On the Decidability of Role Mappings between Modular Ontologies

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Abstract

Many semantic web applications require support for mappings between roles (or properties) defined in multiple independently developed ontology modules. Distributed Description Logics (DDL) and Package-based Description Logics (P-DL) offer alternative logical formalisms that support such mappings. We prove that (a) variants of DDL that allow negated roles or cardinality restrictions in bridge rules or inverse bridge rules that connect $\mathcal{ALC}$ ontologies are undecidable; (b) a variant of P-DL $\mathcal{ALCHIO}(\neg)\mathcal{P}$ that support role mappings between ontology modules in $\mathcal{ALCHIO}(\neg)$ (an extension of $\mathcal{ALC}$ that allows general role inclusions, inverse roles, nominals and negated roles) is decidable.

Introduction

Ontologies play a central role in current efforts aimed at developing a semantic web by enriching the web with machine interpretable content and interoperable resources and services. In such a setting, instead of a single, centralized ontology, it is much more realistic to have multiple, independently developed, distributed ontology modules that cover different, perhaps partially overlapping, domains of expertise. However, many application scenarios require selective, and perhaps context-sensitive use of knowledge from multiple ontology modules with the help of *ontology mappings*. For example, consider two ontology modules O_1 and O_2 describing monarchies and people respectively. Suppose O_1 contains the concept King and a role (binary relation between individuals) $\mathsf{marriedto}$; and O_2 contains the concept Male and a role knows. We may want to assert mappings like "any individual who is a King (as defined in O_1) is a Male (as defined in O_2)", and "if any pair of individuals that belongs the $\mathsf{marriedto}$ relation (as defined in O_1) is also a member of the knows relation (as defined in O_2)".

Against this background, several frameworks for mappings between ontology modules have been explored in the literature. Typically such a framework is expressed in some decidable family of description logics (DL) augmented with constructs that allow some way of connecting symbols across ontology modules . Examples include distributed description logics (DDL)(Borgida & Serafini 2002; Ghidini & Serafini 2006), $\mathcal{E}$-Connections (Grau, Parsia, & Sirin 2004), package-based description logics (P-DL) (Bao, Slutzki, & Honavar 2007) and semantic binding (Zhao *et al.* 2007). Among those proposals, DDL and P-DL currently provide the ability for mappings between roles.

The focus of this paper is on variants of modular ontologies that support mappings (e.g., inter-module inclusion relationships) between *roles* defined in different ontology modules. In particular, we explore the decidability of a network of ontology modules interconnected via role mappings, as decidability is a necessary prerequisite for automated reasoning. Recent work has explored several decidable fragments of DDL (Ghidini & Serafini 2006) and P-DL (Bao, Slutzki, & Honavar 2007). However, little is known about the the precise conditions under which a collection of ontology modules (each expressed in a decidable fragment of DL) linked by role mappings between ontology modules is decidable. Against this background, in this paper we explore the decidability of several variants of DDL wherein each of the individual ontology modules is expressible in $\mathcal{ALC}$, a useful, decidable fragment of DL. We show that if role mappings are combined with some otherwise useful features including negated roles, cardinality restrictions in bridge rules, or inverse bridge rules, they yield an undecidable DDL. We also establish the decidability of P-DL with unrestricted role inclusion between ontology modules when each module is in $\mathcal{ALCHIO}(\neg)$, a language that extends $\mathcal{ALC}$ with general role inclusions, inverse roles, nominals and negated roles.

Preliminaries: DDL and P-DL

In this section, we briefly introduce the syntax and semantics of DDL and P-DL.

DDL

Given a non empty set I of indices, a DDL distributed TBox is of the form $\langle\{\mathcal{T}_i\},\{B_{ij}\}_{i\neq j}\rangle$, where each $\mathcal{T}_i$ is a DL TBox, and each B_{ij} is the collection of bridge rules from $\mathcal{T}_i$ to $\mathcal{T}_j$. In (Ghidini & Serafini 2006), each module $\mathcal{T}_i$ is assumed to be in $\mathcal{SHIQ}$. A bridge rule from i to j is an expression in either one of the two forms:

- (into bridge rule) $i : X \xrightarrow{\sqsubseteq} j : Y$

- (onto bridge rule) $i : X \xrightarrow{\sqsupseteq} j : Y$

where $i : X$ is a concept of $\mathcal{T}_i$, $j : Y$ is a concept of $\mathcal{T}_j$, or $i : X$ is a role of $\mathcal{T}_i$, $j : Y$ is a role of $\mathcal{T}_j$.

For example, a role mapping in DDL could be i : $\mathsf{marries} \xrightarrow{\sqsubseteq} j : \mathsf{knows}$ to indicate that every pair in the relation $\mathsf{marries}$ is also in the relation knows. The two roles $\mathsf{marries}$ and knows are in different ontologies.

The semantics of DDL assigns to each $\mathcal{T}_i$ a *local interpretation domain* $\Delta^{\mathcal{I}_i}$. A *domain relation* r_{ij} is a subset of $\Delta^{\mathcal{I}_i} \times \Delta^{\mathcal{I}_j}$. For $d \in \Delta^{\mathcal{I}_i}$, we use $r_{ij}(d)$ to denote $\{d' \in \Delta^{\mathcal{I}_j} | \langle d, d' \rangle \in r_{ij}\}$. For any subset D of $\Delta^{\mathcal{I}_i}$, we use $r_{ij}(D)$ to denote $\bigcup_{d \in D} r_{ij}(d)$. For any $R \in \Delta^{\mathcal{I}_i} \times \Delta^{\mathcal{I}_i}$, we use $r_{ij}(R)$ to denote $\bigcup_{\langle d,d' \rangle \in R} r_{ij}(d) \times r_{ij}(d')$. For any x, y, if $y \in r_{ij}(x)$, we say that x is a preimage of y and y is an image of x.

The domain relation r_{ij} satisfies a bridge rule in B_{ij} according to the following rules:

- $i : X \xrightarrow{\sqsubseteq} j : Y$: if $r_{ij}(X^{\mathcal{I}_i}) \subseteq Y^{\mathcal{I}_j}$
- $i : X \xrightarrow{\sqsupseteq} j : Y$: if $r_{ij}(X^{\mathcal{I}_i}) \supseteq Y^{\mathcal{I}_j}$

A distributed interpretation $\mathcal{J} = \langle \{\mathcal{I}_i\}_{i \in I}, \{r_{ij}\}_{i \neq j} \rangle$ satisfies a DDL distributed TBox $\Sigma = \langle \{\mathcal{T}_i\}, \{B_{ij}\}_{i \neq j} \rangle$, denoted $\mathcal{J} \models \Sigma$, if, for every i, $\mathcal{I}_i \models \mathcal{T}_i$ and for every $i \neq j$, r_{ij} satisfies all bridge rules in B_{ij}. Concept $i : C$ is satisfiable with respect to Σ if there is a $\mathcal{J}$ such that $\mathcal{J} \models \Sigma$, and $C^{\mathcal{I}_i} \neq \emptyset$.

For convenience, we introduce the naming system of DDL languages. For each DDL language, its name is the concatenation of a DL language, of which each local TBox is a subset, followed by the letter $\mathcal{D}$. In particular, we use $\mathcal{D}_{\mathcal{C}}$ to denote DDLs that allow bridge rules between concepts and $\mathcal{D}_{\mathcal{CR}}$ to denote DDLs that allow bridge rules between concepts and between roles. For example, $\mathcal{ALCD}_{\mathcal{CR}}$ stands for a DDL language that supports bridge rules between concepts and between roles, and each module of which is in a language weaker or equivalent to the DL $\mathcal{ALC}$.

Reductions from $\mathcal{SHIQD}_{\mathcal{C}}$ and $\mathcal{SHIQD}_{\mathcal{CR}}$ to $\mathcal{SHIQ}$ have been given in (Borgida & Serafini 2002) and (Ghidini & Serafini 2006), respectively. The decidability of $\mathcal{SHIQ}$, combined with these reductions, immediately implies

Proposition 1 *The* DDL*s $\mathcal{SHIQD}_{\mathcal{C}}$ and $\mathcal{SHIQD}_{\mathcal{CR}}$ are decidable.*

P-DL

P-DL allows role mappings by using a semantic importing approach. A P-DL ontology is a set $\{\mathcal{T}_i\}$, where each $\mathcal{T}_i$ is a *package*. The signature of each package P_i is divided into two disjoint sets: its local signature $\mathsf{Loc}(P_i)$ and its external signature $\mathsf{Ext}(P_i)$. If a name $X \in \mathsf{Loc}(P_i) \cap \mathsf{Ext}(P_j)$, we say that P_j imports (X from) P_i. P_i's importing transitive closure, including itself, is denoted as P^*.

Each package may contain a set of concept inclusions and a set of role inclusions. Concepts and roles in each package may be constructed starting from atomic concepts and atomic roles in the usual recursive way. The major difference from ordinary DL is that, for a P-DL package P_i, the top concept ($\top$) and negation ($\neg$) are replaced by a contextualized top $\top_i$ and a contextualized negation $\neg_i$. A package P_i may use $\top_k$ and $\neg_k$ in constructing its concept expressions only if P_i imports P_k.

Role mappings are supported by P-DL with unrestricted role inclusions and role importing. For example, suppose package P_j imports the role i : $\mathsf{marries}$ from package P_i. Then a role mapping can be represented as a local role inclusion $i : \mathsf{marries} \sqsubseteq j : \mathsf{knows}$ in P_j.

The naming of P-DL languages is similar to that of DDL. We use $\mathcal{P}$ to denote the package extension. For example, $\mathcal{ALCOP}$ is a P-DL language that allows the importing of concept, role and nominal names between $\mathcal{ALCO}$ modules.

For a P-DL ontology $\Sigma = \{\mathcal{T}_i\}$, an interpretation of Σ is a pair $\mathcal{I} = \langle \{\mathcal{I}_i\}, \{r_{ij}\}_{P_i \in P_j^*} \rangle$. Each of the local interpretations $\mathcal{I}_i = \langle \Delta^{\mathcal{I}_i}, \cdot^{\mathcal{I}_i} \rangle$ interprets each concept expression in P_i starting from assigned interpretations of atomic (concept, role and nominal) names. For example, concept negation and existential restriction are interpreted as

$$
\begin{aligned}
(\neg_j C)^{\mathcal{I}_i} &= r_{ji}(\Delta^{\mathcal{I}_j}) \backslash C^{\mathcal{I}_i} \\
(\exists R.C)^{\mathcal{I}_i} &= \{x \in r_{ki}(\Delta^{\mathcal{I}_i}) | \exists y \in \Delta^{\mathcal{I}_i}, (x, y) \in R^{\mathcal{I}_i} \\
&\qquad \wedge\, y \in C^{\mathcal{I}_i}\},
\end{aligned}
$$

where R is a k-role name and C, D are concepts.

An interpretation $\mathcal{I}$ is a *model* of $\Sigma = \{P_i\}$ if $\bigcup_i \Delta^{\mathcal{I}_i} \neq \emptyset$ and the following conditions are satisfied.

1. For all i, j, such that $P_i \in P_j^*$, r_{ij} is one-to-one;
2. *Compositional Consistency*: For all i, j, k, $P_i \in P_k^*$ and $P_k \in P_j^*$, we have $\rho_{ij} = r_{ij} = r_{kj} \circ r_{ik}$, where ρ_{ij} is the projection on $\Delta^{\mathcal{I}_i} \times \Delta^{\mathcal{I}_j}$ of the equivalence relation on $\bigcup_i \Delta^{\mathcal{I}_i}$ generated by $\bigcup_{i \in P_j^*} r_{ij}$;
3. For every name $X \in \mathsf{Loc}(P_i) \cap \mathsf{Ext}(P_j)$, $r_{ij}(X^{\mathcal{I}_i}) = X^{\mathcal{I}_j}$;
4. *Cardinality Preservation*: For every role name $R \in \mathsf{Loc}(P_i) \cap \mathsf{Ext}(P_j)$ and every $(x, x') \in r_{ij}$, we have $(x, y) \in R^{\mathcal{I}_i}$ iff $(x', r_{ij}(y)) \in R^{\mathcal{I}_j}$;
5. $\mathcal{I}_i \models P_i$, for every i.

A concept C is *satisfiable as witnessed by* P_w if there is a model of P_w^*, such that $C^{\mathcal{I}_w} \neq \emptyset$.

In (Bao, Slutzki, & Honavar 2007), a P-DL $\mathcal{SHOIQP}$ was presented, which allows both role importing and role inclusions. However, the decidability proof of $\mathcal{SHOIQP}$ relies on a reduction to the DL $\mathcal{SHOIQ}$, which is only possible when imported roles do not appear in role inclusions. We will denote this restricted version of $\mathcal{SHOIQP}$, i.e., in which role inclusion may be applied only between two local role names (or their inverses), as $\mathcal{SH}^{-}\mathcal{OIQP}$. The reduction presented in (Bao, Slutzki, & Honavar 2007) shows that

Proposition 2 P-DL *$\mathcal{SH}^{-}\mathcal{OIQP}$ is decidable.*

However, the decidability of P-DLs that support unrestricted role inclusion ("$\mathcal{H}$") is still an open problem.

Undecidable Extensions of DDL

In this section, we will investigate the decidability of several useful extensions of DDL. Each extension is obtained by alternatively considering each of the following features:

- Role negation (denoted as $(\neg)$): Allows negated roles to be used in local TBoxes and bridge rules. A negated role $\neg R$ in a TBox $\mathcal{T}_i$ is interpreted as $(\Delta^{\mathcal{I}_i} \times \Delta^{\mathcal{I}_i}) \backslash R^{\mathcal{I}_i}$.
- Inverse bridge rules (denoted as $\mathcal{D}_{\mathcal{I}}$): Allow bridge rules in both directions. An into inverse bridge rule $i : X \xleftarrow{\sqsubseteq} j : Y$ has the semantics $X^{\mathcal{I}_i} \subseteq r_{ij}^{-}(Y^{\mathcal{I}_j})$, and an onto inverse bridge rule $i : X \xleftarrow{\sqsupseteq} j : Y$ has the semantics $X^{\mathcal{I}_i} \supseteq r_{ij}^{-}(Y^{\mathcal{I}_j})$.
- Cardinality restrictions on domain relations (denoted as $\mathcal{D}_{\mathcal{N}}$): Allow bridge rules of the form $\xrightarrow{\bowtie n} G$ (where $\bowtie \in \{\leq, \geq, =\}$) in B_{ij} to indicate that for any $x \in G^{\mathcal{I}_j}$, $|r_{ij}^{-}(x)| \bowtie n$.

We show that $\mathcal{ALC}(\neg)\mathcal{D}_{\mathcal{CR}}$, $\mathcal{ALCD}_{\mathcal{CRN}}$ and $\mathcal{ALCD}_{\mathcal{CRI}}$ are all undecidable. All proofs are by a reduction of the undecidable domino tiling problem (Berger 1966) to a concept satisfiability problem in DDL.

Definition 1 (Domino System) *A domino system $\mathfrak{D} = (D, H, V)$ consists of a non-empty set of domino types $D = \{D_1, ..., D_n\}$, a horizontal matching condition $H \subseteq D \times D$ and a vertical matching condition $V \subseteq D \times D$. The problem is to determine if, for a given $\mathfrak{D}$, there exists a tiling of the infinite $\mathbb{N} \times \mathbb{N}$ grid, such that each of its points is covered with a domino type in D and all horizontally and vertically adjacent pairs of domino types are in H and in V, respectively. In other words, a solution to the problem is a mapping $t : \mathbb{N} \times \mathbb{N} \to D$, such that, for all $m, n \in \mathbb{N}$, $\langle t(m,n), t(m+1,n)\rangle \in H$ and $\langle t(m,n), t(m,n+1)\rangle \in V$.*

Undecidability of $\mathcal{ALC}(\neg)\mathcal{D}_{\mathcal{CR}}$

We first show that the DDL $\mathcal{ALC}(\neg)\mathcal{D}_{\mathcal{CR}}$, i.e., $\mathcal{ALCD}_{\mathcal{CR}}$ extended with role negations, is undecidable. The reduction is accomplished by the construction of an $\mathcal{ALC}(\neg)\mathcal{D}_{\mathcal{CR}}$ ontology Σ_1, such that a solution to the domino system can be constructed from a model of Σ_1 and vice versa. Let $\mathfrak{D} = (D, H, V)$ be a domino system. Construct an $\mathcal{ALC}(\neg)\mathcal{D}_{\mathcal{CR}}$ ontology $\Sigma_1 = \langle\{\mathcal{T}_1, \mathcal{T}_2\}, \{\mathcal{B}_{12}, \mathcal{B}_{21}\}\rangle$, where the local signature of $\mathcal{T}_k$ consists of a role name v_k and a concept name D_i^k for each $D_i \in D$, $k = 1, 2$. $\mathcal{T}_k$ consists of the following concept inclusions:

$$\top_k \sqsubseteq \bigsqcup_{1\leq i\leq n}\big(D_i^k \sqcap (\bigsqcap_{j\neq i} \neg D_j^k)\big) \qquad (1)$$

$$D_i^k \sqsubseteq \exists v_k.\top_k \sqcap \forall v_k. \bigsqcup_{(D_i,D_j)\in V} D_j^k, \quad \forall i \qquad (2)$$

$\mathcal{B}_{k,3-k}$ contains bridge rules:

$$\neg\big(\bigsqcup_{(D_i,D_j)\in H} D_j^k\big) \xrightarrow{\sqsubseteq} \neg D_i^{3-k} \quad \forall i \qquad (3)$$

$$v_k \xrightarrow{\sqsupseteq} v_{3-k} \qquad (4)$$

$$\neg v_k \xrightarrow{\sqsubseteq} \neg v_{3-k} \qquad (5)$$

Intuitively, Σ_1 contains subsumptions and bridge rules that ensure that each of its models encodes a grid structure corresponding to a solution of the tiling problem $\mathfrak{D}$. The structure (see Figure 1) has alternating columns that belong to the local domains of $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. All vertical edges represent interpretations of local roles (v_1 and v_2) and all horizontal edges represent domain relations (r_{12} and r_{21}). More precisely, Axiom (1) states that, in each local domain, every individual belongs to one and only one type. Axiom (2) ensures that every individual has a vertical successor and all vertical successor relations satisfy the vertical matching condition V. Axiom (3) enforces the horizontal matching condition H. Axiom (4) ensures that every individual has a horizontal successor. Finally, Axiom (5) puts a finishing touch to the grid by closing some gaps.

Lemma 1 *$\mathfrak{D}$ has a solution iff $\top_1$ is satisfiable in Σ_1.*

Proof sketch: Clearly, if $\mathfrak{D}$ has a solution, it corresponds to a model of Σ_1 with $\top_1^{\mathcal{I}_1} \neq \emptyset$. We only need to show the other direction. Suppose there is a model of Σ_1 such that $\top_1^{\mathcal{I}_1} \neq \emptyset$ (see Figure 1). Let $x_{0,0} \in \top_1^{\mathcal{I}_1}$. Then, according to Σ_1, $x_{0,0}$ belongs to one and only one type $D_{0,0}^1$ (Axiom 1) and has a v_1 (vertical) successor $x_{0,1}$ (Axiom 2), which belongs to one and only one type $D_{0,1}^1$ (Axiom 1), and $(D_{0,0}, D_{0,1}) \in V$, i.e., the vertical matching condition is satisfied (Axiom 2). According to Axiom 4, there must be a pair $\langle x_{1,0}, x_{1,1}\rangle \in v_2^{\mathcal{I}_2}$ in domain $\Delta^{\mathcal{I}_2}$, such that $\langle x_{1,0}, x_{0,0}\rangle, \langle x_{1,1}, x_{0,1}\rangle \in r_{21}$. Let $D_{1,0}^2$ be the type of $x_{1,0}$. Then, according to Axiom 3, $(D_{0,0}, D_{1,0}) \in H$, i.e., the horizontal matching condition is satisfied. By a similar analysis, it is easy to see that all edges in the grid structure satisfy the vertical and the horizontal matching conditions. According to Axiom 1, $x_{0,1}$ has a v_1 successor $x_{0,2}$, and $\langle x_{0,1}, x_{0,2}\rangle$ has a preimage $\langle x'_{1,1}, x_{1,2}\rangle \in v_2^{\mathcal{I}_2}$. Note that $x'_{1,1}$ and $x_{1,1}$ are not required to be same. Let us assume that $\langle x_{1,1}, x_{1,2}\rangle \notin v_2^{\mathcal{I}_2}$, i.e., $\langle x_{1,1}, x_{1,2}\rangle \in (\neg v_2)^{\mathcal{I}_2}$. Then, according to Axiom 5, $\langle x_{0,1}, x_{0,2}\rangle \in (\neg v_1)^{\mathcal{I}_1}$, which contradicts that $\langle x_{0,1}, x_{0,2}\rangle \in v_1^{\mathcal{I}_1}$. Therefore, $\langle x_{1,1}, x_{1,2}\rangle \in v_2^{\mathcal{I}_2}$. Thus, the second square in the grid is finished. By similar constructions along both the vertical and the horizontal direction, we can extract from a model of Σ_1, with $\top_1^{\mathcal{I}_1} \neq \emptyset$, a grid that corresponds to a solution of $\mathfrak{D}$. Q.E.D.

An immediate consequence of Lemma 1 is that:

Theorem 1 *The* DDL *$\mathcal{ALC}(\neg)\mathcal{D}_{\mathcal{CR}}$ is undecidable.*

Undecidability of $\mathcal{ALCD}_{\mathcal{CRI}}$

Inverse bridge rules are useful when backward propagation of knowledge between ontology modules is needed. They may also help in avoiding several modeling problems related to the fact that, in $(\cdot)\mathcal{D}_{\mathcal{CR}}$, domain relations can be empty sets and, hence, do not transfer information (Stuckenschmidt, Serafini, & Wache 2006). For example, an inverse bridge rule $\top_i \xleftarrow{\sqsubseteq} \top_j$ requires that every individual in the local domain i has at least one image in the local domain j.

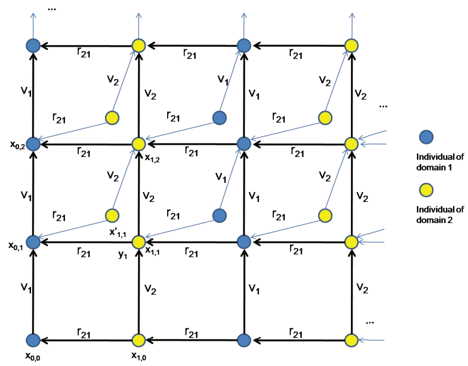

Figure 1: Undecidability of Several DDL Extensions

However, we show that extending the DDL $\mathcal{ALCD}_{\mathcal{CR}}$ with inverse bridge rules between roles also leads to undecidability. The proof is based on a similar reduction from the domino tiling problem $\mathfrak{D}$ to the concept satisfiability problem in an $\mathcal{ALCD}_{\mathcal{CRI}}$ ontology. In fact, such an ontology $\Sigma_2 = \langle\{\mathcal{T}_1, \mathcal{T}_2\}, \{\mathcal{B}_{12}, \mathcal{B}_{21}\}\rangle$ is constructed using Axioms (1)-(4) together with the following two inverse bridge rules that help enforce the grid structure (for $k = 1, 2$):

$$v_k \quad \overset{\sqsupseteq}{\longleftarrow} \quad v_{3-k} \tag{6}$$

Intuitively, Axiom (6) requires that every preimage of an instance of v_k be an instance of v_{3-k}.

Lemma 2 *$\mathfrak{D}$ has a solution iff $\top_1$ is satisfiable in Σ_2.*

Proof sketch: It is easy to see that a solution of $\mathfrak{D}$ corresponds to a model of Σ_2 with $\top_1^{\mathcal{I}_1} \neq \emptyset$. For the other direction, we only need to show that Axiom (6) indeed enforces the grid structure. This can be done by a similar construction to that used for $\mathcal{ALC}(\neg)\mathcal{D}_{\mathcal{CR}}$. In Figure 1, suppose that the boxes $(x_{0,0}, x_{0,1}, x_{1,0}, x_{1,1})$ and $(x_{0,1}, x_{0,2}, x'_{1,1}, x_{1,2})$ have already been constructed. Then, according to Axiom (6), $\langle x_{1,1}, x_{1,2}\rangle \in v_2^{\mathcal{I}_2}$, which completes the box $(x_{0,1}, x_{0,2}, x_{1,1}, x_{1,2})$. Employing a similar argument, we can complete the tiling of the entire plane using the given model of Σ_2. Q.E.D.

Thus, by Lemma 1, we obtain

Theorem 2 *The* DDL $\mathcal{ALCD}_{\mathcal{CRI}}$ *is undecidable.*

Undecidability of $\mathcal{ALCD}_{\mathcal{CRN}}$

Cardinality restrictions on domain relations have been proposed as a useful feature by several authors. In (Serafini, Borgida, & Tamilin 2005), introduction of bridge rules of the form $\xrightarrow{\leq 1} G$ has been proposed to avoid the problem of incomplete modeling by enforcing the partial injectivity of domain relations relative to the concept G. Cardinality restrictions on inter-domain relations are also provided by $\mathcal{E}$-Connections (Grau, Parsia, & Sirin 2004), another modular ontology language, to express number restrictions, such as "1 : DogOwner (a concept in ontology 1) owns at least one 2 : Dog (another concept in ontology 2)". Unfortunately, extending $\mathcal{ALCD}_{\mathcal{CR}}$ with cardinality restrictions on domain relations, thus forming $\mathcal{ALCD}_{\mathcal{CRN}}$, again results in undecidability.

The undecidability proof for $\mathcal{ALCD}_{\mathcal{CRN}}$ is also by a reduction of the domino tiling problem. Let $\mathfrak{D} = (D, H, V)$ be a domino system. An $\mathcal{ALCD}_{\mathcal{CRN}}$ ontology $\Sigma_3 = \langle\{\mathcal{T}_1, \mathcal{T}_2\}, \{\mathcal{B}_{12}, \mathcal{B}_{21}\}\rangle$ contains Axioms (1)-(4) together with the following bridge rule in $\mathcal{B}_{3-k,k}$ (for $k = 1, 2$):

$$\xrightarrow{\leq 1} \top_k \tag{7}$$

Axiom (7) means that every individual in the local domain $\Delta^{\mathcal{I}_k}$ has at most one preimage in the local domain $\Delta^{\mathcal{I}_{3-k}}$. This axiom ensures that a model of Σ_3 contains an encoding of a grid structure, resulting in a tiling of the plane.

Lemma 3 *$\mathfrak{D}$ has a solution iff $\top_1$ is satisfiable in Σ_3.*

Proof sketch: We use again Figure 1 to illustrate the proof. The main difference from the previous two proofs lies in relying on the inverse functionality of domain relations to complete the grid. For example, if $x_{0,1}$ has two preimages $x_{1,1}$ and $x'_{1,1}$, then, according to Axiom (7), they must be the same individual. Hence, the edge $\langle x_{1,1}, x_{1,2}\rangle = \langle x'_{1,1}, x_{1,2}\rangle \in v_2^{\mathcal{I}_2}$. Thus, we can tile an infinite plane using a grid construction in this way. Q.E.D.

This lemma, together with the undecidability of the domino tiling problem, yields

Theorem 3 *The* DDL $\mathcal{ALCD}_{\mathcal{CRN}}$ *is undecidable.*

Decidable P-DL Family $\mathcal{ALCHIO}(\neg)\mathcal{P}$

The P-DL Family $\mathcal{ALCHIO}(\neg)\mathcal{P}$

In this section, we show that the two P-DLs $\mathcal{ALCHIO}(\neg)_{\mathcal{CR}}\mathcal{P}$ and $\mathcal{ALCHIO}(\neg)_{\mathcal{R}}\mathcal{P}$, constituting the family $\mathcal{ALCHIO}(\neg)\mathcal{P}$, extending the P-DL $\mathcal{ALCP}$ with role importing, general role inclusions (thus, supporting role mappings between ontologies), inverse roles, nominals, nominal importing, and negation on roles are decidable. The syntax of both P-DLs in $\mathcal{ALCHIO}(\neg)\mathcal{P}$ can be obtained from $\mathcal{ALCHIOP}$ with (contextualized) negations on roles. Thus, roles of a package P_j in both P-DLs in $\mathcal{ALCHIO}(\neg)\mathcal{P}$ are defined inductively by the following grammar:

$$R := p|R^-|\neg_k R$$

where p is a local or imported role name, and P_j imports P_k. A role of the form $\neg_k R$ is called a *k-negated role*. The semantics of role negation is given by $(\neg_k R)^{\mathcal{I}_j} = (r_{kj}(\Delta^{\mathcal{I}_k}) \times r_{kj}(\Delta^{\mathcal{I}_k}))\backslash R^{\mathcal{I}_j}$.

Depending on whether negated roles can be used or not in concept inclusions, the two members of the family $\mathcal{ALCHIO}(\neg)\mathcal{P}$ are given by:

- $\mathcal{ALCHIO}(\neg)_{\mathcal{CR}}\mathcal{P}$: negated roles can be used in both concept and role inclusions. If an i-role name P is imported by P_j, we require that the cardinality preservation condition holds for both P and $\neg_i P$.
- $\mathcal{ALCHIO}(\neg)_{\mathcal{R}}\mathcal{P}$: negated roles can only be used in role inclusions. In this variant, we only require cardinality preservation for imported role names but not their negations.

Consideration of these two P-DLs and the respective conditions imposed in each case are motivated by the desire to achieve transitive reusability of knowledge using a *minimal* set of restrictions on domain relations between local models.

The decidability proofs of the P-DLs in $\mathcal{ALCHIO}(\neg)\mathcal{P}$ use a reduction to the decidable DL $\mathcal{ALBO}$ (Schmidt & Tishkovsky 2007). The logic $\mathcal{ALBO}$ extends $\mathcal{ALC}$ with boolean role operators, role inclusions, inverses of roles, domain and range restriction operators and nominals.

In $\mathcal{ALBO}$, roles are defined inductively by the following grammar:

$$R := p|R \sqcap R|R^-|\neg R|(R \downharpoonleft C)|(R \upharpoonleft C)$$

where p is a role name and C is a concept. The semantics of $\mathcal{ALBO}$ is defined as an extension of that of $\mathcal{ALCHIO}$ with the following additional constraints on interpretations (where $\Delta^\mathcal{I}$ is the interpretation domain):

$$\begin{aligned}(\neg R)^\mathcal{I} &= (\Delta^\mathcal{I} \times \Delta^\mathcal{I})\backslash R^\mathcal{I}\\ (R \sqcap S)^\mathcal{I} &= R^\mathcal{I} \cap S^\mathcal{I}\\ (R \downharpoonleft C)^\mathcal{I} &= R^\mathcal{I} \cap (\Delta^\mathcal{I} \times C^\mathcal{I})\\ (R \upharpoonleft C)^\mathcal{I} &= R^\mathcal{I} \cap (C^\mathcal{I} \times \Delta^\mathcal{I})\end{aligned}$$

We use the abbreviation $R \updownarrow C = (R \downharpoonleft C) \upharpoonleft C$.

Decidability of P-DL $\mathcal{ALCHIO}(\neg)_{\mathcal{CR}}\mathcal{P}$

A reduction $\Re$ from an $\mathcal{ALCHIO}(\neg)_{\mathcal{CR}}\mathcal{P}$ KB $\Sigma_d = \{P_i\}$ to an $\mathcal{ALBO}$ KB Σ can be established based on the reduction of P-DL $\mathcal{SHOIQP}$ to $\mathcal{SHOIQ}$, as presented in (Bao *et al.* 2008), with a couple of modifications to handle role inclusions: $\#_j()$ is also applied to roles and that a negated local domain and a negated local range axiom for roles are added to the $\mathcal{ALBO}$ KB Σ.

- The signature of Σ is the union of the local signatures of the component packages together with a global top $\top$, a global bottom $\bot$ and local top concepts $\top_i$, for all i, i.e., $\mathsf{Sig}(\Sigma) = \bigcup_i \mathsf{Loc}(P_i) \cup \{\top_i\} \cup \{\top, \bot\}$.
- For all i, j, k such that $P_i \in P_k^*$, $P_k \in P_j^*$, $\top_i \sqcap \top_j \sqsubseteq \top_k$ is added to Σ.
- For each GCI or role inclusion $X \sqsubseteq Y$ in P_j, $\#_j(X) \sqsubseteq \#_j(Y)$ is added to Σ. The mapping $\#_j()$ is defined below.
- For each i-concept name or i-nominal name C in P_i, $i : C \sqsubseteq \top_i$ is added to Σ.
- For each i-role name R in P_i, its domain and range is $\top_i$, i.e., $\top \sqsubseteq \forall R^-.\top_i$ and $\top \sqsubseteq \forall R.\top_i$ are added to Σ.
- For each i-role name R in P_j, the following axioms are added to Σ:
  - $\exists R.\top_j \sqsubseteq \top_j$; (local domain)
  - $\exists R^-.\top_j \sqsubseteq \top_j$; (local range)
  - $\exists((\neg R) \updownarrow \top_i).\top_j \sqsubseteq \top_j$; (negated local domain)
  - $\exists((\neg R) \updownarrow \top_i)^-.\top_j \sqsubseteq \top_j$; (negated local range)

For a formula X used in P_j, $\#_j(X)$ is:

- X, for a j-(concept, role or nominal) name.
- $X \sqcap \top_j$, for an i-concept name or an i-nominal name X.
- $X \updownarrow \top_j$, for an i-role name.
- $\#_j(Y)^-$, for a role $X = Y^-$.
- $\neg\#_j(X) \sqcap \top_i \sqcap \top_j$, for $\neg_i X$, where X is a concept.
- $\neg\#_j(Y) \updownarrow (\top_i \sqcap \top_j)$, for a role $X = \neg_i Y$.
- $(\#_j(X_1) \oplus \#_j(X_2)) \sqcap \top_j$, for a concept $X = X_1 \oplus X_2$, where $\oplus = \sqcap$ or $\oplus = \sqcup$.
- $(\oplus\#_j(R).\#_j(X')) \sqcap \top_i \sqcap \top_j$, for a concept $X = (\oplus R.X')$, where $\oplus \in \{\exists, \forall\}$ and R is an i-role or an i-negated role.

The following lemma shows that the consistency problem in $\mathcal{ALCHIO}(\neg)\mathcal{P}$ can be reduced to the concept satisfiability problem in $\mathcal{ALBO}$:

Lemma 4 *An $\mathcal{ALCHIO}(\neg)_{\mathcal{CR}}\mathcal{P}$ KB Σ is consistent as witnessed by a package P_w if and only if $\top_w$ is satisfiable with respect to $\Re(P_w^*)$.*

Proof: The proof is similar to the proof of Theorem 1 in (Bao *et al.* 2008). The main modification concerns the reduction of role inclusion axioms. The basic idea is that, given a distributed model of Σ, we can construct an ordinary model of $\Re(P_w^*)$ by "merging" individuals connected by domain relations. Given a model of $\Re(P_w^*)$, we can construct a distributed model of Σ by "copying shared individuals" into local interpretation domains.

For the "if" direction, if $\top_w$ is satisfiable with respect to $\Re(P_w^*)$, then $\Re(P_w^*)$ has at least one model $\mathcal{I} = \langle \Delta^\mathcal{I}, \cdot^\mathcal{I} \rangle$, such that $\top_w^\mathcal{I} \neq \emptyset$. Our goal is to construct a model of P_w^* from $\mathcal{I}$, such that $\Delta^{\mathcal{I}_w} \neq \emptyset$. For each package P_i, a local interpretation $\mathcal{I}_i$ is constructed in the following way:

- $\Delta^{\mathcal{I}_i} = \top_i^\mathcal{I}$.
- For every concept name C in P_i, $C^{\mathcal{I}_i} = C^\mathcal{I} \cap \top_i^\mathcal{I}$.
- For every role name R in P_i, $R^{\mathcal{I}_i} = R^\mathcal{I} \cap (\top_i^\mathcal{I} \times \top_i^\mathcal{I})$.
- For every nominal name o that appears in P_i, $o^{\mathcal{I}_i} = o^\mathcal{I}$.

For every pair i, j, such that $P_i \in P_j^*$, we define

$$r_{ij} = \{(x, x)|x \in \Delta^{\mathcal{I}_i} \cap \Delta^{\mathcal{I}_j}\}.$$

Clearly, we have $\Delta^{\mathcal{I}_w} = \top_w^\mathcal{I} \neq \emptyset$, by the hypothesis. So it suffices to show that $\langle\{\mathcal{I}_i\}, \{r_{ij}\}_{P_i \in P_j^*}\rangle$ is a model of the modular ontology P_w^*. The proof is similar to (Bao *et al.* 2008). We will only show that if $\#_j(X) \sqsubseteq \#_j(Y)$ is satisfied by $\mathcal{I}$, then $X \sqsubseteq Y$ is satisfied by $\mathcal{I}_j$. To accomplish this, it suffices to show that for any role X in the signature of P_j, $\#_j(X)^\mathcal{I} = X^{\mathcal{I}_j}$:

- If X is a j-role name, $\#_j(X)^\mathcal{I} = X^{\mathcal{I}_j}$ by definition.
- If X is a i-role name, $i \neq j$, $\#_j(X)^\mathcal{I} = (X \updownarrow \top_j)^\mathcal{I} = X^\mathcal{I} \cap (\Delta^{\mathcal{I}_j} \times \Delta^{\mathcal{I}_j}) = X^{\mathcal{I}_j}$.
- If $X = Y^-$ and $\#_j(Y)^\mathcal{I} = Y^{\mathcal{I}_j}$, then $\#_j(X)^\mathcal{I} = (\#_j(Y)^-)^\mathcal{I} = (\#_j(Y)^\mathcal{I})^- = (Y^{\mathcal{I}_j})^- = (Y^-)^{\mathcal{I}_j} = X^{\mathcal{I}_j}$.
- If $X = \neg_i Y$ and $\#_j(Y)^\mathcal{I} = Y^{\mathcal{I}_j}$, then $\#_j(X)^\mathcal{I} = (\neg\#_j(Y) \updownarrow (\top_i \sqcap \top_j))^\mathcal{I} = ((\Delta^{\mathcal{I}_i} \cap \Delta^{\mathcal{I}_j}) \times (\Delta^{\mathcal{I}_i} \cap \Delta^{\mathcal{I}_j}))\backslash Y^{\mathcal{I}_j} = (\neg_i Y)^{\mathcal{I}_j} = X^{\mathcal{I}_j}$.

For the "only if" direction, suppose that Σ is consistent as witnessed by P_w. Thus, Σ has a distributed model $\langle\{\mathcal{I}_i\},\{r_{ij}\}_{P_i\in P_j^*}\rangle$, such that $\Delta^{\mathcal{I}_w}\neq\emptyset$. We construct a model $\mathcal{I}$ of $\Re(P_w^*)$ by merging individuals that are related via chains of image domain relations or their inverses. More precisely, for every element x in the distributed model, we define its equivalence class $\overline{x}=\{y|(x,y)\in\rho\}$, where ρ is the symmetric and transitive closure of the set $\bigcup_{P_i\in P_j^*} r_{ij}$. For a set S, we define $\overline{S}=\{\bar{x}|x\in S\}$ and, for a binary relation R, we define $\overline{R}=\{(\overline{x},\overline{y})|(x,y)\in R\}$.

Now, let $\mathcal{I}=\langle\Delta^{\mathcal{I}},\cdot^{\mathcal{I}}\rangle$ be defined as follows:

- $\Delta^{\mathcal{I}}=\overline{\bigcup_i\Delta^{\mathcal{I}_i}}$.
- For every i-name X, $X^{\mathcal{I}}:=\overline{X^{\mathcal{I}_i}}$.
- For every i, $\top_i^{\mathcal{I}}=\overline{\Delta^{\mathcal{I}_i}}$.

We denote by $\overline{x}|_i$ the element (if it exists) in $\Delta^{\mathcal{I}_i}$ that belongs to $\overline{x}$, i.e., $\overline{x}|_i\in\Delta^{\mathcal{I}_i}\cap\overline{x}$.

The proof that $\mathcal{I}$ is a model of $\Re(P_w^*)$, with $\top_w^{\mathcal{I}}\neq\emptyset$, is also similar to that of Lemma 4 in (Bao *et al.* 2008). We only show that for every role inclusion $X\sqsubseteq Y\in P_j$, we have that $\mathcal{I}$ satisfies $\#_j(X)\sqsubseteq\#_j(Y)$. We prove this by showing that, for any role R the appears in P_j, $\overline{R^{\mathcal{I}_j}}=\#_j(R)^{\mathcal{I}}$. We again use induction on the structure of R. When R is a j-role name or an inverse role, the statement is straightforward. When R is an imported role, we show that

$$\overline{R^{\mathcal{I}_j}}=R^{\mathcal{I}}\cap(\top_j^{\mathcal{I}}\times\top_j^{\mathcal{I}})=(R\updownarrow\top_j)^{\mathcal{I}}.$$

The second equation is apparent from the semantics of role domain and range restrictions. We only prove the first equation. Let R be an i-role name.

- $(x,y)\in\overline{R^{\mathcal{I}_j}}\Rightarrow(x,y)\in\overline{r_{ij}(R^{\mathcal{I}_i})}\Rightarrow(x,y)\in\overline{R^{\mathcal{I}_i}}$. In addition, since $(x,y)\in\overline{R^{\mathcal{I}_j}}$, we have that $x,y\in\overline{\Delta^{\mathcal{I}_j}}$. Hence, $(x,y)\in R^{\mathcal{I}}\cap(\top_j^{\mathcal{I}}\times\top_j^{\mathcal{I}})$.
- If $(x,y)\in R^{\mathcal{I}}\cap(\top_j^{\mathcal{I}}\times\top_j^{\mathcal{I}})=\overline{R^{\mathcal{I}_i}}\cap(\overline{\Delta^{\mathcal{I}_j}}\times\overline{\Delta^{\mathcal{I}_j}})$, then $(x|_i,y|_i)\in R^{\mathcal{I}_i}$ and $(x|_j,y|_j)\in\Delta^{\mathcal{I}_j}\times\Delta^{\mathcal{I}_j}$. Since $P_i\in P_j^*$ (because P_j imports R), $(x|_i,x|_j)\in\rho_{ij}=r_{ij}$ and $(y|_i,y|_j)\in\rho_{ij}=r_{ij}$, whence $(x|_j,y|_j)\in r_{ij}(R^{\mathcal{I}_i})=R^{\mathcal{I}_j}$. Thus $(x,y)\in\overline{R^{\mathcal{I}_j}}$.

When $R=\neg_i S$ and $\overline{S^{\mathcal{I}_j}}=\#_j(S)^{\mathcal{I}}$, we can show that $\overline{R^{\mathcal{I}_j}}=\overline{(\neg_i S)^{\mathcal{I}_j}}=\overline{(r_{ij}(\Delta^{\mathcal{I}_i})\times r_{ij}(\Delta^{\mathcal{I}_i}))\backslash S^{\mathcal{I}_j}}=((\top_i\sqcap\top_j)^{\mathcal{I}}\times(\top_i\sqcap\top_j)^{\mathcal{I}})\backslash\overline{S^{\mathcal{I}_j}}=(\neg\#_j(S)\updownarrow(\top_i\sqcap\top_j))^{\mathcal{I}}=\#_j(R)^{\mathcal{I}}$. Q.E.D.

Decidability of P-DL $\mathcal{ALCHIO}(\neg)_{\mathcal{R}}\mathcal{P}$

The decidability proof of $\mathcal{ALCHIO}(\neg)_{\mathcal{R}}\mathcal{P}$ is almost the same as that of $\mathcal{ALCHIO}(\neg)_{\mathcal{CR}}\mathcal{P}$ and uses a reduction to $\mathcal{ALBO}$. Since negated roles appear only in role inclusions and cardinality preservation is not required for negated roles, in the reduction from an $\mathcal{ALCHIO}(\neg)_{\mathcal{R}}\mathcal{P}$ ontology to an $\mathcal{ALBO}$ ontology, the negated local domain and the negated local range axioms are not needed. Note that, in $\mathcal{ALCHIO}(\neg)_{\mathcal{CR}}\mathcal{P}$, if P_j imports a role from P_i, then, due to cardinality preservation on both role names and negated roles, r_{ij} has to be either empty or a total function. In $\mathcal{ALCHIO}(\neg)_{\mathcal{R}}\mathcal{P}$, on the other hand, there is no such requirement. This allows some increased flexibility in role mappings while, at the same time, maintaining the autonomy of ontology modules.

From the above reductions from $\mathcal{ALCHIO}(\neg)_{\mathcal{CR}}\mathcal{P}$ and $\mathcal{ALCHIO}(\neg)_{\mathcal{R}}\mathcal{P}$ to $\mathcal{ALBO}$ and the fact that the complexity of $\mathcal{ALBO}$ is NExpTime-complete (Schmidt & Tishkovsky 2007) we obtain the following decidability and complexity result.

Theorem 4 *$\mathcal{ALCHIO}(\neg)_{\mathcal{CR}}\mathcal{P}$ and $\mathcal{ALCHIO}(\neg)_{\mathcal{R}}\mathcal{P}$ are in* NEXPTIME.

Conclusions

We have explored the decidability of modular ontology languages (specifically, variants of DDL and P-DL). We have shown that if role mappings between ontology modules that are expressible in $\mathcal{ALC}$ are combined with some otherwise useful features such as negated roles, cardinality restrictions in bridge rules, or inverse bridge rules, they yield an undecidable DDL. We also established the decidability of P-DLs ($\mathcal{ALCHIO}(\neg)_{\mathcal{CR}}\mathcal{P}$ and $\mathcal{ALCHIO}(\neg)_{\mathcal{R}}\mathcal{P}$) with unrestricted role inclusion between ontology modules when each module is in $\mathcal{ALCHIO}(\neg)$, a language that extends $\mathcal{ALC}$ with general role inclusions, inverse roles, nominals and negated roles. The fact the restriction that the domain relations in P-DL be *one-to-one* and *compositionally consistent* (Bao, Slutzki, & Honavar 2007) (as opposed to DDL which imposes no such restrictions (Ghidini & Serafini 2006)) turns out to be critical to the decidability of P-DLs $\mathcal{ALCHIO}(\neg)_{\mathcal{CR}}\mathcal{P}$ and $\mathcal{ALCHIO}(\neg)_{\mathcal{R}}\mathcal{P}$. Ongoing work is focused on further exploring the *decidability frontier* of DDL and P-DL, e.g., by investigating the decidability of the DDL $\mathcal{ALCOD}_{\mathcal{CR}}$ (with support for nominals), and of the P-DL $\mathcal{SHQP}$ (i.e., with support for transitive roles and number restrictions).

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