained by D is, by definition, the codomain $\mathcal{I}' \amalg_D \mathcal{I}''$ of the cocartesian lifting of the coequalizer $\nu : U(\mathcal{I}' \amalg \mathcal{I}'') \to C^{\triangleright}$ in **Con** of

$$C^{\mathcal{ALC}} \underset{\iota'' \circ \eta''}{\overset{\iota' \circ \eta'}{\rightrightarrows}} U(\mathcal{I}' \amalg \mathcal{I}'')$$

where $\iota' : \mathcal{I}' \to \mathcal{I}' \amalg \mathcal{I}''$ and $\iota'' : \mathcal{I}'' \to \mathcal{I}' \amalg \mathcal{I}''$ are the canonical injections of the coproduct $\mathcal{I}' \amalg \mathcal{I}''$ in **Ins** of $\mathcal{I}'$ and $\mathcal{I}''$.

By the general construction of Section 4, $C^{\triangleright}$ will consist of a binary operation $\sqcap$, unary operations $\neg$ and $\exists R, R \in \mathcal{R}$, which are inherited by both C' and C'' under the identification through D, and additional operations induced by the corresponding ones in C' and C''. Moreover, the consequence system $\vdash^{\triangleright}$ of $\mathcal{I}' \amalg_D \mathcal{I}''$ will be the least closure system on $L(C^{\triangleright})^2$, such that, for all $\mathcal{S} \cup \{E \sqsubseteq F\} \subseteq L(C')^2$,

$$\mathcal{S} \vdash'_C E \sqsubseteq F \quad \text{implies} \quad \mathcal{S} \vdash^{\triangleright}_C E \sqsubseteq F$$

and, for all $\mathcal{S} \cup \{E \sqsubseteq F\} \subseteq L(C'')^2$,

$$\mathcal{S} \vdash''_C E \sqsubseteq F \quad \text{implies} \quad \mathcal{S} \vdash^{\triangleright}_C E \sqsubseteq F.$$

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