Categorical Abstract Algebraic Logic
Local Deduction Theorems for π-Institutions

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September 10, 2007

Abstract

In this paper, some of the results of Blok and Pigozzi on the local deduction-detachment theorems (LDDT) in Abstract Algebraic Logic, which followed pioneering work of Czelakowski on the same topic, are abstracted to cover logics formalized as π-institutions. The relationship between the LDDT and the property of various classes of $\mathcal{I}$-matrices having locally definable principal $\mathcal{I}$-filters is investigated. Moreover, it is shown that the LDDT implies various forms of the principal and the local filter extension properties that are mutually equivalent. Finally, the notion of algebraic equivalence for π-institutions, a strengthening of the notion of deductive equivalence, previously introduced by the author, is formulated and it is shown that the property of having the LDDT is preserved under both bilogical morphisms and algebraic equivalence.

1 Introduction

In this introduction, an effort will be made to review some of the results due to Czelakowski [6, 7] and Blok and Pigozzi [4, 2], that inspired, among others, the work that started in [18] and culminated in the material to be presented in this paper. The origin of these results goes back to Czelakowski's work in [7], which studies the local deduction-detachment theorem (LDDT) in the context of deductive systems in the traditional sense. Czelakowski reveals a close connection between the property of a deductive system having the LDDT, on the one hand, and that of its corresponding class of matrices having the filter extension property, on the other. In a sequel to this work, Czelakowski joins forces with Dziobiak in [9] to investigate a similar relationship between quasivarieties with the property of having a weak version

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[0] *Keywords:* N-rule based π-institutions, N-structurality, $\mathcal{I}$-matrix, $\mathcal{I}$-morphism, lifting of N-quotients, N-protoalgebraic π-institutions, N-correspondence property, local deduction-detachment theorem, locally N-definable principal filters, filter extension property, bilogical morphisms, deductively equivalent π-institutions

1991 AMS Subject Classification: Primary: 03B22, 03C05, 03G25 Secondary: 08B05, 08C05

of equationally definable principal congruences and those having the congruence extension property. Blok and Pigozzi, after having introduced k-dimensional deductive systems to cover under one umbrella both ordinary deductive systems and the theory of quasivarieties of universal algebras, showed in [2] that both results mentioned above are "manifestations of a single theorem" stating, roughly speaking, that a k-dimensional deductive system $\mathcal{S}$ has the local deduction-detachment theorem if and only if the class of matrices of $\mathcal{S}$ has the filter extension property. This is part of the content of Corollary 3.6 of [2]. Another important result of [2], that will also be adapted to the framework of π-institutions in this paper, is Theorem 2.4, which relates the LDDT with the property of a class of matrices having locally formula definable principal filters, which forms a matrix-theoretic analog of the LDDT.

In what follows, an attempt will be made to describe more formally these important results of Blok and Pigozzi, appearing in [2], that formed the inspiration for the results presented in this paper. Besides illuminating the background and origin of the present work, this will, hopefully, help the reader acquire an appreciation for the generality of the corresponding relationships established in the categorical framework.

A k-deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ is *protoalgebraic* if the *Leibniz operator*, which associates with each theory T of the deductive system the largest congruence $\Omega_{\mathcal{S}}(T)$ on the formula algebra, that is compatible with the theory T, is monotone on the lattice of theories. This is equivalent to the condition that the *generalized Leibniz operator*, which associates with each $\mathcal{S}$-filter F on an algebra $\mathbf{A}$ the largest congruence $\Omega_{\mathbf{A}}(F)$ on $\mathbf{A}$ compatible with F, is monotone on the lattice of all $\mathcal{S}$-filters on any algebra $\mathbf{A}$ of the same similarity type as $\mathcal{S}$. On the other hand, a k-dimensional deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ has the *correspondence property* if, for every surjective matrix homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ between $\mathcal{S}$-matrices,

$$h^{-1}(h^*(F)) = F \vee^{\mathbf{Fi}_{\mathcal{S}}(\mathfrak{A})} h^{-1}(F_{\mathfrak{B}}), \text{ for all } F \in \mathrm{Fi}_{\mathcal{S}}(\mathfrak{A}),$$

where by $h^*(F)$ is denoted the $\mathcal{S}$-filter $\mathrm{Fg}^{\mathcal{S}}_{\mathfrak{B}}(h(F))$ of $\mathfrak{B}$ that is generated by the set $h(F)$, that may not be itself an $\mathcal{S}$-filter. In Theorem 1.4.1 of [2], it is shown that a k-dimensional deductive system $\mathcal{S}$ is protoalgebraic if and only if it has the correspondence property. An analog of this result in the categorical framework was shown to hold in Theorem 20 of [18].

Suppose, next that $E_i(p,q), i \in I$, is a finite set of k-formulas in two k-variables p and q and $\mathcal{E} = \{E_i : i \in I\}$. The system $\mathcal{E}$ is a *local deduction-detachment system for* $\mathcal{S}$ if, for all $\Gamma \cup \{\phi, \psi\} \subseteq \mathrm{Fm}^k_{\mathcal{L}}(V)$, we have that

$$\Gamma, \phi \vdash_{\mathcal{S}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathcal{S}} E_i(\phi, \psi), \text{ for some } i \in I.$$

If there exists a local deduction-detachment system $\mathcal{E}$ for $\mathcal{S}$, then $\mathcal{S}$ is said to *have the local deduction-detachment theorem* (LDDT) *with respect to* the system $\mathcal{E}$. On the other hand, a class M of $\mathcal{S}$-matrices is said to have *locally formula definable principal filters* (LFDPF) *with defining system* $\mathcal{E}$ if, for all $\mathfrak{A} = \langle \mathbf{A}, F_{\mathfrak{A}} \rangle \in M$ and all $a, b \in A^k$,

$$b \in \mathrm{Fg}^{\mathcal{S}}_{\mathfrak{A}}(a) \quad \text{iff} \quad E_i^{\mathbf{A}}(a,b) \subseteq F_{\mathfrak{A}}, \text{ for some } i \in I.$$

In Theorem 2.4 of [2], which will be abstracted in Theorem 6, in the sequel, to cover the π-institution framework, Blok and Pigozzi show that $\mathcal{S}$ has the LDDT if and only if $\mathrm{Mat}(\mathcal{S})$ has LFDPF and this is true if and only if the same holds for $\mathrm{Mat}^*(\mathcal{S})$.

Turning now to the results of Czelakowski [7] and Czelakowski and Dziobiak [9], Blok and Pigozzi introduce the (principal) filter extension property. Given a k-dimensional deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$, an $\mathcal{S}$-matrix $\mathfrak{A}$ has the *(principal) filter extension property* ((P)FEP) if, for every submatrix $\mathfrak{B}$ of $\mathfrak{A}$ and every (principal) $\mathcal{S}$-filter F of $\mathfrak{B}$, there is an $\mathcal{S}$-filter F' of $\mathfrak{A}$, such that $F' \cap B^k = F$. A class of $\mathcal{S}$-matrices is said to *have the* (P)FEP if each of its members does. Blok and Pigozzi also show in Theorem 3.1 and Corollary 3.5 that $\mathcal{S}$ has the LDDT if and only if the class $\mathrm{Mat}(\mathcal{S})$ has the PFEP and, moreover, that this condition is equivalent to $\mathrm{Mat}(\mathcal{S})$ having the FEP, to $\mathrm{Mat}^*(\mathcal{S})$ having the PFEP and to $\mathrm{Mat}^*(\mathcal{S})$ having the FEP. Applying these results to 2-deductive systems, they are able to obtain the results of Czelakowski and Dziobiak for quasivarieties of universal algebras as special cases. They also obtain several results connecting the LDDT, the ordinary deduction-detachment theorem and filter distributivity, but those will not be at the focus of our investigations here and, therefore, we refer the interested reader to [2] for more details.

Finally, we provide a brief overview of the contents of the present paper.

Section 2 provides a review of the basic definitions and results from [18]. Since the work in [18] is a precursor to the study of the local deduction-detachment theorems that is presented in this paper, this review will be helpful in making the paper more self-contained. Among the basic notions discussed in Section 2 one may find that of N-structurality, lifting of N-quotients and transferability of N-rules. One of the main results states that, if a π-institution $\mathcal{I}$ has transferable N-rules, admits lifting of N-quotients and has an N-implication system, then it has the N-correspondence property. Moreover, under the hypotheses that $\mathcal{I}$ is N-structural and admits lifting of N-quotients, if $\mathcal{I}$ has the N-correspondence property, then it is N-protoalgebraic. This constitutes the main result, Theorem 20, of Section 5 of [18].

Section 3 begins with the definition of the local deduction-detachment theorem (LDDT) for a π-institution $\mathcal{I}$ with respect to a local deduction-detachment system. In [18], the notion of the $\mathcal{I}$-filter generated by families of sentences of the underlying sentence functor of an N-algebraic system was defined. In Section 3, this notion is specialized to that of a local $\mathcal{I}$-filter. Local $\mathcal{I}$-filters are $\mathcal{I}$-filters that are generated by a single set of sentences over a single signature. In this paper, local $\mathcal{I}$-filters play a very important role because they are used to abstract many properties of ordinary filters from the theory of logical matrices of sentential logics. An $\mathcal{I}$-matrix is called local if its filter is a local $\mathcal{I}$-filter. Various classes of $\mathcal{I}$-matrices are singled out, based on the notion of a local $\mathcal{I}$-matrix. $\mathrm{Mat}(\mathcal{I})$ is the entire class of $\mathcal{I}$-matrices. $\mathrm{Mat}^{is}(\mathcal{I})$ is the class of all those $\mathcal{I}$-matrices, for which there exists at least one surjective algebraic morphism, with an isomorphic functor component, from the underlying algebraic system of $\mathcal{I}$ onto their own underlying algebraic system. $\mathrm{Mat}^{lis}(\mathcal{I})$ denotes the subclass of $\mathrm{Mat}^{is}(\mathcal{I})$ consisting of all local $\mathcal{I}$-matrices. The notations $\mathrm{Mat}^{N}(\mathcal{I}), \mathrm{Mat}^{Nis}(\mathcal{I})$ and $\mathrm{Mat}^{Nlis}(\mathcal{I})$ are used to denote the classes of all reduced members of the aforementioned three classes of $\mathcal{I}$-matrices, respectively. Given a collection M of local $\mathcal{I}$-matrices, the definition of M having locally N-definable principal filters (LDPF) is provided next. Moreover, it is shown that, for a specific class of π-institutions, the property of having the LDDT implies that the class $\mathrm{Mat}^{lis}(\mathcal{I})$ has LDPF. Furthermore, it is shown that, if $\mathrm{Mat}^{Nlis}(\mathcal{I})$ has LDPF, then $\mathcal{I}$ has the LDDT. As a consequence of these two results and the obvious fact that, if $\mathrm{Mat}^{lis}(\mathcal{I})$ has LDPF, then its subclass $\mathrm{Mat}^{Nlis}(\mathcal{I})$ also has LDPF, one obtains, that the conditions of $\mathcal{I}$ having the LDDT, $\mathrm{Mat}^{lis}(\mathcal{I})$ having LDPF and $\mathrm{Mat}^{Nlis}(\mathcal{I})$ having LDPF are equivalent. This is the content of the main result, Theorem 6, of Section 3. It parallels Theorem 2.4 of [2], which is applicable in the special case of k-dimensional deductive systems.

Section 4 studies the filter extension property in connection with the local deduction-detachment theorem. It defines the notion of a submatrix of a given $\mathcal{I}$-matrix and asserts that, every submatrix of an $\mathcal{I}$-matrix is itself an $\mathcal{I}$-matrix. Then, it introduces the property of a class of $\mathcal{I}$-matrices having the principal filter extension property (PFEP). It is shown that, roughly speaking, if $\mathcal{I}$ has the LDDT, then the class $\mathrm{Mat}^{lis}(\mathcal{I})$ has the PFEP. The local filter extension property (LFEP) is introduced next, that parallels the ordinary filter extension property in the sentential logic framework. Informally stated, it is shown that the class $\mathrm{Mat}^{lis}(\mathcal{I})$ has the PFEP if and only if it has the LFEP, under the hypothesis that $\mathcal{I}$ is finitary. Moreover, it is shown that, if $\mathrm{Mat}^{Nlis}(\mathcal{I})$ has the PFEP, then $\mathrm{Mat}^{lis}(\mathcal{I})$ also has the PFEP, provided that $\mathcal{I}$ satisfies the N-correspondence property. As a result, in Corollary 14 of Section 4, it is proven that, under suitable hypotheses on the π-institution $\mathcal{I}$, the conditions that $\mathrm{Mat}^{lis}(\mathcal{I})$ has the PFEP, $\mathrm{Mat}^{lis}(\mathcal{I})$ has the LFEP, $\mathrm{Mat}^{Nlis}(\mathcal{I})$ has the PFEP and $\mathrm{Mat}^{Nlis}(\mathcal{I})$ has the LFEP are equivalent conditions.

In Section 5, it is asserted that the property of having the LDDT is preserved under bilogical morphisms, whereas in Section 6, the final section of the paper, preservation of the property of having the LDDT under algebraic equivalence of π-institutions is proven. Bilogical morphisms were introduced in the categorical framework in [15], taking after the work of Font and Jansana [10] in the sentential logic framework, whereas algebraic equivalence is a stronger form of deductive equivalence, which was introduced in [13]. This latter work was inspired by the pioneering work of Blok and Pigozzi on algebraizable logics [3].

The current state-of-art in Abstract Algebraic Logic is detailed in the review article [11]. For more details the monograph [10] and the book [8] are recommended. For all unexplained categorical notation, the reader is referred to any of the standard references [1, 5, 12].

2 Protoalgebraicity and the Correspondence Property

In this section, some of the main definitions and results of [18] will be reviewed. Since many of these results will be used in the present paper, the goal of this section is to make the present work as self-contained as possible. For the proofs the reader will be referred to [18].

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a functor and N a category of natural transformations on SEN. An N-**algebraic system** is a triple $\mathbf{A}=\langle \mathrm{SEN}', \langle N', F' \rangle\rangle$, with

- $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ a functor,

- N' a category of natural transformations on SEN' and
- $F' : N \to N'$ a surjective functor that preserves projections.

Given two N-algebraic systems $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$ and $\mathbf{B} = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$, an N-**(algebraic) morphism** $\langle F, \alpha\rangle : \mathbf{A} \to \mathbf{B}$ is an (N', N'')-epimorphic translation $\langle F, \alpha\rangle : \mathrm{SEN}' \to^{se} \mathrm{SEN}''$, such that the following triangle commutes

$$\begin{tikzcd} & N \arrow[dl, "F'"'] \arrow[dr, "F''"] & \\ N' \arrow[rr, dash, dashed, "\langle F, \alpha\rangle"'] & & N'' \end{tikzcd}$$

where the dotted line represents the two-way correspondence established by the (N', N'')-epimorphic property of $\langle F, \alpha\rangle$.

A π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is said to be N-**structural** if, for all N-(endo)morphisms $\langle F, \alpha\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \to \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle$, $\langle F, \alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}$ is an (N, N)-logical morphism, i.e., an (N, N)-epimorphic translation that is a semi-interpretation. More explicitly, this means that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma)$,

$$\alpha_\Sigma(C_\Sigma(\Phi)) \subseteq C_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

Given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, and an N-algebraic system $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, an axiom family $T' = \{T'_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}'|}$ of SEN', i.e., a family $T' = \{T'_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}'|}$, such that $T'_{\Sigma'} \subseteq \mathrm{SEN}'(\Sigma')$, for all $\Sigma' \in |\mathbf{Sign}'|$, is said to be an $\mathcal{I}$-**filter on A** if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$,

$$\alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq T'_{F(\Sigma')} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \in T'_{F(\Sigma')},$$

for all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$ and every N-morphism $\langle F, \alpha\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \to \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$. The collection of all $\mathcal{I}$-filters on $\mathbf{A}$ is denoted by $\mathrm{Fi}^{\mathcal{I}}(\mathbf{A})$. The pair $\mathbf{Fi}^{\mathcal{I}}(\mathbf{A}) := \langle \mathrm{Fi}^{\mathcal{I}}(\mathbf{A}), \leq\rangle$ is a complete lattice, where $\leq$ denotes signature-wise inclusion.

Given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, an $\mathcal{I}$-**matrix** is a pair $\mathfrak{A} = \langle \mathbf{A}, T'\rangle$, where

- $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$ is an N-algebraic system and
- T' is an $\mathcal{I}$-filter on $\mathbf{A}$.

The collection of all $\mathcal{I}$-matrices will be denoted by $\mathrm{Mat}(\mathcal{I})$.

It is shown in Proposition 3 of [18] that, given an N-structural π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, the collection of $\mathcal{I}$-filters on the N-algebraic system $\langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle$ coincides with the collection of all theory families of $\mathcal{I}$. Moreover, Proposition 5 of [18] asserts that the class $\mathrm{Mat}(\mathcal{I})$ of all $\mathcal{I}$-matrices forms a complete semantics for the π-institution $\mathcal{I}$ in a precise technical sense.

Consider a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ and a category N of natural transformations on SEN. Let $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$ be an N-algebraic system and $\mathfrak{A} = \langle \mathbf{A}, T'\rangle \in \mathrm{Mat}(\mathcal{I})$ an $\mathcal{I}$-matrix. A collection $U = \{U_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}'|}$ is called an **$\mathcal{I}$-filter of $\mathfrak{A}$** if U is an $\mathcal{I}$-filter on $\mathbf{A}$ such that $T' \leq U$. The collection of all $\mathcal{I}$-filters of $\mathfrak{A}$, denoted by $\mathrm{Fi}^{\mathcal{I}}\mathfrak{A}$, also forms a complete lattice $\mathbf{Fi}^{\mathcal{I}}\mathfrak{A} = \langle \mathrm{Fi}^{\mathcal{I}}\mathfrak{A}, \leq\rangle$.

Given $\Sigma \in |\mathbf{Sign}'|$ and $\Phi \subseteq \mathrm{SEN}'(\Sigma)$, denote by $\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(\langle \Sigma, \Phi\rangle)$ the **$\mathcal{I}$-filter of $\mathfrak{A}$ generated by** $\langle \Sigma, \Phi\rangle$, defined by

$$\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(\langle \Sigma, \Phi\rangle) = \bigcap\{U \in \mathrm{Fi}^{\mathcal{I}}\mathfrak{A} : \Phi \subseteq U_\Sigma\},$$

where, of course, intersection is taken signature-wise. More generally, given a collection of pairs $X = \{\langle \Sigma_i, \Phi_i\rangle : i \in I\}$, with $\Sigma_i \in |\mathbf{Sign}'|$ and $\Phi_i \subseteq \mathrm{SEN}'(\Sigma_i)$, $i \in I$, denote by $\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(X)$ the **$\mathcal{I}$-filter of $\mathfrak{A}$ generated by** X, defined by

$$\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(X) = \bigcap\{U \in \mathrm{Fi}^{\mathcal{I}}\mathfrak{A} : \Phi_i \subseteq U_{\Sigma_i}, \text{ for all } i \in I\}.$$

It is shown in Proposition 6 of [18] that if $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is an N-structural π-institution, $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, $\mathbf{A} = \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle$ and $\mathfrak{A} = \langle \mathbf{A}, \mathrm{Thm}^{[\langle \Sigma, \Phi\rangle]}\rangle$, then $C_\Sigma(\Phi, \phi) = \mathrm{Fg}^{\mathcal{I},\mathfrak{A}}_\Sigma(\langle \Sigma, \phi\rangle)$, where, by $\mathrm{Thm}^{[\langle \Sigma, \Phi\rangle]}$ is denoted the least theory family of $\mathcal{I}$ including the theorem system Thm of $\mathcal{I}$ and the set of Σ-sentences Φ.

A functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is said to **admit lifting of N-quotients** if, for every N-algebraic system $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, every N'-congruence system θ on SEN' and every N-morphism $\langle F, \alpha\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \to \mathbf{A}/\theta$, there exists an N-morphism $\langle F, \beta\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \to \mathbf{A}$ that makes the following diagram commute:

$$\begin{tikzcd}
& \mathrm{SEN} \arrow[dl, "\langle F, \beta\rangle"'] \arrow[dr, "\langle F, \alpha\rangle"] & \\
\mathrm{SEN}' \arrow[rr, "\langle \mathrm{I}, \pi^\theta\rangle"'] & & \mathrm{SEN}'/\theta
\end{tikzcd}$$

A π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, will be said to **admit lifting of N-quotients** if the functor SEN admits lifting of N-quotients.

Let $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle, \mathbf{B} = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$ be two N-algebraic systems and $\mathfrak{A} = \langle \mathbf{A}, T'\rangle, \mathfrak{B} = \langle \mathbf{B}, T''\rangle$ two $\mathcal{I}$-matrices. An N-morphism $\langle F, \alpha\rangle : \mathbf{A} \to \mathbf{B}$ is said to be an **$\mathcal{I}$-(matrix) morphism** from $\mathfrak{A}$ to $\mathfrak{B}$, written $\langle F, \alpha\rangle : \mathfrak{A} \to \mathfrak{B}$, if, for all $\Sigma \in |\mathbf{Sign}'|$, $\alpha_\Sigma(T'_\Sigma) \subseteq T''_{F(\Sigma)}$ or, equivalently, if $T'_\Sigma \subseteq \alpha^{-1}_\Sigma(T''_{F(\Sigma)})$, for all $\Sigma \in |\mathbf{Sign}'|$, which may also be written as $T' \leq \alpha^{-1}(T'')$.

If $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is a π-institution that admits lifting of N-quotients, $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$ is an N-algebraic system, $\mathfrak{A} = \langle \mathbf{A}, T'\rangle$ an $\mathcal{I}$-matrix and θ an N'-congruence system on SEN', that is compatible with T', then $\mathbf{A}/\theta = \langle \mathrm{SEN}'^\theta, \langle N'^\theta, F'^\theta\rangle\rangle$ is also an N-algebraic system and $\mathfrak{A}/\theta =$

$\langle \mathbf{A}/\theta, T'/\theta\rangle$ is an $\mathcal{I}$-matrix. Moreover, in Proposition 10 of [18], it is shown that, under the same hypotheses, $\langle \mathrm{I}, \pi^\theta\rangle : \mathfrak{A} \to \mathfrak{A}/\theta$ is an $\mathcal{I}$-morphism, such that $T' = {\pi^\theta}^{-1}(T'/\theta)$.

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on SEN. Let $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle, \mathbf{B} = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$ be two N-algebraic systems, $\mathfrak{A} = \langle \mathbf{A}, T'\rangle, \mathfrak{B} = \langle \mathbf{B}, T''\rangle$ two $\mathcal{I}$-matrices and $\langle F, \alpha\rangle : \mathfrak{A} \to \mathfrak{B}$ an $\mathcal{I}$-morphism from $\mathfrak{A}$ to $\mathfrak{B}$. Given an $\mathcal{I}$-filter U of $\mathfrak{A}$, define $\alpha^*(U) = \mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\alpha(U))$. The π-institution $\mathcal{I}$ is said to have the N-**correspondence property** if, for all $\mathcal{I}$-matrices $\mathfrak{A}, \mathfrak{B}$, as above, and every surjective $\mathcal{I}$-morphism $\langle F, \alpha\rangle : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$, with F an isomorphism,

$$\alpha^{-1}(\alpha^*(U)) = U \vee^{\mathbf{Fi}^{\mathcal{I}}(\mathfrak{A})} \alpha^{-1}(T''), \quad \text{for every } U \in \mathrm{Fi}^{\mathcal{I}}(\mathfrak{A}).$$

In this case, α^* induces an isomorphism between the sublattice of $\mathbf{Fi}^{\mathcal{I}}\mathfrak{A}$ with universe $\{U \in \mathrm{Fi}^{\mathcal{I}}\mathfrak{A} : U \geq \alpha^{-1}(T'')\}$ and $\mathbf{Fi}^{\mathcal{I}}\mathfrak{B}$. This is the content of Proposition 13 of [18]. Proposition 14 of [18], on the other hand, asserts that, if $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is an N-structural π-institution, that admits lifting of N-quotients, and has the N-correspondence property, then $\mathcal{I}$ is N-protoalgebraic.

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, be a π-institution. $\mathcal{I}$ is said to have **transferable** N-**rules** if, for every N-rule $\langle\{\sigma^0, \ldots, \sigma^{n-1}\}, \tau\rangle$ of $\mathcal{I}$, the rule $\langle\{\sigma^0, \ldots, \sigma^{n-1}\}, \tau\rangle$ holds in all $\mathcal{I}$-matrices. This means that, for every $\mathcal{I}$-matrix $\mathfrak{A} = \langle \mathbf{A}, T'\rangle$, with $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, all $\Sigma \in |\mathbf{Sign}'|, \vec{\chi} \in \mathrm{SEN}'(\Sigma)^k$,

$$\sigma'^0_\Sigma(\vec{\chi}), \ldots, \sigma'^{n-1}_\Sigma(\vec{\chi}) \subseteq T'_\Sigma \quad \text{imply that} \quad \tau'_\Sigma(\vec{\chi}) \in T'_\Sigma.$$

A collection $E = \{\epsilon^i : i \in I\}$ of natural transformations $\epsilon^i : \mathrm{SEN}^2 \to \mathrm{SEN}$ in N is said to be an N-**implication system for** $\mathcal{I}$ if, for every $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

- $\epsilon^i_\Sigma(\phi, \phi) \in C_\Sigma(\emptyset)$, for all $i \in I$, (E-Reflexivity)
- $\psi \in C_\Sigma(\{\epsilon^i_\Sigma(\phi, \psi) : i \in I\} \cup \{\phi\})$ (E-Modus Ponens)

Usually, these two properties will be abbreviated, respectively, as

$$E_\Sigma(\phi, \phi) \subseteq C_\Sigma(\emptyset), \quad \psi \in C_\Sigma(E_\Sigma(\phi, \psi), \phi). \tag{1}$$

In Proposition 19 of [18], it is shown that, if a π-institution $\mathcal{I}$, with transferable N-rules, that admits lifting of N-quotients, has an N-implication system, then it also has the N-correspondence property.

3 Local Deduction Theorems

Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, with N a category of natural transformations on SEN. Let $E^i = \{\epsilon^{i,j} : \mathrm{SEN}^2 \to \mathrm{SEN} : j < n_i\}, n_i < \omega, i \in I$, where, for all $i \in I, j < n_i$, $\epsilon^{i,j} : \mathrm{SEN}^2 \to \mathrm{SEN}$ is a binary natural transformation in N. The collection

$\mathcal{E} = \{E^i : i \in I\}$ is a **local deduction-detachment system for** $\mathcal{I}$ if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Gamma \cup \{\phi, \psi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\psi \in C_\Sigma(\Gamma, \phi) \quad \text{iff} \quad E^i_\Sigma(\phi, \psi) \subseteq C_\Sigma(\Gamma), \text{ for some } i \in I.$$

In this case $\mathcal{I}$ is said to have the **local deduction-detachment theorem** (LDDT) with respect to the local deduction-detachment system $\mathcal{E}$.

As usual, in the sequel, we use the notation $\mathrm{Mat}(\mathcal{I})$ to denote the collection of all $\mathcal{I}$ matrices. Recall that these are pairs $\mathfrak{A} = \langle \mathbf{A}, T' \rangle$, where $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F' \rangle\rangle$ is an N-algebraic system and T' is an $\mathcal{I}$-filter on $\mathbf{A}$.

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, with N a category of natural transformations on SEN, be a π-institution that admits lifting of N-quotients. This means (see [18]) that, for every N-algebraic system $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F' \rangle\rangle$, every N'-congruence system θ on SEN' and every N-morphism $\langle F, \alpha \rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I} \rangle\rangle \to \mathbf{A}/\theta$, there exists an N-morphism $\langle F, \beta \rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I} \rangle\rangle \to \mathbf{A}$ that makes the following diagram commute:

$$\begin{tikzcd} \mathrm{SEN} \arrow[rr, "\langle F, \alpha \rangle"] \arrow[dr, "\langle F, \beta \rangle"'] & & \mathrm{SEN}'/\theta \\ & \mathrm{SEN}' \arrow[ur, "\langle \mathrm{I}, \pi^\theta \rangle"'] & \end{tikzcd}$$

Let, also, $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F' \rangle\rangle$ be an N-algebraic system and $\mathfrak{A} = \langle \mathbf{A}, T' \rangle \in \mathrm{Mat}(\mathcal{I})$. Recall from [17] that $\Omega^{N'}(T')$ denotes the Leibniz N'-congruence system on SEN', $\mathrm{SEN}'^{\Omega^{N'}(T')}$ denotes the quotient functor $\mathrm{SEN}'/\Omega^{N'}(T')$ and $\langle \mathrm{I}_{\mathbf{Sign}'}, \pi^{\Omega^{N'}(T')} \rangle : \mathrm{SEN}' \to \mathrm{SEN}'^{\Omega^{N'}(T')}$ denotes the $(N', N'^{\Omega^{N'}(T')})$-epimorphic projection.

$$\mathrm{SEN}' \xrightarrow{\langle \mathrm{I}_{\mathbf{Sign}'}, \pi^{\Omega^{N'}(T')} \rangle} \mathrm{SEN}'^{\Omega^{N'}(T')}$$

Since $\mathcal{I}$ was assumed to admit lifting of N-quotients, according to Corollary 11 of [18], the tuple $\mathfrak{A}/\Omega^{N'}(T') = \langle\langle \mathrm{SEN}'^{\Omega^{N'}(T')}, \langle N'^{\Omega^{N'}(T')}, F'^{\Omega^{N'}(T')} \rangle\rangle, T'/\Omega^{N'}(T') \rangle$ is also an $\mathcal{I}$-matrix.

Now, let

$$\begin{array}{rcl} \mathrm{Mat}^N(\mathcal{I}) & = & \{\langle\langle \mathrm{SEN}'^{\Omega^{N'}(T')}, \langle N'^{\Omega^{N'}(T')}, F'^{\Omega^{N'}(T')} \rangle\rangle, T'/\Omega^{N'}(T') \rangle : \\ & & \langle\langle \mathrm{SEN}', \langle N', F' \rangle\rangle, T' \rangle \in \mathrm{Mat}(\mathcal{I})\}, \end{array}$$

i.e., $\mathrm{Mat}^N(\mathcal{I})$ is the class of N'-reduced $\mathcal{I}$-matrices. Of course, this definition generalizes that of a reduced matrix from Abstract Algebraic Logic.

Using Proposition 9 of [18], we obtain the following

Proposition 1 *For every π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, with N a category of natural transformations on* SEN*, that admits lifting of N-quotients,* $\mathrm{Mat}^N\mathcal{I} \subseteq \mathrm{Mat}\mathcal{I}$.

In working with the local deduction-detachment theorem, it will be important to focus attention to a special subclass of the class of all $\mathcal{I}$-matrices for an N-structural π-institution $\mathcal{I}$. The $\mathcal{I}$-matrices in this subclass are defined next.

Definition 2 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be a π-institution and $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$ an N-algebraic system. An $\mathcal{I}$-filter $T' \in \mathrm{Fi}^{\mathcal{I}}(\mathbf{A})$ on $\mathbf{A}$ is said to be* **local** *if it is of the form*

$$T' = \mathrm{Fg}^{\mathcal{I},\mathbf{A}}(\langle \Sigma', \Phi'\rangle) := \bigcap\{T'' \in \mathrm{Fi}^{\mathcal{I}}(\mathbf{A}) : \Phi' \subseteq T''_{\Sigma'}\},$$

for some $\Sigma' \in |\mathbf{Sign}'|$ and $\Phi' \subseteq \mathrm{SEN}'(\Sigma')$. An $\mathcal{I}$-matrix $\mathfrak{A} = \langle \mathbf{A}, T'\rangle$, with $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, will be said to be **local** *if T' is a local $\mathcal{I}$-filter on $\mathbf{A}$.*

Let, in what follows, $\mathrm{Mat}^{is}(\mathcal{I})$ denote the subclass of $\mathrm{Mat}(\mathcal{I})$ consisting of all those $\mathcal{I}$-matrices $\mathfrak{A} = \langle \mathbf{A}, T'\rangle$, such that there exists at least one surjective N-morphism $\langle F, \alpha\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \to \mathbf{A}$, with F an isomorphism. The ^{is} superscript is supposed to remind the reader of the requirement of the existence of an $\langle F, \alpha\rangle$, with F an isomorphism and with $\langle F, \alpha\rangle$ surjective. Similarly, $\mathrm{Mat}^{Nis}(\mathcal{I})$ denotes the corresponding subclass of all N-reduced $\mathcal{I}$-matrices. Note, here, that this class can either be taken to be the class of all reductions of members of $\mathrm{Mat}^{is}(\mathcal{I})$ or the class of all members $\mathfrak{A} = \langle \mathbf{A}, T'\rangle$ of $\mathrm{Mat}^{N}(\mathcal{I})$, such that there exists at least one surjective N-morphism $\langle F, \alpha\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \to \mathbf{A}$, with isomorphic functor component. Furthermore, the notations $\mathrm{Mat}^{lis}(\mathcal{I})$ and $\mathrm{Mat}^{Nlis}(\mathcal{I})$ will be used to denote the subclass of $\mathrm{Mat}^{is}(\mathcal{I})$ consisting of its local members and the one consisting of the reduced counterparts of members of $\mathrm{Mat}^{lis}(\mathcal{I})$, respectively.

Recall from [17] the notation $\mathrm{Thm}^{[\langle \Sigma_0, \Phi_0\rangle]}$, which denotes the least theory family of a π-institution $\mathcal{I}$ that includes its theorem system and the set of Σ_0-sentences Φ_0. It is given, for all $\Sigma \in |\mathbf{Sign}|$, by

$$\mathrm{Thm}^{[\langle \Sigma_0, \Phi_0\rangle]}_{\Sigma} = \begin{cases} C_{\Sigma_0}(\Phi_0), & \text{if } \Sigma = \Sigma_0 \\ C_{\Sigma}(\emptyset), & \text{otherwise} \end{cases}$$

Lemma 3 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be an N-structural π-institution with the N-correspondence property, $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$ an N-algebraic system and $\mathfrak{A} = \langle \mathbf{A}, T'\rangle$ a local $\mathcal{I}$-matrix in $\mathrm{Mat}^{lis}(\mathcal{I})$, with $T' = \mathrm{Fg}^{\mathcal{I},\mathbf{A}}(\langle \Sigma', \Phi'\rangle)$. Then, for every surjective N-morphism $\langle F, \alpha\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \to \mathbf{A}$, with F an isomorphism, there exist $\Sigma \in |\mathbf{Sign}|$ and $\Phi \subseteq \mathrm{SEN}(\Sigma)$, such that $\alpha^{-1}(T') = \mathrm{Thm}^{[\langle \Sigma, \Phi\rangle]}$.*

Proof:

Let $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$ be an N-algebraic system and $\mathfrak{A} = \langle \mathbf{A}, T'\rangle$ a local $\mathcal{I}$-matrix in $\mathrm{Mat}^{lis}(\mathcal{I})$, with $T' = \mathrm{Fg}^{\mathcal{I},\mathbf{A}}(\langle \Sigma', \Phi'\rangle)$, and consider a surjective N-morphism $\langle F, \alpha\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \to \mathbf{A}$, with F an isomorphism. Then, there exist $\Sigma \in |\mathbf{Sign}|$ and $\Phi \subseteq \mathrm{SEN}(\Sigma)$, such that $F(\Sigma) = \Sigma'$ and $\alpha_{\Sigma}(\Phi) = \Phi'$. Thus, it suffices to show that

$$\alpha^{-1}(\mathrm{Fg}^{\mathcal{I},\mathbf{A}}(\langle F(\Sigma), \alpha_{\Sigma}(\Phi)\rangle)) = \mathrm{Thm}^{[\langle \Sigma, \Phi\rangle]}.$$

We do have, indeed,

$$\begin{array}{rcl}\alpha^{-1}(\mathrm{Fg}^{\mathcal{I},\mathbf{A}}(\langle F(\Sigma),\alpha_\Sigma(\Phi)\rangle)) & = & \alpha^{-1}(\bigcap\{T''\in \mathrm{Fi}^{\mathcal{I}}(\mathbf{A}):\alpha_\Sigma(\Phi)\subseteq T''_{F(\Sigma)}\})\\ & & \text{(by the definition of } \mathrm{Fg}^{\mathcal{I},\mathbf{A}}(\langle F(\Sigma),\alpha_\Sigma(\Phi)\rangle))\\ & = & \bigcap\{\alpha^{-1}(T''):T''\in \mathrm{Fi}^{\mathcal{I}}(\mathbf{A}),\alpha_\Sigma(\Phi)\subseteq T''_{F(\Sigma)}\}\\ & = & \bigcap\{\alpha^{-1}(T''):T''\in \mathrm{Fi}^{\mathcal{I}}(\mathbf{A}),\Phi\subseteq \alpha_\Sigma^{-1}(T''_{F(\Sigma)})\}\\ & = & \bigcap\{T\in \mathrm{ThFam}(\mathcal{I}):\Phi\subseteq T_\Sigma\}\\ & & \text{(by Propositions 2,3,13 of [18], } N\text{-structurality}\\ & & \text{and the } N\text{-correspondence property)}\\ & = & \mathrm{Thm}^{[\langle\Sigma,\Phi\rangle]}\\ & & \text{(by the definition of } \mathrm{Thm}^{[\langle\Sigma,\Phi\rangle]}).\end{array}$$

■

Let $\mathbf{A}^k=\langle \mathrm{SEN}^k,\langle N^k,F^k\rangle\rangle$, $k\in K$, be a collection of N-algebraic systems and $\mathfrak{A}^k=\langle \mathbf{A}^k,T^k\rangle$, $k\in K$, a collection of local $\mathcal{I}$-matrices. Set $M:=\{\mathfrak{A}^k:k\in K\}$. The collection M is said to have **locally N-definable principal filters** (LDPF) with **defining system** $\mathcal{E}$ if, for all $k\in K$, such that $T^k=\mathrm{Fg}^{\mathcal{I},\mathbf{A}^k}(\langle\Sigma,\Phi\rangle)$, and all $\phi,\psi\in \mathrm{SEN}^k(\Sigma)$,

$$\psi\in \mathrm{Fg}^{\mathcal{I},\mathfrak{A}^k}_\Sigma(\langle\Sigma,\phi\rangle)\quad\text{iff}\quad {E^i}^k_\Sigma(\phi,\psi)\subseteq T^k_\Sigma,\ \text{for some } i\in I.$$

It is shown next that, if an N-structural π-institution $\mathcal{I}$, that admits lifting of N-quotients and has the N-correspondence property, has the LDDT, then the class $\mathrm{Mat}^{lis}(\mathcal{I})$ has the LDPF. This result extends the implication $(i)\Rightarrow(ii)$ of Theorem 2.4 of [2].

Lemma 4 *Let $\mathcal{I}=\langle \mathbf{Sign},\mathrm{SEN},C\rangle$, with N a category of natural transformations on* SEN*, be an N-structural π-institution that admits lifting of N-quotients and has the N-correspondence property. If $\mathcal{I}$ has the* LDDT *with respect to the local deduction-detachment system $\mathcal{E}$, then* $\mathrm{Mat}^{lis}(\mathcal{I})$ *has* LDPF *with defining system $\mathcal{E}$.*

Proof:
Suppose that $\mathcal{I}$ has the local deduction-detachment theorem with respect to $\mathcal{E}$. Let $\mathbf{A}=\langle \mathrm{SEN}',\langle N',F'\rangle\rangle$ be an N-algebraic system and $\mathfrak{A}=\langle\mathbf{A},T'\rangle$, with $T'=\mathrm{Fg}^{\mathcal{I},\mathbf{A}}(\langle\Sigma',\Phi'\rangle)$, an $\mathcal{I}$-matrix in $\mathrm{Mat}^{lis}(\mathcal{I})$. Let, also, $\phi',\psi'\in \mathrm{SEN}'(\Sigma')$. We must show that $\psi'\in \mathrm{Fg}^{\mathcal{I},\mathfrak{A}}_{\Sigma'}(\langle\Sigma',\phi'\rangle)$ if and only if ${E^i}'_{\Sigma'}(\phi',\psi')\subseteq T'_{\Sigma'}$, for some $i\in I$.

Since $\mathfrak{A}\in \mathrm{Mat}^{lis}(\mathcal{I})$, there exist a surjective $\langle F,\alpha\rangle:\langle \mathrm{SEN},\langle N,\mathrm{I}\rangle\rangle\to\mathbf{A}$, with F an isomorphism, $\Sigma\in|\mathbf{Sign}|$ and $\Phi\cup\{\phi,\psi\}\subseteq \mathrm{SEN}(\Sigma)$, such that $F(\Sigma)=\Sigma'$, $\alpha_\Sigma(\phi)=\phi'$, $\alpha_\Sigma(\psi)=\psi'$ and $\alpha_\Sigma(\Phi)=\Phi'$. Moreover, by Lemma 3, $T:=\alpha^{-1}(T')=\mathrm{Thm}^{[\langle\Sigma,\Phi\rangle]}$. Therefore, by Proposition 3 of [18], since T is a theory family of $\mathcal{I}$, $\mathfrak{B}=\langle\langle \mathrm{SEN},\langle N,\mathrm{I}\rangle\rangle,T\rangle$ is an $\mathcal{I}$-matrix and, by Proposition 6 of [18], $\mathrm{Fg}^{\mathcal{I},\mathfrak{B}}_\Sigma(\langle\Sigma,\phi\rangle)=C_\Sigma(T_\Sigma,\phi)$. Since $\langle F,\alpha\rangle$ is a surjective $\mathcal{I}$-morphism, we have that

$$\begin{array}{lll}\psi' \in \mathrm{Fg}^{\mathcal{I},\mathfrak{A}}_{\Sigma'}(\langle\Sigma',\phi'\rangle) & \text{iff} & \alpha_\Sigma(\psi) \in \mathrm{Fg}^{\mathcal{I},\mathfrak{A}}_{F(\Sigma)}(\langle F(\Sigma),\alpha_\Sigma(\phi)\rangle) \quad \text{(by hypothesis)} \\ & \text{iff} & \psi \in \mathrm{Fg}^{\mathcal{I},\mathfrak{B}}_{\Sigma}(\langle\Sigma,\phi\rangle) \quad \text{(by Lemma 16 of [18])} \\ & \text{iff} & \psi \in C_\Sigma(T_\Sigma,\phi) \quad \text{(by Proposition 6 of [18])} \\ & \text{iff} & E^i_\Sigma(\phi,\psi) \subseteq C_\Sigma(T_\Sigma) = T_\Sigma = \alpha_\Sigma^{-1}(T'_{F(\Sigma)}),\ \text{for some } i \in I, \\ & & \text{(by hypothesis)} \\ & \text{iff} & {E^{i}}'_{F(\Sigma)}(\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)) \subseteq T'_{F(\Sigma)},\ \text{for some } i \in I, \\ & \text{iff} & {E^{i}}'_{\Sigma'}(\phi',\psi') \subseteq T'_{\Sigma'},\ \text{for some } i \in I.\end{array}$$

■

Lemma 5, on the other hand, asserts that, given an N-structural π-institution $\mathcal{I}$, that admits lifting of N-quotients and has the N-correspondence property, if the class $\mathrm{Mat}^{Nlis}(\mathcal{I})$ has the LDPF with defining system $\mathcal{E}$, then $\mathcal{I}$ has the LDDT with local deduction-detachment system $\mathcal{E}$. This result is the analog of the implication $(iii) \Rightarrow (i)$ of Theorem 2.4 of [2].

Lemma 5 *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$*, with* N *a category of natural transformations on* SEN*, be an* N*-structural* π*-institution that admits lifting of* N*-quotients and has the* N*-correspondence property. If* $\mathrm{Mat}^{Nlis}(\mathcal{I})$ *has* LDPF *with defining system* $\mathcal{E}$*, then* $\mathcal{I}$ *has the* LDDT *with respect to the local deduction-detachment system* $\mathcal{E}$*.*

Proof:

Suppose that $\mathrm{Mat}^{Nlis}(\mathcal{I})$ has locally N-definable principal filters with defining system $\mathcal{E}$. Let $\Sigma \in |\mathbf{Sign}|$ and $\Gamma \cup \{\phi,\psi\} \subseteq \mathrm{SEN}(\Sigma)$. It must be shown that $\psi \in C_\Sigma(\Gamma,\phi)$ if and only if $E^i_\Sigma(\phi,\psi) \subseteq C_\Sigma(\Gamma)$, for some $i \in I$.

To this end, let $T = \mathrm{Thm}^{[\langle\Sigma,\Gamma\rangle]}$ be the theory family on SEN generated by the collection Γ of Σ-sentences. Then, by Proposition 3 of [18] and N-structurality, T is a local $\mathcal{I}$-filter on $\langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle$ and, hence, $\mathfrak{A} = \langle\langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle, T\rangle$ is a local $\mathcal{I}$-matrix. Thus, by taking into account lifting of N-quotients, Proposition 9 of [18] yields that $\mathfrak{A}/\Omega^N(T)$ is an N-reduced local $\mathcal{I}$-matrix and $\langle \mathrm{I}, \pi^{\Omega^N(T)}\rangle : \mathfrak{A} \to \mathfrak{A}/\Omega^N(T)$ is a surjective $\mathcal{I}$-morphism that satisfies ${\pi^{\Omega^N(T)}}^{-1}(T/\Omega^N(T)) = T$. Thus we get that

$$\begin{array}{lll}\psi \in C_\Sigma(\Gamma,\phi) & \text{iff} & \psi \in \mathrm{Fg}^{\mathcal{I},\mathfrak{A}}_\Sigma(\langle\Sigma,\phi\rangle) \\ & & \text{(by Proposition 6 of [18])} \\ & \text{iff} & \psi/\Omega^N_\Sigma(T) \in \mathrm{Fg}^{\mathcal{I},\mathfrak{A}/\Omega^N(T)}_\Sigma(\langle\Sigma,\phi/\Omega^N_\Sigma(T)\rangle) \\ & & \text{(by Lemma 16 of [18])} \\ & \text{iff} & {E^{i}}^{\Omega^N(T)}_\Sigma(\phi/\Omega^N_\Sigma(T),\psi/\Omega^N_\Sigma(T)) \subseteq T_\Sigma/\Omega^N_\Sigma(T),\ \text{for some } i \in I, \\ & & \text{(by the hypothesis)} \\ & \text{iff} & E^i_\Sigma(\phi,\psi) \subseteq {\pi^{\Omega^N(T)}_\Sigma}^{-1}(T_\Sigma/\Omega^N_\Sigma(T)) = T_\Sigma,\ \text{for some } i \in I.\end{array}$$

■

Theorem 6 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be an N-structural π-institution admitting lifting of N-quotients and having the N-correspondence property. Furthermore, let $\mathcal{E}$ be a system of finite sets of binary natural transformations in N. The following statements are equivalent:*

1. *$\mathcal{I}$ has the* LDDT *with respect to $\mathcal{E}$.*
2. $\mathrm{Mat}^{lis}(\mathcal{I})$ *has the* LDPF *with defining system $\mathcal{E}$.*
3. $\mathrm{Mat}^{Nlis}(\mathcal{I})$ *has the* LDPF *with defining system $\mathcal{E}$.*

Proof:

The implication $1 \to 2$ is the content of Lemma 4. The implication $2 \to 3$ is trivial. The implication $3 \to 1$ is the content of Lemma 5. ■

4 The Filter Extension Property

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on SEN. Suppose that $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle, \mathbf{B} = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$ are two N-algebraic systems and $\mathfrak{A} = \langle \mathbf{A}, T'\rangle \in \mathrm{Mat}(\mathcal{I})$. A pair $\mathfrak{B} = \langle \mathbf{B}, T''\rangle$ is said to be a **submatrix** of $\mathfrak{A}$, written $\mathfrak{B} \leq \mathfrak{A}$, if

- $\mathbf{B}$ is a simple N-algebraic subsystem of $\mathbf{A}$, i.e., an N-algebraic subsystem over the same signature category $\mathbf{Sign}'$, and
- $T'' = T' \cap \mathrm{SEN}''$, i.e., $T''_\Sigma = T'_\Sigma \cap \mathrm{SEN}''(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}'|$.

Note that, if $\mathfrak{A} \in \mathrm{Mat}(\mathcal{I})$ and $\mathfrak{B} \leq \mathfrak{A}$, then $\mathfrak{B} \in \mathrm{Mat}(\mathcal{I})$, as well. This is the content of the following proposition.

Proposition 7 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be a π-institution. Suppose that $\mathfrak{A} = \langle\langle \mathrm{SEN}', \langle N', F'\rangle\rangle, T'\rangle \in \mathrm{Mat}(\mathcal{I})$ and that $\mathfrak{B} = \langle\langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle, T''\rangle \leq \mathfrak{A}$. Then $\mathfrak{B} \in \mathrm{Mat}(\mathcal{I})$.*

Proof:

Let $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$ and $\mathbf{B} = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$ be the underlying N-algebraic systems of $\mathfrak{A}$ and $\mathfrak{B}$, respectively. To prove the conclusion of the proposition, suppose that $\langle F, \alpha\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \to \mathbf{B}$ is an N-morphism, $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, with $\phi \in C_\Sigma(\Phi)$, and $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$, such that $\alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq T''_{F(\Sigma')}$. Therefore, since $T''_{F(\Sigma')} = T'_{F(\Sigma')} \cap \mathrm{SEN}''(F(\Sigma'))$, we obtain that $\alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq T'_{F(\Sigma')}$. Now, notice that, by the hypothesis, $\mathfrak{A} \in \mathrm{Mat}(\mathcal{I})$ and, also, that $\langle F, \alpha\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \to \mathbf{B}$ may be viewed as an N-morphism $\langle F, \alpha\rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I}\rangle\rangle \to \mathbf{A}$. Hence we obtain that $\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \in T'_{F(\Sigma')}$. Thus, since, in addition, $\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \in \mathrm{SEN}''(F(\Sigma'))$ and $T'_{F(\Sigma')} \cap \mathrm{SEN}''(F(\Sigma')) = T''_{F(\Sigma')}$, we finally get that $\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) \in T''_{F(\Sigma')}$, showing that $\mathfrak{B}$ is indeed an $\mathcal{I}$-matrix. ■

A local $\mathcal{I}$-matrix $\mathfrak{A} = \langle \mathbf{A}, T' \rangle$, with $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F' \rangle \rangle$ and $T' = \mathrm{Fg}^{\mathcal{I},\mathbf{A}}(\langle \Sigma', \Phi' \rangle)$, has the **principal filter extension property (PFEP)** if, for every local submatrix $\mathfrak{B} = \langle \mathbf{B}, T'' \rangle$, with $\mathbf{B} = \langle \mathrm{SEN}'', \langle N'', F'' \rangle \rangle$ and $T'' = \mathrm{Fg}^{\mathcal{I},\mathbf{B}}(\langle \Sigma', \Phi'' \rangle)$, of $\mathfrak{A}$ and every principal $\mathcal{I}$-filter $V'' = \mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\langle \Sigma', \psi'' \rangle) \in \mathrm{Fi}^{\mathcal{I}}(\mathfrak{B})$, there exists an $\mathcal{I}$-filter $V' \in \mathrm{Fi}^{\mathcal{I}}(\mathfrak{A})$, such that $V''_{\Sigma'} = V'_{\Sigma'} \cap \mathrm{SEN}''(\Sigma')$.

A class M of matrices of $\mathcal{I}$ is said to **have the PFEP** if every member of M has the PFEP.

It is shown, next, that the property of having the LDDT implies the PFEP for the class $\mathrm{Mat}^{lis}(\mathcal{I})$. To show this, we must, however, further restrict attention to those π-institutions $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, with N a category of natural transformations on SEN, that are such that the class of all N-algebraic systems $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F' \rangle \rangle$, for which there exists at least one surjective N-morphism $\langle F, \alpha \rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I} \rangle \rangle \to \mathbf{A}$, with an isomorphic functor component, is closed under simple N-algebraic subsystems. We introduce the following:

Definition 8 *Let* SEN : $\mathbf{Sign} \to \mathbf{Set}$ *be a functor and N a category of natural transformations on* SEN. SEN *is called* **downward N-closed** *if, every simple N-algebraic subsystem of an N-algebraic system $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F' \rangle \rangle$, for which there exists at least one surjective N-morphism $\langle F, \alpha \rangle : \langle \mathrm{SEN}, \langle N, \mathrm{I} \rangle \rangle \to \mathbf{A}$, with an isomorphic functor component, is also an N-algebraic system satisfying the same property.*

A π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, with N a category of natural transformations on SEN*, will be said to be* **downward N-closed** *if the functor* SEN *is downward N-closed.*

The following lemma, showing, roughly speaking, that the property of $\mathcal{I}$ having the LDDT implies the PFEP for the class $\mathrm{Mat}^{lis}(\mathcal{I})$, is an analog for π-institutions of the implication $(i) \Rightarrow (ii)$ of Theorem 3.1 of [2].

Lemma 9 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, with N a category of natural transformations on* SEN*, be an N-structural, downward N-closed π-institution, that admits lifting of N-quotients, has the N-correspondence property and the* LDDT *with local deduction-detachment system $\mathcal{E}$. Then, every $\mathfrak{A} = \langle \langle \mathrm{SEN}', \langle N', F' \rangle \rangle, T' \rangle \in \mathrm{Mat}^{lis}(\mathcal{I})$ has the* PFEP.

Proof:

Suppose that $\mathcal{I}$ has the LDDT with local deduction-detachment system $\mathcal{E} = \{E^i : i \in I\}$ and let $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F' \rangle \rangle$ be an N-algebraic system and $\mathfrak{A} = \langle \mathbf{A}, T' \rangle \in \mathrm{Mat}^{lis}(\mathcal{I})$, with $T' = \mathrm{Fg}^{\mathcal{I},\mathbf{A}}(\langle \Sigma', \Phi' \rangle)$. We need to show that $\mathfrak{A}$ has the PFEP. To do this, suppose that $\mathfrak{B} = \langle \mathbf{B}, T'' \rangle$, with $\mathbf{B} = \langle \mathrm{SEN}'', \langle N'', F'' \rangle \rangle$ and $T'' = \mathrm{Fg}^{\mathcal{I},\mathbf{B}}(\langle \Sigma', \Phi'' \rangle)$, is a local submatrix of $\mathfrak{A}$, and $V'' = \mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\langle \Sigma', \phi'' \rangle) \in \mathrm{Fi}^{\mathcal{I}}(\mathfrak{B})$ is a principal $\mathcal{I}$-filter of $\mathfrak{B}$. To see that there exists $V' \in \mathrm{Fi}^{\mathcal{I}}(\mathfrak{A})$, such that $V''_{\Sigma'} = V'_{\Sigma'} \cap \mathrm{SEN}''(\Sigma')$, we show, first, that

$$\mathrm{Fg}^{\mathcal{I},\mathfrak{B}}_{\Sigma'}(\langle \Sigma', \phi'' \rangle) = \mathrm{Fg}^{\mathcal{I},\mathfrak{A}}_{\Sigma'}(\langle \Sigma', \phi'' \rangle) \cap \mathrm{SEN}''(\Sigma'). \tag{2}$$

To see this, let $\psi'' \in \mathrm{SEN}''(\Sigma')$. Since, by Theorem 6, $\mathrm{Mat}^{lis}(\mathcal{I})$ has the LDPF, we have, using the downward N-closedness of $\mathcal{I}$, that $\psi'' \in \mathrm{Fg}^{\mathcal{I},\mathfrak{B}}_{\Sigma'}(\langle \Sigma', \phi'' \rangle)$ if and only if, there exists

$i \in I$, such that $E^{i''}_{\Sigma'}(\phi'', \psi'') \subseteq T''_{\Sigma'}$. Therefore, since $E^{i''}_{\Sigma'}(\phi'', \psi'') = E^{i'}_{\Sigma'}(\phi'', \psi'')$ and $T''_{\Sigma'} = T'_{\Sigma'} \cap \mathrm{SEN}''(\Sigma')$, we get that $E^{i''}_{\Sigma'}(\phi'', \psi'') \subseteq T''_{\Sigma'}$ if and only if $E^{i'}_{\Sigma'}(\phi'', \psi'') \subseteq T'_{\Sigma'}$. But, by the LDPF property for $\mathrm{Mat}^{lis}(\mathcal{I})$, we also have that $\psi'' \in \mathrm{Fg}^{\mathcal{I},\mathfrak{A}}_{\Sigma'}(\langle \Sigma', \phi'' \rangle)$ if and only if $E^{i'}_{\Sigma'}(\phi'', \psi'') \subseteq T'_{\Sigma'}$, for some $i \in I$. Taken together these three statements yield that

$$\psi'' \in \mathrm{Fg}^{\mathcal{I},\mathfrak{B}}_{\Sigma'}(\langle \Sigma', \phi'' \rangle) \quad \text{iff} \quad \psi'' \in \mathrm{Fg}^{\mathcal{I},\mathfrak{A}}_{\Sigma'}(\langle \Sigma', \phi'' \rangle).$$

This finishes the proof of Equation (2). We use Equation (2) now to finish the proof of the lemma. We consider $V'' = \mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\langle \Sigma', \phi'' \rangle) \in \mathrm{Fi}^{\mathcal{I}}(\mathfrak{B})$ and take $V' = \mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(\langle \Sigma', \phi'' \rangle)$. Then, by Equation (2), we readily get that $V''_{\Sigma'} = V'_{\Sigma'} \cap \mathrm{SEN}''(\Sigma')$, whence $\mathrm{Mat}^{lis}(\mathcal{I})$ has in fact the PFEP. ∎

A local $\mathcal{I}$-matrix $\mathfrak{A} = \langle \mathbf{A}, T' \rangle$, with $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F' \rangle \rangle$ and $T' = \mathrm{Fg}^{\mathcal{I},\mathbf{A}}(\langle \Sigma', \Phi' \rangle)$, has the **local filter extension property (LFEP)** if, for every local submatrix $\mathfrak{B} = \langle \mathbf{B}, T'' \rangle$, with $\mathbf{B} = \langle \mathrm{SEN}'', \langle N'', F'' \rangle \rangle$ and $T'' = \mathrm{Fg}^{\mathcal{I},\mathbf{B}}(\langle \Sigma', \Phi'' \rangle)$, of $\mathfrak{A}$ and every local $\mathcal{I}$-filter $V'' = \mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\langle \Sigma', \Psi'' \rangle) \in \mathrm{Fi}^{\mathcal{I}}(\mathfrak{B})$, there exists an $\mathcal{I}$-filter $V' \in \mathrm{Fi}^{\mathcal{I}}(\mathfrak{A})$, such that $V''_{\Sigma'} = V'_{\Sigma'} \cap \mathrm{SEN}''(\Sigma')$.

A class M of matrices of $\mathcal{I}$ is said to **have the LFEP** if every member of M has the LFEP.

It is shown, next, that the local filter extension property is equivalent to the principal filter extension property for finitary, downward N-closed π-institutions.

Theorem 10 *For a finitary, downward N-closed π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, where N is a category of natural transformations on* SEN, $\mathrm{Mat}^{lis}(\mathcal{I})$ *has the* PFEP *iff it has the* LFEP*.*

Proof:

Since every principal $\mathcal{I}$-filter is also a local $\mathcal{I}$-filter, it is obvious that, if $\mathrm{Mat}^{lis}(\mathcal{I})$ has the LFEP, then it has the PFEP.

So it suffices to show that if $\mathrm{Mat}^{lis}(\mathcal{I})$ has the PFEP, then it has the LFEP. In fact, it is enough to show that, if $\mathfrak{A} = \langle \mathbf{A}, T' \rangle$, with $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F' \rangle \rangle$ and $T' = \mathrm{Fg}^{\mathcal{I},\mathbf{A}}(\langle \Sigma', \Phi' \rangle)$, and $\mathfrak{B} = \langle \mathbf{B}, T'' \rangle$, with $\mathbf{B} = \langle \mathrm{SEN}'', \langle N'', F'' \rangle \rangle$ and $T'' = \mathrm{Fg}^{\mathcal{I},\mathbf{B}}(\langle \Sigma', \Phi'' \rangle)$, are two local $\mathcal{I}$-matrices, such that $\mathfrak{B} \leq \mathfrak{A}$, then, for every finite $\Gamma'' \subseteq \mathrm{SEN}''(\Sigma')$, we have that

$$\mathrm{Fg}^{\mathcal{I},\mathfrak{B}}_{\Sigma'}(\langle \Sigma', \Gamma'' \rangle) = \mathrm{Fg}^{\mathcal{I},\mathfrak{A}}_{\Sigma'}(\langle \Sigma', \Gamma'' \rangle) \cap \mathrm{SEN}''(\Sigma'). \tag{3}$$

Indeed, if this equation holds, then, for $V'' = \mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\langle \Sigma', \Psi'' \rangle) \in \mathrm{Fi}^{\mathcal{I}}(\mathfrak{B})$ a local $\mathcal{I}$-filter of $\mathfrak{B}$, we have

$$\begin{array}{rcl}
V''_{\Sigma'} & = & \mathrm{Fg}^{\mathcal{I},\mathfrak{B}}_{\Sigma'}(\langle \Sigma', \Psi'' \rangle) \\
& = & \bigcup\{\mathrm{Fg}^{\mathcal{I},\mathfrak{B}}_{\Sigma'}(\langle \Sigma', \Gamma'' \rangle) : \Gamma'' \subseteq_\omega \Psi''\} \quad (\text{since } \mathcal{I} \text{ is finitary}) \\
& = & \bigcup\{\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}_{\Sigma'}(\langle \Sigma', \Gamma'' \rangle) \cap \mathrm{SEN}''(\Sigma') : \Gamma'' \subseteq_\omega \Psi''\} \\
& = & \bigcup\{\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}_{\Sigma'}(\langle \Sigma', \Gamma'' \rangle) : \Gamma'' \subseteq_\omega \Psi''\} \cap \mathrm{SEN}''(\Sigma') \\
& = & V'_{\Sigma'} \cap \mathrm{SEN}''(\Sigma'),
\end{array}$$

where $V' := \bigcup\{\mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(\langle\Sigma',\Gamma''\rangle) : \Gamma'' \subseteq_\omega \Psi''\}$ is also an $\mathcal{I}$-filter on $\mathfrak{A}$.

We now turn to the proof of Equation (3), which will be carried out by induction on the cardinality of $\Gamma'' \subseteq_\omega \mathrm{SEN}''(\Sigma')$. If $\Gamma'' = \emptyset$, then $\mathrm{Fg}^{\mathcal{I},\mathfrak{B}}_{\Sigma'}(\langle\Sigma',\emptyset\rangle) = T'' = T' \cap \mathrm{SEN}''(\Sigma')$ follows by the fact that $\mathfrak{B} \leq \mathfrak{A}$. Assume, next, that Equation (3) holds whenever $|\Gamma''| \leq n$ and let $\Gamma'' = \{\gamma_0,\ldots,\gamma_n\} \subseteq \mathrm{SEN}''(\Sigma')$. Set $X'' = \mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\langle\Sigma',\Gamma''\backslash\{\gamma_n\}\rangle)$. Then, by the induction hypothesis, we get that, if $X' = \mathrm{Fg}^{\mathcal{I},\mathfrak{A}}(\langle\Sigma',\Gamma''\backslash\{\gamma_n\}\rangle)$, then $X''_{\Sigma'} = X'_{\Sigma'} \cap \mathrm{SEN}''(\Sigma')$. Thus, we also get that $\mathfrak{B}' := \langle\langle\mathrm{SEN}'',\langle N'',F''\rangle\rangle,X''\rangle \leq \langle\langle\mathrm{SEN}',\langle N',F'\rangle\rangle,X'\rangle =: \mathfrak{A}'$. By hypothesis, there exists $W' \in \mathrm{Fi}^{\mathcal{I}}(\mathfrak{A}')$, such that $\mathrm{Fg}^{\mathcal{I},\mathfrak{B}'}_{\Sigma'}(\langle\Sigma',\gamma_n\rangle) = W'_{\Sigma'} \cap \mathrm{SEN}''(\Sigma')$. Since $\mathrm{Fi}^{\mathcal{I}}(\mathfrak{A}') \subseteq \mathrm{Fi}^{\mathcal{I}}(\mathfrak{A})$, we get that $W' \in \mathrm{Fi}^{\mathcal{I}}(\mathfrak{A})$ and $\mathrm{Fg}^{\mathcal{I},\mathfrak{B}'}_{\Sigma'}(\langle\Sigma',\gamma_n\rangle) = \mathrm{Fg}^{\mathcal{I},\mathfrak{B}}_{\Sigma'}(\langle\Sigma',\Gamma''\backslash\{\gamma_n\} \cup \{\gamma_n\}\rangle) = \mathrm{Fg}^{\mathcal{I},\mathfrak{B}}_{\Sigma'}(\langle\Sigma',\Gamma''\rangle)$, whence $W'_{\Sigma'} \cap \mathrm{SEN}''(\Sigma') = \mathrm{Fg}^{\mathcal{I},\mathfrak{B}}_{\Sigma'}(\langle\Sigma',\Gamma''\rangle)$. ■

Let $\mathcal{I} = \langle\mathbf{Sign},\mathrm{SEN},C\rangle$, with N a category of natural transformations on SEN, be a π-institution and $\mathbf{A} = \langle\mathrm{SEN}',\langle N',F'\rangle\rangle, \mathbf{B} = \langle\mathrm{SEN}'',\langle N'',F''\rangle\rangle$ be N-algebraic systems over the same signature category $\mathbf{Sign}'$, with $\mathbf{B} \leq \mathbf{A}$. Suppose that $\theta = \{\theta_\Sigma\}_{\Sigma\in|\mathbf{Sign}'|} \in \mathrm{Con}^{N'}(\mathrm{SEN}')$. Define, for every $\Sigma \in |\mathbf{Sign}|$, $\mathrm{SEN}''^{\theta}(\Sigma) = \mathrm{SEN}''(\Sigma)/\theta_\Sigma$ and, for all $\Sigma,\Sigma' \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma,\Sigma')$, $\mathrm{SEN}''^{\theta}(f) = \mathrm{SEN}'^{\theta}(f) \restriction_{\mathrm{SEN}''^{\theta}(\Sigma)}$. This definition makes sense because, if $\phi''/\theta_\Sigma \in \mathrm{SEN}''(\Sigma)/\theta_\Sigma$, then, there exists $\psi'' \in \mathrm{SEN}''(\Sigma)$, with $\langle\phi'',\psi''\rangle \in \theta_\Sigma$. Therefore,

$$\begin{array}{rcl}\mathrm{SEN}''^{\theta}(f)(\phi''/\theta_\Sigma) & = & \mathrm{SEN}''^{\theta}(f)(\psi''/\theta_\Sigma)\\ & = & \mathrm{SEN}'(f)(\psi'')/\theta_{\Sigma'}\\ & = & \mathrm{SEN}''(f)(\psi'')/\theta_{\Sigma'}\\ & \in & \mathrm{SEN}''(\Sigma')/\theta_{\Sigma'}.\end{array}$$

Defined, thus, on objects and morphisms, $\mathrm{SEN}''^{\theta} : \mathbf{Sign} \to \mathbf{Set}$ is a functor.

Next, given $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, define $\sigma''^{\theta} : \mathrm{SEN}''^{\theta^k} \to \mathrm{SEN}''^{\theta}$ by setting, for all $\Sigma \in |\mathbf{Sign}|$ and $\phi''_0,\ldots,\phi''_{k-1} \in \mathrm{SEN}''(\Sigma)$,

$$\sigma''^{\theta}_\Sigma(\phi''_0/\theta_\Sigma,\ldots,\phi''_{k-1}/\theta_\Sigma) := \sigma'^{\theta}_\Sigma(\phi''_0/\theta_\Sigma,\ldots,\phi''_{k-1}/\theta_\Sigma) = \sigma'_\Sigma(\phi''_0,\ldots,\phi''_{k-1})/\theta_\Sigma.$$

This is also a well-defined mapping, since, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi''_0/\theta_\Sigma,\ldots,\phi''_{k-1}/\theta_\Sigma \in \mathrm{SEN}''^{\theta}(\Sigma)$, there exist $\psi''_0,\ldots,\psi''_{k-1} \in \mathrm{SEN}''(\Sigma)$, such that $\langle\phi''_i,\psi''_i\rangle \in \theta_\Sigma, i < k$. Therefore,

$$\begin{array}{rcl}\sigma''^{\theta}_\Sigma(\phi''_0/\theta_\Sigma,\ldots,\phi''_{k-1}/\theta_\Sigma) & = & \sigma'^{\theta}_\Sigma(\phi''_0/\theta_\Sigma,\ldots,\phi''_{k-1}/\theta_\Sigma)\\ & = & \sigma'^{\theta}_\Sigma(\psi''_0/\theta_\Sigma,\ldots,\psi''_{k-1}/\theta_\Sigma)\\ & = & \sigma'_\Sigma(\psi''_0,\ldots,\psi''_{k-1})/\theta_\Sigma\\ & = & \sigma''_\Sigma(\psi''_0,\ldots,\psi''_{k-1})/\theta_\Sigma\\ & \in & \mathrm{SEN}''(\Sigma)/\theta_\Sigma.\end{array}$$

Lemma 11 *Let $\mathcal{I} = \langle\mathbf{Sign},\mathrm{SEN},C\rangle$, with N a category of natural transformations on* SEN*, be a π-institution and $\mathbf{A} = \langle\mathrm{SEN}',\langle N',F'\rangle\rangle, \mathbf{B} = \langle\mathrm{SEN}'',\langle N'',F''\rangle\rangle$ be N-algebraic systems over the same signature category $\mathbf{Sign}'$, with $\mathbf{B} \leq \mathbf{A}$. Assume $\theta = \{\theta_\Sigma\}_{\Sigma\in|\mathbf{Sign}'|} \in \mathrm{Con}^{N'}(\mathrm{SEN}')$. Then $\mathbf{B}^\theta = \langle\mathrm{SEN}''^{\theta},\langle N''^{\theta},F''^{\theta}\rangle\rangle$ is also an N-algebraic system.*

Proof:

The proof follows easily from the definitions and the comments preceding the statement of the lemma. For instance, by the fact that $\sigma'^{\theta} : \mathrm{SEN}'^{\theta^k} \to \mathrm{SEN}'^{\theta}$ is a natural transformation, i.e., by the commutativity of the rectangle

$$\begin{tikzcd}
\mathrm{SEN}'^{\theta}(\Sigma)^k \arrow[r, "\sigma'^{\theta}_{\Sigma}"] \arrow[d, "\mathrm{SEN}'^{\theta}(f)^k"'] & \mathrm{SEN}'^{\theta}(\Sigma) \arrow[d, "\mathrm{SEN}'^{\theta}(f)"] \\
\mathrm{SEN}'^{\theta}(\Sigma')^k \arrow[r, "\sigma'^{\theta}_{\Sigma'}"'] & \mathrm{SEN}'^{\theta}(\Sigma')
\end{tikzcd}$$

and the definitions involved, it follows directly that the following rectangle commutes

$$\begin{tikzcd}
\mathrm{SEN}''^{\theta}(\Sigma)^k \arrow[r, "\sigma''^{\theta}_{\Sigma}"] \arrow[d, "\mathrm{SEN}''^{\theta}(f)^k"'] & \mathrm{SEN}''^{\theta}(\Sigma) \arrow[d, "\mathrm{SEN}''^{\theta}(f)"] \\
\mathrm{SEN}''^{\theta}(\Sigma')^k \arrow[r, "\sigma''^{\theta}_{\Sigma'}"'] & \mathrm{SEN}''^{\theta}(\Sigma')
\end{tikzcd}$$

i.e., that $\sigma''^{\theta} : \mathrm{SEN}''^{\theta^k} \to \mathrm{SEN}''^{\theta}$ is also a natural transformation. ■

Based on Lemma 11, we may now show that given two $\mathcal{I}$-matrices $\mathfrak{A}, \mathfrak{B}$, with $\mathfrak{B} \leq \mathfrak{A}$, and a congruence system θ on the underlying N-algebraic system of $\mathfrak{A}$, that is included in the Leibniz congruence system of $\mathfrak{A}$, $\mathfrak{B}^{\theta}$ is a submatrix of the $\mathcal{I}$-matrix $\mathfrak{A}^{\theta}$. Then, as a consequence of Proposition 7, we also obtain that $\mathfrak{B}^{\theta}$ is an $\mathcal{I}$-matrix on its own right. This result forms an analog in the π-institution framework of Lemma 3.3 of [2].

Lemma 12 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN, *be a π-institution, that admits lifting of N-quotients, and $\mathfrak{A} = \langle\langle \mathrm{SEN}', \langle N', F'\rangle\rangle, T'\rangle, \mathfrak{B} = \langle\langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle, T''\rangle \in \mathrm{Mat}^{is}(\mathcal{I})$, with $\mathfrak{B} \leq \mathfrak{A}$. If $\theta = \{\theta_{\Sigma}\}_{\Sigma \in |\mathbf{Sign}'|} \in \mathrm{Con}^{N'}(\mathrm{SEN}')$, such that $\theta \leq \Omega^{N'}(T')$, then the tuple $\mathfrak{B}^{\theta} := \langle\langle \mathrm{SEN}''^{\theta}, \langle N''^{\theta}, F''^{\theta}\rangle\rangle, T''/\theta\rangle$ is a submatrix of the $\mathcal{I}$-matrix $\mathfrak{A}^{\theta} := \langle\langle \mathrm{SEN}'^{\theta}, \langle N'^{\theta}, F'^{\theta}\rangle\rangle, T'/\theta\rangle$.*

Proof:

It has already been shown in Lemma 11 that $\mathbf{B}^{\theta} = \langle \mathrm{SEN}''^{\theta}, \langle N''^{\theta}, F''^{\theta}\rangle\rangle$ is an N-algebraic system. So it suffices to show that, for every $\Sigma \in |\mathbf{Sign}|$, $T''_{\Sigma}/\theta_{\Sigma} = T'_{\Sigma}/\theta_{\Sigma} \cap \mathrm{SEN}''^{\theta}(\Sigma)$. Then, it will follow, by Proposition 7, that T''/θ is an $\mathcal{I}$-filter on $\mathbf{B}^{\theta} = \langle \mathrm{SEN}''^{\theta}, \langle N''^{\theta}, F''^{\theta}\rangle\rangle$.

Indeed, if $\phi''/\theta_{\Sigma} \in T''_{\Sigma}/\theta_{\Sigma}$, then, there exists $\psi'' \in T''_{\Sigma}$, such that $\langle \phi'', \psi''\rangle \in \theta_{\Sigma}$. But, we have $\mathfrak{B} \leq \mathfrak{A}$, whence $T'' = T' \cap \mathrm{SEN}''$, and, hence, $\psi'' \in T'_{\Sigma} \cap \mathrm{SEN}''(\Sigma)$ and $\langle \phi'', \psi''\rangle \in \theta_{\Sigma}$. This shows that $\phi''/\theta_{\Sigma} \in T'_{\Sigma}/\theta_{\Sigma} \cap \mathrm{SEN}''(\Sigma)/\theta_{\Sigma}$. If, conversely, $\phi''/\theta_{\Sigma} \in T'_{\Sigma}/\theta_{\Sigma} \cap \mathrm{SEN}''(\Sigma)/\theta_{\Sigma}$, then $\phi''/\theta_{\Sigma} \in T'_{\Sigma}/\theta_{\Sigma}$ and $\phi''/\theta_{\Sigma} \in \mathrm{SEN}''(\Sigma)/\theta_{\Sigma}$. Thus, there exist $\psi'' \in T'_{\Sigma}$ and $\chi'' \in \mathrm{SEN}''(\Sigma)$, such that $\langle \phi'', \psi''\rangle, \langle \phi'', \chi''\rangle \in \theta_{\Sigma}$. Now, since $\theta \leq \Omega^{N'}(T')$, θ is compatible with T'. Therefore, $\chi'' \in T'_{\Sigma} \cap \mathrm{SEN}''(\Sigma) = T''_{\Sigma}$ and $\langle \phi'', \chi''\rangle \in \theta_{\Sigma}$. This shows that $\phi''/\theta_{\Sigma} \in T''_{\Sigma}/\theta_{\Sigma}$. ■

Finally, in Theorem 13, it is shown that, if the Leibniz quotient of a local $\mathcal{I}$-matrix in $\mathrm{Mat}^{is}(\mathcal{I})$ has the principal filter extension property, then so does the $\mathcal{I}$-matrix itself. $\mathcal{I}$, in this case, is assumed to be an N-structural, downward N-closed π-institution, that admits lifting of N-quotients and has the N-correspondence property. This theorem extends to the present framework Theorem 3.4 of [2].

Theorem 13 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be an N-structural, downward N-closed π-institution, that admits lifting of N-quotients, has the N-correspondence property. Let $\mathfrak{A} = \langle \mathbf{A}, T'\rangle \in \mathrm{Mat}^{lis}(\mathcal{I})$, with $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$ and $T' = \mathrm{Fg}^{\mathcal{I},\mathbf{A}}(\langle \Sigma', \Phi'\rangle)$. If $\mathfrak{A}^{\Omega^{N'}(T')}$ has the* PFEP*, then so does $\mathfrak{A}$.*

Proof:

Let $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle$ and $\mathbf{B} = \langle \mathrm{SEN}'', \langle N'', F''\rangle\rangle$ be N-algebraic systems over the same signature category $\mathbf{Sign}'$, with $\mathbf{B} \leq \mathbf{A}$, and $\mathfrak{A} = \langle \mathbf{A}, T'\rangle, \mathfrak{B} = \langle \mathbf{B}, T''\rangle$ two local $\mathcal{I}$-matrices, with $T' = \mathrm{Fg}^{\mathcal{I},\mathbf{A}}(\langle \Sigma', \Phi'\rangle)$ and $T'' = \mathrm{Fg}^{\mathcal{I},\mathbf{B}}(\langle \Sigma', \Phi''\rangle)$, such that $\mathfrak{B} \leq \mathfrak{A}$. Assume $\mathfrak{A}^{\Omega^{N'}(T')}$ has the PFEP and that $V'' = \mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\langle \Sigma', \phi''\rangle)$ is a principal $\mathcal{I}$-filter of $\mathfrak{B}$. We must show that V'' can be extended to an $\mathcal{I}$-filter V' of $\mathfrak{A}$, such that $V''_{\Sigma'} = V'_{\Sigma'} \cap \mathrm{SEN}''(\Sigma')$.

Notice that, due to the lifting of N-quotients, taking into account Corollary 11 of [18], the canonical projection N-morphism $\langle \mathrm{I}_{\mathbf{Sign}'}, \pi^{\Omega^{N'}(T')}\rangle : \mathfrak{A} \to \mathfrak{A}^{\Omega^{N'}(T')}$ is a surjective $\mathcal{I}$-morphism, such that ${\pi^{\Omega^{N'}(T')}}^{-1}(T'/\Omega^{N'}(T')) = T'$. Let

$$\mathbf{B}^{\Omega^{N'}(T')} = \langle \mathrm{SEN}''^{\Omega^{N'}(T')}, \langle N''^{\Omega^{N'}(T')}, F''^{\Omega^{N'}(T')}\rangle\rangle$$

be the sub-N-algebraic system of $\mathbf{A}^{\Omega^{N'}(T')}$, as postulated by Lemma 11, and $\mathfrak{B}^{\Omega^{N'}(T')} = \langle \mathbf{B}^{\Omega^{N'}(T')}, T''^{\Omega^{N'}(T')}\rangle$ the corresponding $\mathcal{I}$-matrix, which is, by Lemma 12, a submatrix of $\mathfrak{A}^{\Omega^{N'}(T')}$. Consider, now, the restriction $\langle \mathrm{I}, \pi\rangle := \langle \mathrm{I}_{\mathbf{Sign}'}, \pi^{\Omega^{N'}(T')}\rangle \restriction_{\mathrm{SEN}''}: \mathrm{SEN}'' \to \mathrm{SEN}''^{\Omega^{N'}(T')}$. Then $\langle \mathrm{I}, \pi\rangle : \mathfrak{B} \to \mathfrak{B}^{\Omega^{N'}(T')}$ is a surjective $\mathcal{I}$-morphism and $\pi^{-1}(T''^{\Omega^{N'}(T')}) = T''$. Since $\mathcal{I}$ has the N-correspondence property, we have, using Proposition 21 of [18], that

$$\begin{array}{rcl}\pi^{-1}(\mathrm{Fg}^{\mathcal{I},\mathfrak{B}^{\Omega^{N'}(T')}}(\langle \Sigma', \pi_{\Sigma'}(\phi'')\rangle)) & = & \mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\langle \Sigma', \phi''\rangle) \vee \pi^{-1}(T''^{\Omega^{N'}(T')}) \\ & = & \mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\langle \Sigma', \phi''\rangle) \vee T''^{\Omega^{N'}(T')} \\ & = & \mathrm{Fg}^{\mathcal{I},\mathfrak{B}}(\langle \Sigma', \phi''\rangle) \\ & = & V''.\end{array}$$

Since $\mathfrak{A}^{\Omega^{N'}(T')}$ has, by hypothesis, the PFEP, there exists $W' \in \mathrm{Fi}^{\mathcal{I}}(\mathfrak{A}^{\Omega^{N'}(T')})$, such that $W'_{\Sigma'} \cap \mathrm{SEN}''(\Sigma')/\Omega^{N'}_{\Sigma'}(T') = \mathrm{Fg}^{\mathcal{I},\mathfrak{B}^{\Omega^{N'}(T')}}_{\Sigma'}(\langle \Sigma', \pi_{\Sigma'}(\phi'')\rangle)$. But, then ${\pi^{\Omega^{N'}(T')}}^{-1}(W') \in \mathrm{Fi}^{\mathcal{I}}(\mathfrak{A})$ and

$${\pi_{\Sigma'}^{\Omega^{N'}(T')}}^{-1}(W'_{\Sigma'}) \cap \mathrm{SEN}''(\Sigma') =$$

$$\begin{array}{rl}
= & \pi_{\Sigma'}^{\Omega^{N'}(T')^{-1}}(W'_{\Sigma'}) \cap \pi_{\Sigma'}^{\Omega^{N'}(T')^{-1}}(\mathrm{SEN}''(\Sigma')/\Omega^{N'}_{\Sigma'}(T')) \cap \mathrm{SEN}''(\Sigma') \\
= & \pi_{\Sigma'}^{\Omega^{N'}(T')^{-1}}(W'_{\Sigma'} \cap \mathrm{SEN}''(\Sigma')/\Omega^{N'}_{\Sigma'}(T')) \cap \mathrm{SEN}''(\Sigma') \\
= & \pi_{\Sigma'}^{-1}(W'_{\Sigma'} \cap \mathrm{SEN}''(\Sigma')/\Omega^{N'}_{\Sigma'}(T')) \\
= & \pi_{\Sigma'}^{-1}(\mathrm{Fg}_{\Sigma'}^{\mathcal{I},\mathfrak{B}^{\Omega^{N'}(T')}}(\langle\Sigma', \pi_{\Sigma'}(\phi'')\rangle)) \\
= & V''_{\Sigma'}.
\end{array}$$

Therefore, $\mathfrak{A}$ has indeed the PFEP. ■

The results presented in Theorems 10 and 13 immediately imply the equivalence of several conditions concerning the principal and the local filter extension properties on the classes $\mathrm{Mat}^{lis}(\mathcal{I})$ and $\mathrm{Mat}^{Nlis}(\mathcal{I})$ for a finitary, N-structural, downward N-closed π-institution $\mathcal{I}$, that admits lifting of N-quotients and has the N-correspondence property.

Corollary 14 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, is a finitary, N-structural, downward N-closed π-institution $\mathcal{I}$, that admits lifting of N-quotients and has the N-correspondence property. Then, the following conditions are equivalent:*

1. $\mathrm{Mat}^{Nlis}(\mathcal{I})$ *has* PFEP*;*
2. $\mathrm{Mat}^{lis}(\mathcal{I})$ *has the* PFEP*;*
3. $\mathrm{Mat}^{lis}(\mathcal{I})$ *has the* LFEP*;*
4. $\mathrm{Mat}^{Nlis}(\mathcal{I})$ *has the* LFEP.

Proof:
$1 \to 2$ is the content of Theorem 13. $2 \to 3$ is the content of Theorem 10 and the implications $3 \to 4$ and $4 \to 1$ are obvious. ■

Finally, considering Theorem 6, together with Corollary 14 and Lemma 9, the following corollary, summarizing the results presented in Sections 3 and 4 of the present paper, may be formulated. Note that this result forms, in the context of π-institutions, a partial analog of Corollary 3.6 of [2]. It is only partial, since in the context of k-deductive systems the deduction-detachment theorem, formula definability of principal filters and the filter extension property are all shown to be equivalent. Thus, in that more restricted context, the implication of the following corollary can be replaced by an equivalence.

Corollary 15 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, is a finitary, N-structural, downward N-closed π-institution $\mathcal{I}$, that admits lifting of N-quotients and has the N-correspondence property. The the following conditions are related by* $(1 \leftrightarrow 2 \leftrightarrow 3) \to (4 \leftrightarrow 5 \leftrightarrow 6 \leftrightarrow 7)$*:*

1. *$\mathcal{I}$ has the* LDDT*;*
2. $\mathrm{Mat}^{lis}(\mathcal{I})$ *has the* LDPF*;*

3. $\mathrm{Mat}^{Nlis}(\mathcal{I})$ *has the* LDPF*;*

4. $\mathrm{Mat}^{lis}(\mathcal{I})$ *has the* PFEP*;*

5. $\mathrm{Mat}^{Nlis}(\mathcal{I})$ *has the* PFEP*;*

6. $\mathrm{Mat}^{lis}(\mathcal{I})$ *has the* LFEP*;*

7. $\mathrm{Mat}^{Nlis}(\mathcal{I})$ *has the* LFEP.

Proof:
The equivalences $1 \leftrightarrow 2 \leftrightarrow 3$ are the content of Theorem 6. The equivalences $4 \leftrightarrow 5 \leftrightarrow 6 \leftrightarrow 7$ are the content of Corollary 14 and the implication of the statement is based on Lemma 9. ■

It should be emphasized that, despite the fact that we were unable to replace the only implication appearing in Corollary 15 by an equivalence, no known counterexample exists of the converse implication. Hence, it is still an *open problem* whether equivalence can, in fact, replace implication. It is conjectured that this cannot be done in general. The reasons for this conjecture are similar to the reasons that lead to the syntactical-semantical dichotomy in the classes of the categorical abstract algebraic hierarchy of π-institutions. In a nutshell, they stem from the loose association between the formulas of a π-institution and the natural transformations in N. This association is much stronger in the special case of π-institutions arising from sentential logics.

5 Bilogical Morphisms and LDDT

In this section, the aim is to show that if $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ and $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ are two π-institutions, with N and N' categories of natural transformations on SEN and SEN$'$, respectively, such that $\langle F, \alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$ is an (N, N')-bilogical morphism, then $\mathcal{I}$ has the LDDT with respect to a local deduction-detachment system $\mathcal{E}$ if and only if $\mathcal{I}'$ has the LDDT with respect to $\mathcal{E}'$. The formal definition of an (N, N')-bilogical morphism will be recalled before formulating the relevant theorem, but the reader is encouraged for more details to consult [15], where the concept of an (N, N')-bilogical morphism was first introduced. It should be noted, at this point, that bilogical morphisms for sentential logics were first introduced in [10] and that the work of Font and Jansana formed the main inspiration for the subsequent developments in [15].

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle, \mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ be two π-institutions, with N, N' categories of natural transformations on $\mathrm{SEN}, \mathrm{SEN}'$, respectively. An (N, N')-**bilogical morphism** $\langle F, \alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$ **from** $\mathcal{I}$ **to** $\mathcal{I}'$ consists of

- a surjective functor $F : \mathbf{Sign} \to \mathbf{Sign}'$ and
- a surjective natural transformation $\alpha : \mathrm{SEN} \to \mathrm{SEN}' \circ F$, i.e., such that $\alpha_\Sigma : \mathrm{SEN}(\Sigma) \to \mathrm{SEN}'(F(\Sigma))$ is surjective, for all $\Sigma \in |\mathbf{Sign}|$,

such that

- there exists a one-to-one correspondence $\sigma \mapsto \sigma'$ between natural transformations in N and in N', satisfying, for all $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$,
$$\alpha_\Sigma(\sigma_\Sigma(\vec{\phi})) = \sigma'_{F(\Sigma)}(\alpha^k_\Sigma(\vec{\phi}));$$
- for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,
$$\phi \in C_\Sigma(\Phi) \quad \text{iff} \quad \alpha_\Sigma(\phi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

It is shown, now, that the property of having the LDDT is preserved by (N, N')-bilogical morphisms. This result has many precursor results in categorical abstract algebraic logic. It is related to Theorem 2.17 of [14], but [14] belongs to the era preceding the introduction of the categories of natural transformations on sentence functors [15] and, as a result, does not consider at all the relevant algebraic structure. It is also related to Lemma 5.4 of [16], but that result deals only with the finitary uniterm deduction-detachment theorem. Theorem 16 adds locality, the multi-term property and does not require finitarity. In this sense, it improves Theorem 5.4 of [16].

Theorem 16 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ and $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ be two π-institutions, with N and N' categories of natural transformations on* SEN *and* SEN′*, respectively. Suppose $\langle F, \alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$ is an (N, N')-bilogical morphism. Then, $\mathcal{I}$ has the* LDDT *with respect to a local deduction-detachment system $\mathcal{E}$ if and only if $\mathcal{I}'$ has the* LDDT *with respect to $\mathcal{E}'$.*

Proof:

Suppose, first, that $\mathcal{I}$ has the LDDT with respect to a local deduction-detachment system $\mathcal{E} = \{E^i : i \in I\}$. Let $\Sigma' \in |\mathbf{Sign}'|$ and $\Gamma' \cup \{\phi', \psi'\} \subseteq \mathrm{SEN}'(\Sigma')$. Then, since $\langle F, \alpha\rangle$ is a bilogical morphism, it is surjective, whence, there exist $\Sigma \in |\mathbf{Sign}|$ and $\Gamma \cup \{\phi, \psi\} \in \mathrm{SEN}(\Sigma)$, such that $F(\Sigma) = \Sigma'$ and $\alpha_\Sigma(\Gamma) = \Gamma', \alpha_\Sigma(\phi) = \phi'$ and $\alpha_\Sigma(\psi) = \psi'$. Hence, we get

$$\begin{array}{lll}
\psi' \in C'_{\Sigma'}(\Gamma', \phi') & \text{iff} & \alpha_\Sigma(\psi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Gamma), \alpha_\Sigma(\phi)) \\
& \text{iff} & \psi \in C_\Sigma(\Gamma, \phi) \\
& \text{iff} & E^i_\Sigma(\phi, \psi) \subseteq C_\Sigma(\Gamma), \text{ for some } i \in I, \\
& \text{iff} & E'^i_{F(\Sigma)}(\alpha_\Sigma(\phi), \alpha_\Sigma(\psi)) \subseteq C'_{F(\Sigma)}(\alpha_\Sigma(\Gamma)), \text{ for some } i \in I, \\
& \text{iff} & E'^i_{\Sigma'}(\phi', \psi') \subseteq C'_{\Sigma'}(\Gamma'), \text{ for some } i \in I.
\end{array}$$

Hence $\mathcal{I}'$ also has the LDDT with respect to the local deduction-detachment system $\mathcal{E}' = \{E'^i : i \in I\}$.

Suppose, conversely, that $\mathcal{I}'$ has the LDDT with respect to a local deduction-detachment system $\mathcal{E}' = \{E'^i : i \in I\}$. Let $\Sigma \in |\mathbf{Sign}|, \Gamma \cup \{\phi, \psi\} \subseteq \mathrm{SEN}(\Sigma)$. We have

$$\begin{array}{lll}
\psi \in C_\Sigma(\Gamma, \phi) & \text{iff} & \alpha_\Sigma(\psi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Gamma), \alpha_\Sigma(\phi)) \\
& \text{iff} & E'^i_{F(\Sigma)}(\alpha_\Sigma(\phi), \alpha_\Sigma(\psi)) \subseteq C'_{F(\Sigma)}(\alpha_\Sigma(\Gamma)), \text{ for some } i \in I, \\
& \text{iff} & \alpha_\Sigma(E^i_\Sigma(\phi, \psi)) \subseteq C'_{F(\Sigma)}(\alpha_\Sigma(\Gamma)), \text{ for some } i \in I, \\
& \text{iff} & E^i_\Sigma(\phi, \psi) \subseteq C_\Sigma(\Gamma), \text{ for some } i \in I.
\end{array}$$

Therefore, $\mathcal{I}$ also has the LDDT with respect to $\mathcal{E} = \{E^i : i \in I\}$. ■

6 Equivalent π-Institutions and LDDT

This section has a goal similar to that of Section 5. It aims at showing that, if two π-institutions are deductively equivalent in a sense slightly stronger than that defined in [13], then one of the two has the LDDT if and only if the other does. This result generalizes Theorem 5.2 of [2] from the k-deductive system to the π-institution level. Its proof, as expected, follows along similar lines as that of Theorem 5.2 of [2]. Before stating formally Theorem 19, which contains the main result, we will recall the definition of deductively equivalent π-institutions from [13] and, then, formulate a lemma, similar in content to Lemma 5.1 of [2], which will be crucial in the proof of the main Theorem 19.

Let, as before, $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ be two π-institutions, with N, N' categories of natural transformations on $\mathrm{SEN}, \mathrm{SEN}'$, respectively. An **interpretation** $\langle G, \alpha\rangle : \mathcal{I} \vdash \mathcal{I}'$ **from** $\mathcal{I}$ **to** $\mathcal{I}'$ consists of a functor $F : \mathbf{Sign} \rightarrow \mathbf{Sign}'$ and a natural transformation $\alpha : \mathrm{SEN} \rightarrow \mathcal{P}\mathrm{SEN}' \circ F$, such that, for every $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\phi \in C_\Sigma(\Phi) \quad \text{iff} \quad \alpha_\Sigma(\phi) \subseteq C'_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

An interpretation $\langle G, \beta\rangle : \mathcal{I}' \vdash \mathcal{I}$ is called an **inverse** of the interpretation $\langle F, \alpha\rangle : \mathcal{I} \vdash \mathcal{I}'$ if there exists an adjoint equivalence $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign} \rightarrow \mathbf{Sign}'$, such that, for all $\Sigma \in |\mathbf{Sign}|, \Sigma' \in |\mathbf{Sign}'|$ and all $\phi \in \mathrm{SEN}(\Sigma), \psi \in \mathrm{SEN}'(\Sigma')$,

$$C_{G(F(\Sigma))}(\mathrm{SEN}(\eta_\Sigma)(\phi)) = C_{G(F(\Sigma))}(\beta_{F(\Sigma)}(\alpha_\Sigma(\phi)))$$

and

$$C'_{\Sigma'}(\mathrm{SEN}'(\epsilon_{\Sigma'})(\alpha_{G(\Sigma')}(\beta_{\Sigma'}(\psi)))) = C'_{\Sigma'}(\psi).$$

An interpretation $\langle F, \alpha\rangle : \mathcal{I} \vdash \mathcal{I}'$ is said to be **invertible** if it has an inverse $\langle G, \beta\rangle : \mathcal{I}' \vdash \mathcal{I}$. Two π-institutions $\mathcal{I}$ and $\mathcal{I}'$ are **(deductively) equivalent** if there exists an invertible interpretation $\langle F, \alpha\rangle : \mathcal{I} \vdash \mathcal{I}'$.

Finally, $\mathcal{I}$ and $\mathcal{I}'$, with N, N' categories of natural transformations on $\mathrm{SEN}, \mathrm{SEN}'$, respectively, are **algebraically equivalent** if they are deductively equivalent and, in addition,

1. the unit and the counit of the adjoint equivalence $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign} \rightarrow \mathbf{Sign}'$, witnessing the deductive equivalence, are identities, i.e., the functors $F : \mathbf{Sign} \rightarrow \mathbf{Sign}'$ and $G : \mathbf{Sign}' \rightarrow \mathbf{Sign}$ are inverses of one another;
2. there exist n (N, N')-epimorphic translations $\langle F, \alpha^0\rangle, \ldots, \langle F, \alpha^{n-1}\rangle : \mathrm{SEN} \rightarrow \mathrm{SEN}' \circ F$ and m (N', N)-epimorphic translations $\langle G, \beta^0\rangle, \ldots, \langle G, \beta^{m-1}\rangle : \mathrm{SEN}' \rightarrow \mathrm{SEN} \circ G$, such that, for all $\Sigma \in |\mathbf{Sign}|, \Sigma' \in |\mathbf{Sign}'|, \phi \in \mathrm{SEN}(\Sigma), \phi' \in \mathrm{SEN}'(\Sigma')$, $\alpha_\Sigma(\phi) = \{\alpha^0_\Sigma(\phi), \ldots, \alpha^{n-1}_\Sigma(\phi)\}$ and $\beta_{\Sigma'}(\phi') = \{\beta^0_{\Sigma'}(\phi'), \ldots, \beta^{m-1}_{\Sigma'}(\phi')\}$;
3. the compositions $\alpha^i \circ \beta^j : \mathrm{SEN} \rightarrow \mathrm{SEN}$, $i < n, j < m$, and $\beta^j \circ \alpha^i : \mathrm{SEN}' \rightarrow \mathrm{SEN}'$, $i < n, j < m$, are natural transformations in N and N', respectively.

Suppose, next, that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is a π-institution with the LDDT with respect to a local deduction-detachment system $\mathcal{E} = \{E^i : i \in I\}$. Following [2], we will define inductively $\mathcal{E}^n = \{E^{n,i} : i \in I^{(n)}\}, n \geq 1$, where $E^{n,i}$ will be a finite set of natural transformation $\mathrm{SEN}^{n+1} \to \mathrm{SEN}$ in N. The definition goes as follows:

$$\begin{aligned} &I^{(1)} = I\\ &E^{1,i} = E^i, i \in I,\\ &\mathcal{E}^1 = \{E^{1,i} : i \in I\}.\end{aligned}$$

Assuming, now, that $I^{(n)}$ and $\mathcal{E}^n = \{E^{n,i} : \mathrm{SEN}^{n+1} \to \mathrm{SEN}, i \in I^{(n)}\}$ have already been defined, we proceed as follows: For $i \in I^{(n)}$, we let $I^{E^{n,i}}$ be the set of all maps from $E^{n,i}$ to I and set

$$I^{(n+1)} = \bigcup\{I^{E^{n,i}} : i \in I^{(n)}\}.$$

Moreover, for every $f \in I^{(n+1)}$, i.e., $f : E^{n,i} \to I$, for some $i \in I^{(n)}$, we set

$$E^{n+1,f} = \bigcup_{\epsilon \in E^{n,i}} E^{f(\epsilon)}(p^{n+2,n}, \epsilon(p^{n+2,0}, \ldots, p^{n+2,n-1}, p^{n+2,n+1})),$$

i.e., for all $\Sigma \in |\mathbf{Sign}|, \phi_0, \ldots, \phi_n, \psi \in \mathrm{SEN}(\Sigma)$,

$$E^{n+1,f}_\Sigma(\phi_0, \ldots, \phi_n, \psi) = \bigcup_{\epsilon \in E^{n,i}} E^{f(\epsilon)}_\Sigma(\phi_n, \epsilon_\Sigma(\phi_0, \ldots, \phi_{n-1}, \psi)).$$

Finally, set

$$\mathcal{E}^{n+1} = \{E^{n+1,f} : f \in I^{(n+1)}\}.$$

This step completes the inductive construction of $\mathcal{E}^n, n \geq 1$.

It is now shown that, for every $n \geq 1$ and all $f \in I^{(n)}$, the set $E^{n,f}$ of natural transformations is a finite set and, moreover, all its members are natural transformations in N.

Lemma 17 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be a π-institution with the LDDT with respect to a local deduction-detachment system $\mathcal{E} = \{E^i : i \in I\}$. Then, for every $n \geq 1$ and all $f \in I^{(n)}$, the set $E^{n,f}$ is a finite set of natural transformations in N.*

Proof:

The fact that, for every n and all $f \in I^{(n)}$, the set $E^{n,f}$ is finite is established by an easy induction on $n \geq 1$. For $n = 1$ and every $i \in I^{(1)} = I$, we have that $E^{1,i} = E^i$, which is finite by the definition of a local deduction-detachment system. Assume, now, as the induction hypothesis, that for $n = k \geq 1$ and all $f \in I^{(k)}$, we have that $E^{k,f}$ is finite. Then, for $n = k + 1$ and $f \in I^{(k+1)} = \bigcup\{I^{E^{k,i}} : i \in I^{(k)}\}$, say $f : E^{k,i} \to I$, we have $E^{k+1,f} = \bigcup_{\epsilon \in E^{k,i}} E^{f(\epsilon)}(p^{k+2,k}, \epsilon(p^{k+2,0}, \ldots, p^{k+2,k-1}, p^{k+2,k+1}))$, which is finite, since $E^{f(\epsilon)}$ is finite, by definition, and $E^{k,i}$ is finite, by the induction hypothesis. This concludes the inductive step and shows that $E^{n,f}$ is finite, for all $n \geq 1$ and all $f \in I^{(n)}$.

Finally, we set out to show that, for every $n \geq 1$ and all $f \in I^{(n)}$, the set $E^{n,f}$ consists of natural transformations in N. This will also be established by induction on $n \geq 1$. For $n = 1$ and all $i \in I^{(1)} = I$, we have that $E^{1,i} = E^i$, which is, by definition, a finite set of binary natural transformations in N. Assume, as the induction hypothesis, that for $n = k \geq 1$ and $f \in I^{(k)}$, the set $E^{k,f}$ is a set of $(k+1)$-ary natural transformations in N. Now consider $n = k+1$ and $f \in I^{(k+1)} = \bigcup\{I^{E^{k,i}} : i \in I^{(k)}\}$, say $f : E^{k,i} \to I$. Then, every $\sigma \in E^{k+1,f} = \bigcup_{\epsilon \in E^{k,i}} E^{f(\epsilon)}(p^{k+2,k}, \epsilon(p^{k+2,0}, \ldots, p^{k+2,k-1}, p^{k+2,k+1}))$ is of the form

$$\tau(p^{k+2,k}, \epsilon(p^{k+2,0}, \ldots, p^{k+2,k-1}, p^{k+2,k+1})), \text{ for some } \tau \in E^{f(\epsilon)}, \epsilon \in E^{k,i}.$$

$\epsilon : \mathrm{SEN}^{k+1} \to \mathrm{SEN}$ is in $E^{k,i}$, whence it is also in N, by the induction hypothesis. $p^{k+2,0}, \ldots, p^{k+2,k-1}, p^{k+2,k+1} : \mathrm{SEN}^{k+2} \to \mathrm{SEN}$ are $k+1$ natural transformations in N, by the definition of N, whence, also by the definition of N, we get that $\epsilon(p^{k+2,0}, \ldots, p^{k+2,k-1}, p^{k+2,k+1}) : \mathrm{SEN}^{k+2} \to \mathrm{SEN}$ is also a natural transformation in N. This, combined with the fact that $\tau : \mathrm{SEN}^2 \to \mathrm{SEN}$ is in $E^{f(\epsilon)}$ and, thus, in N, by the definition of $E^{f(\epsilon)}$, and that $p^{k+2,k} : \mathrm{SEN}^{k+2} \to \mathrm{SEN}$ is also in N, yields that $\tau(p^{k+2,k}, \epsilon(p^{k+2,0}, \ldots, p^{k+2,k-1}, p^{k+2,k+1})) : \mathrm{SEN}^{k+2} \to \mathrm{SEN}$ is also in N.

This concludes the induction step and shows that, for all $n \geq 1$ and all $f \in I^{(n)}$, $E^{n,f}$ is a finite set of natural transformations in N. ■

With these definitions and Lemma 17 in place, an analog of Lemma 5.1 of [2] will now be formulated. It states that, if $\mathcal{E}$ is a local deduction-detachment system for $\mathcal{I}$, then the collection $\mathcal{E}^n$ acts as an n-premiss local deduction-detachment system for $\mathcal{I}$. This lemma will be crucial in the proof of the main theorem of this section, Theorem 19, stating that having the LDDT is preserved under the property of algebraic equivalence of π-institutions. Despite the fact that the proof of Lemma 18 follows by essentially imitating the proof of Lemma 5.1 of [2], it will be included here for the sake of completeness.

Lemma 18 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be a π-institution, having the* LDDT *with respect to a local deduction-detachment system $\mathcal{E}$. Then, for all $\Sigma \in |\mathbf{Sign}|, \Gamma \cup \{\phi_0, \ldots, \phi_{n-1}, \psi\} \subseteq \mathrm{SEN}(\Sigma)$,*

$$\psi \in C_\Sigma(\Gamma, \phi_0, \ldots, \phi_{n-1}) \quad \textit{iff} \quad E^{n,i}_\Sigma(\phi_0, \ldots, \phi_{n-1}, \psi) \subseteq C_\Sigma(\Gamma), \textit{ for some } i \in I^{(n)}.$$

Proof:

Induction on n will be employed. In fact, for $n = 1$, the claim reduces to the LDDT. Assume that the claim is true for $n \geq 1$ and suppose that $\Sigma \in |\mathbf{Sign}|, \Gamma \cup \{\phi_0, \ldots, \phi_n, \psi\} \subseteq \mathrm{SEN}(\Sigma)$.

For the "if" direction, suppose that

$$E^{n+1,f}_\Sigma(\phi_0, \ldots, \phi_n, \psi) \subseteq C_\Sigma(\Gamma), \text{ for some } f \in I^{(n+1)}. \tag{4}$$

By construction $f : E^{n,i} \to I$, for some $i \in I^{(n)}$. By the induction hypothesis, we get that

$$\psi \in C_\Sigma(E^{n,i}_\Sigma(\phi_0, \ldots, \phi_{n-1}, \psi), \phi_0, \ldots, \phi_{n-1}). \tag{5}$$

Furthermore, by the properties of a local deduction-detachment system, we have that $\epsilon_\Sigma(\phi_0,\ldots,\phi_{n-1},\psi) \in C_\Sigma(E^{f(\epsilon)}_\Sigma(\phi_n,\epsilon_\Sigma(\phi_0,\ldots,\phi_{n-1},\psi)),\phi_n)$, for all $\epsilon \in E^{n,i}$, which yields that

$$\begin{array}{lll} E^{n,i}_\Sigma(\phi_0,\ldots,\phi_{n-1},\psi) & \subseteq & C_\Sigma(\bigcup_{\epsilon\in E^{n,i}} E^{f(\epsilon)}_\Sigma(\phi_n,\epsilon_\Sigma(\phi_0,\ldots,\phi_{n-1},\psi)),\phi_n) \\ & = & C_\Sigma(E^{n+1,f}_\Sigma(\phi_0,\ldots,\phi_n,\psi),\phi_n). \end{array} \tag{6}$$

Finally, we get

$$\begin{array}{lll} \psi & \in & C_\Sigma(E^{n,i}_\Sigma(\phi_0,\ldots,\phi_{n-1},\psi),\phi_0,\ldots,\phi_{n-1}) \quad \text{(by Membership (5))} \\ & \subseteq & C_\Sigma(E^{n+1,f}_\Sigma(\phi_0,\ldots,\phi_n,\psi),\phi_0,\ldots,\phi_n) \quad \text{(by Inclusion (6))} \\ & \subseteq & C_\Sigma(\Gamma,\phi_0,\ldots,\phi_n) \quad \text{(by Inclusion (4))}. \end{array}$$

For the "only if" direction, assume that $\psi \in C_\Sigma(\Gamma,\phi_0,\ldots,\phi_n)$. Then, by the induction hypothesis, we get that

$$E^{n,i}_\Sigma(\phi_0,\ldots,\phi_{n-1},\psi) \subseteq C_\Sigma(\Gamma,\phi_n), \text{ for some } i \in I^{(n)}. \tag{7}$$

Now, for $\epsilon \in E^{n,i}$, let $f(\epsilon) \in I$ be such that

$$E^{f(\epsilon)}_\Sigma(\phi_n,\epsilon_\Sigma(\phi_0,\ldots,\phi_{n-1},\psi)) \subseteq C_\Sigma(\Gamma), \tag{8}$$

which exists, by Inclusion (7), since $\mathcal{E}$ is a local deduction-detachment system for $\mathcal{I}$. This defines a function $f : E^{n,i} \to I \in I^{(n+1)}$. We then have

$$\begin{array}{lll} E^{n+1,f}_\Sigma(\phi_0,\ldots,\phi_n,\psi) & = & \bigcup_{\epsilon\in E^{n,i}} E^{f(\epsilon)}_\Sigma(\phi_n,\epsilon_\Sigma(\phi_0,\ldots,\phi_{n-1},\psi)) \\ & \subseteq & C_\Sigma(\Gamma) \quad \text{(by Inclusion (8))}. \end{array}$$

■

Theorem 19 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$, with N, N' categories of natural transformations on* SEN, SEN′*, respectively, be two algebraically equivalent π-institutions. Then $\mathcal{I}$ has the* LDDT *if and only if $\mathcal{I}'$ does.*

Proof:

Assume that $\mathcal{I}'$ has the LDDT with respect to a local deduction-detachment system $\mathcal{E} = \{E^i : i \in I\}$ and let $\langle F,\alpha\rangle : \mathcal{I} \vdash \mathcal{I}'$ and $\langle G,\beta\rangle : \mathcal{I}' \vdash \mathcal{I}$ be the two interpretations witnessing the algebraic equivalence of $\mathcal{I}$ and $\mathcal{I}'$, such that, for all $\Sigma \in |\mathbf{Sign}|, \Sigma' \in |\mathbf{Sign}'|$ and all $\phi \in \mathrm{SEN}(\Sigma)$ and $\phi' \in \mathrm{SEN}'(\Sigma')$,

$$\alpha_\Sigma(\phi) = \{\alpha^0_\Sigma(\phi),\ldots,\alpha^{n-1}_\Sigma(\phi)\} \quad \text{and} \quad \beta_{\Sigma'}(\phi') = \{\beta^0_{\Sigma'}(\phi'),\ldots,\beta^{m-1}_{\Sigma'}(\phi')\}.$$

for (N,N')-epimorphic translations $\langle F,\alpha^i\rangle, i < n$, and (N',N)-epimorphic translations $\langle G,\beta^j\rangle, j < m$.

Let $\Sigma \in |\mathbf{Sign}|, \Gamma \cup \{\phi, \psi\} \subseteq \mathrm{SEN}(\Sigma)$. We have $\psi \in C_\Sigma(\Gamma, \phi)$ if and only if, since $\langle F, \alpha \rangle : \mathcal{I} \vdash \mathcal{I}'$ is an interpretation, $\alpha_\Sigma(\psi) \subseteq C'_{F(\Sigma)}(\alpha_\Sigma(\Gamma), \alpha_\Sigma(\phi))$ if and only if, by Lemma 18 and the hypothesis, there exists, for all $j < n$, $i_j \in I^{(n)}$, such that

$$E^{n,i_j}_{F(\Sigma)}(\alpha^0_\Sigma(\phi), \ldots, \alpha^{n-1}_\Sigma(\phi), \alpha^j_\Sigma(\psi)) \subseteq C'_{F(\Sigma)}(\alpha_\Sigma(\Gamma)),$$

if and only if, since $\langle G, \beta \rangle : \mathcal{I}' \vdash \mathcal{I}$ is an interpretation, for all $j < n$, there exists $i_j \in I^{(n)}$, such that

$$\beta_{G(F(\Sigma))}(E^{n,i_j}_{F(\Sigma)}(\alpha^0_\Sigma(\phi), \ldots, \alpha^{n-1}_\Sigma \phi), \alpha^j_\Sigma(\psi))) \subseteq C_{G(F(\Sigma))}(\beta_{F(\Sigma)}(\alpha_\Sigma(\Gamma))),$$

if and only if, by the algebraic invertibility conditions on the interpretations $\langle F, \alpha \rangle, \langle G, \beta \rangle$, for all $j < n$, there exists $i_j \in I^{(n)}$, such that

$$\beta_{G(F(\Sigma))}(E^{n,i_j}_{F(\Sigma)}(\alpha^0_\Sigma(\phi), \ldots, \alpha^{n-1}_\Sigma(\phi), \alpha^j_\Sigma(\psi))) \subseteq C_\Sigma(\Gamma).$$

Therefore, if we define $\mathcal{D} = \{D^i : i \in {I^{(n)}}^n\}$, with, for all $i = \langle i_0, \ldots, i_n \rangle \in {I^{(n)}}^n$, $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$D^i_\Sigma(\phi, \psi) = \beta_{G(F(\Sigma))}(\bigcup_{j<n} E^{n,i_j}_{F(\Sigma)}(\alpha^0_\Sigma(\phi), \ldots, \alpha^{n-1}_\Sigma(\phi), \alpha^j_\Sigma(\psi))),$$

we will have $\psi \in C_\Sigma(\Gamma, \phi)$ if and only if $D^i_\Sigma(\phi, \psi) \subseteq C_\Sigma(\Gamma)$, for some $i \in {I^{(n)}}^n$. Thus, it only remains to show that D^i is a finite set of natural transformations in N, for every $i \in {I^{(n)}}^n$. But this follows immediately by Lemma 17 together with Condition 3 of the definition of algebraic equivalence.

That, if $\mathcal{I}$ has the LDDT, then $\mathcal{I}'$ also has the LDDT, can now be proven by a symmetric argument. ■

Acknowledgements

Thanks go to Janusz Czelakowski and to Don Pigozzi, whose joint work with Wim Blok on the deduction theorems in abstract algebraic logic inspired the current developments on the categorical side of the field. Thanks go also to Josep Maria Font and to Ramon Jansana for providing inspiration, encouragement and support over many years. Finally, warm thanks to Charles Wells and Giora Slutzki for guidance and support.

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