Categorical Abstract Algebraic Logic:
Equivalence of Closure Systems

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Abstract

In their famous "Memoirs" monograph, Blok and Pigozzi defined algebraizable deductive systems as those whose consequence relation is equivalent to the algebraic consequence relation associated with a quasivariety of universal algebras. In characterizing this property, they showed that it is equivalent with the existence of an isomorphism between the lattices of theories of the two consequence relations that commutes with inverse substitutions. Thus emerged the prototypical and paradigmatic result relating an equivalence between two consequence relations established by means of syntactic translations and the isomorphism between corresponding lattices of theories. This result was subsequently generalized in various directions. Blok and Pigozzi themselves extended it to cover equivalences between k-deductive systems. Rebagliato and Verdú and, later, also Pynko and Raftery, considered equivalences between consequence relations on associative sequents. The author showed that it holds for equivalences between two term π-institutions. Blok and Jónsson considered equivalences between structural closure operations on regular M-sets. Gil-Férez lifted the author's results to the case of multi-term π-institutions. Finally, Galatos and Tsinakis considered the case of equivalences between closure operators on **A**-modules and provided an exact characterization of those that are induced by syntactic translations. In this paper, we contribute to this line of research by further abstracting the results of Galatos and Tsinakis to the case of consequence systems on **Sign**-module systems, which are set-valued functors SEN : **Sign** $\rightarrow$ **Set** on complete residuated categories **Sign**.

1 Introduction

In this paper the order-theoretic and categorical framework developed by Galatos and Tsinakis [10], based on previous work of Blok and Jónsson [3, 4], to study

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logical consequence relations on modules over complete residuated lattices is generalized to encompass consequence systems on **Sign**-module systems, which are set-valued functors SEN : **Sign** $\rightarrow$ **Set**, where **Sign** is an arbitrary complete residuated category. This extension deals, apart with the equivalence of consequence relations on various systems based on propositional languages, also with equivalences of logics formalized as π-institutions. In particular, it encompasses previous results obtained by the author [20] and later extended by Gil-Férez [11].

Blok and Pigozzi in [5] made, for the first time, precise the notion of an algebraizable deductive system. Let $\mathcal{L}$ be a language type, thought of as a set of logical connectives or as a set of algebraic operation symbols of finite arities, depending on the context. A *deductive system* $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ over $\mathcal{L}$ is composed of a finitary and structural consequence relation on the set $\mathrm{Fm}_{\mathcal{L}}(V)$ of all formulas constructed in the ordinary recursive way starting from variables in a fixed denumerable set V and using the connectives in $\mathcal{L}$. Blok and Pigozzi called $\mathcal{S}$ *algebraizable* if there exist mutually inverse interpretations between the consequence $\vdash_{\mathcal{S}}$ and the equational consequence relation $\models_{\mathsf{K}}$ associated with a quasivariety K of $\mathcal{L}$-algebras. They provided a characterization of algebraizability by showing that $\mathcal{S}$ is algebraizable iff there exists an isomorphism between the lattices $\mathbf{Th}(\vdash_{\mathcal{S}})$ and $\mathbf{Th}(\models_{\mathsf{K}})$ of the theories of $\vdash_{\mathcal{S}}$ and $\models_{\mathsf{K}}$, that commutes with inverse substitutions. In an effort to capture the symmetry in the definition of algebraizability, Blok and Pigozzi generalized this framework by defining equivalence between k-deductive systems [6, 7]. Given a positive integer k, a k-*deductive system* over a language type $\mathcal{L}$ consists of a finitary and structural consequence relation on the set $\mathrm{Fm}^k_{\mathcal{L}}(V)$, i.e., the set of all k-tuples of $\mathcal{L}$-formulas. Deductive systems are captured by 1-deductive systems, in this sense, and equational consequence relations are captured by 2-deductive systems, where a 2-formula $\langle \phi, \psi \rangle \in \mathrm{Fm}^2_{\mathcal{L}}(V)$ is perceived as an $\mathcal{L}$-equation $\phi \approx \psi$. In this context, a 1-deductive system is algebraizable in the original sense of [5] iff it is equivalent to the 2-deductive system corresponding to the equational consequence associated with a quasivariety K of $\mathcal{L}$-algebras. Blok and Pigozzi show in [7] that, in this context as well, equivalence of a k-deductive system with an l-deductive system is tantamount to the existence of an isomorphism between their lattices of theories that commutes with inverse substitutions.

The next development, chronologically almost parallel to [6], occurred in Barcelona in the context of studies pertaining to the algebraizability of Gentzen systems. Rebagliato and Verdú [17] defined the *algebraizability of a Gentzen system*, following the lead of [5], and in subsequent work [18] established a characterization of algebraizability in terms of the existence of an isomorphism between the theories of the algebraizable Gentzen system and that of an equational deductive system associated with a class of algebras.

In the mid 90's, under the supervision of Don Pigozzi, the author initiated his studies in the categorical side of abstract algebraic logic. The goal was to widen the scope of definitions, methods and results pertaining to the algebraization of deductive systems and make them available to logical systems that are not defined necessarily as consequence relations on sets of propositional formulas.

Using the structure of π-*institution* [9], which derives from that of an institution [12, 13], as the underlying framework, the author defined the notion of *deductive equivalence* between two π-institutions [19, 20]. This concept is inspired by, and abstracts, the equivalence between k-deductive systems. It is based, in essence, on mutually inverse transformations between the sets of sentences of the π-institutions involved that can be perceived as analogs of the syntactic mutually inverse translations between a k- and an l-deductive system. Although, in general, the notion of a π-institution is too general for a characterization theorem along the lines of the one established in [5, 6] and [18] to hold (contrary to the erroneous claim in [21], which was rescinded in [22]), the author was able to obtain a characterization for the special case of *term* π-*institutions*. These are, informally speaking, π-institutions in which a distinguished sentence that behaves like an ordinary variable in relation to substitutions, is singled out. Again informally speaking, it is shown in the main result, Theorem 10.5, of [20] that two term π-institutions are deductively equivalent iff there is an isomorphism between their categories of theories, which abstract the theory lattices in the categorical context, that commutes with substitutions.

Also taking after the work of [5, 6], Blok and Jónsson, in a joint presentation at the 23rd Holiday Mathematics Symposium at New Mexico State University in 1999 [3] (later published by Jónsson after Wim Blok's death [4]), revisited the equivalence underlying the algebraizability of a deductive system. They recast the notion as that of an *equivalence* between two (structural) *consequence operations on* M-*sets*, where M is a monoid acting on the sets. This action is intended to abstract the action of the monoid of substitutions on the set of formulas of a sentential logic. It is also not the case that every equivalence between closure operations on M-sets is induced by syntactic transformations between the corresponding sets of formulas. Blok and Jónsson were able, however, similarly with the case in [20], to obtain a characterization theorem to that effect in the case of *regular* M-*sets*. These are M-sets having a base, i.e., a set of elements that behave, roughly speaking, as ordinary variables with respect to substitutions. In their main theorem, Theorem 5.5 of [4], they were able to characterize the equivalence between two structural closure operations on regular M-sets as one induced by appropriate syntactic translations between the two underlying M-sets.

Next came two almost parallel developments in the line of research on the equivalence between consequence relations. On the more classical side, Raftery undertook the study of correspondences between Hilbert-style and Gentzen-style deductive systems [16]. In it, inspired by both the work of Blok and Jónsson on the equivalence of closure operators on M-sets and by the work of Rebagliato and Verdú on the algebraization of Gentzen systems (which was also developed further by Pynko [15]), he establidhed a general result on the equivalence of Gentzen systems (Theorem 6.8 of [16]). Raftery defined two Gentzen systems to be *equivalent* if two mutually inverse syntactic interpretations exist between their consequence relations. He was able to show that two Gentzen systems are equivalent iff there exists a lattice isomorphism between their lattices of theories which commutes with substitutions. This result encompasses, but is more general than, the characterization of algebraizability of Gentzen systems, previously proven by Rebagliato and Verdú in [18]. On the other hand, at around the same time, on the categorical side, Gil-Férez [11], inspired by [20] and noticing the error in [22], as well as the fact that, despite their generality, term π-institutions were unable to capture Gentzen-style systems, introduced the notion of a *multi-term* π*-institution*. This notion generalizes that of a term π-institution and, moreover, is wide enough to encompass Gentzen systems in appropriate equivalent reformulations. In the main theorem of [11], Theorem 8.9, the characterization theorem of deductive equivalence of [20] is extended to cover deductive equivalence between two multi-term π-institutions.

The latest development on the studies of equivalences between consequence relations was in the form of a further abstraction of the work of Blok and Jónsson [3, 4]. Namely, Galatos and Tsinakis [10] studied the equivalence of two structural consequence relations between $\mathbf{A}$-modules. Roughly speaking, given a complete residuated lattice $\mathbf{A}$, an $\mathbf{A}$-*module* $\mathbf{P}$ is a complete lattice, together with a residuated action $\star$ of the monoid reduct of $\mathbf{A}$ on the complete lattice $\mathbf{P}$. In Theorem 5.1 of [10], it is shown that the *equivalences of structural closure operations on two* $\mathbf{A}$*-modules* may be characterized as being induced by mutually inverse syntactic translations between the underlying $\mathbf{A}$-modules iff the $\mathbf{A}$-modules involved are *projective* objects in the category of $\mathbf{A}$-modules. In Theorem 5.7, they are able to exactly pinpoint conditions that characterize the *projective cyclic* $\mathbf{A}$*-modules*, i.e., those that are generated by a single element. This characterization enables them to obtain as a corollary the characterization of algebraizability of deductive systems, since both a consequence relation on a set of formulas, as well as that on a set of equations, can be naturally viewed as consequence relations on modules. Taking a further step, Galatos and Tsinakis show in Lemma 5.12 of [10] that the coproduct in the category of $\mathbf{A}$-modules of projective objects is also projective. This enables them to also cover the case of consequence relations based on sequents, which cannot be viewed as consequences based on cylcic modules but can be captured as consequences on coproducts of projective cyclic modules which are, as a result, also projective. Thus, the work in [10] captures all known results concerning the equivalence of consequence relations that have appeared on classical studies in abstract algebraic logic.

A review of all aforementioned developments concerning equivalences between consequence relations studied previously in the literature, including further details, as well as formal relevant definitions and all main theorems, will be presented in Section 2.

In this paper, inspired by the framework of Galatos and Tsinakis, we provide a slightly more general platform in which, not only the study of the classical results can be carried out, but also all of the known categorical analogs may be obtained. Namely, instead of using $\mathbf{A}$-modules over complete residuated lattices, we study consequence systems over $\mathbf{Sign}$-module systems, where $\mathbf{Sign}$ is an arbitrary complete residuated category. A *complete residuated category* $\mathbf{Sign}$ is a category each of whose sets of morphisms $\mathbf{Sign}(\Sigma, \Sigma')$ is endowed with the structure of a complete lattice and whose composition operations are bi-residuated functions. Clearly, complete residuated lattices in the sense of [10] are special complete residuated categories having one object and collection of morphisms corresponding to the elements of the residuated lattice. Moreover, the ordering of the morphisms is inherited by the ordering of the lattice elements and composition is exactly the monoid operation of the residuated lattice. Given such a complete residuated category **Sign**, a **Sign**-*module system* is a set-valued functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{set}$, such that, every set $\mathrm{SEN}(\Sigma)$ has the structure of a complete lattice, that is endowed with the natural action of morphisms on sentences, i.e., $f \star^{\Sigma,\Sigma'} \phi = \mathrm{SEN}(f)(\phi)$, for every $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\phi \in \mathrm{SEN}(\Sigma)$, that is postulated to be bi-residuated. Note that this action automatically satisfies the properties of a monoid action, i.e., that $i_\Sigma \star^{\Sigma,\Sigma} \phi = \phi$ and that $g \star^{\Sigma',\Sigma''} (f \star^{\Sigma,\Sigma'} \phi) = (g \circ f) \star^{\Sigma,\Sigma''} \phi$, for all $\Sigma, \Sigma', \Sigma'' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma'), g \in \mathbf{Sign}(\Sigma', \Sigma'')$ and $\phi \in \mathrm{SEN}(\Sigma)$. We use the notation $\mathcal{M}$ to refer to the category with objects all **Sign**-module systems, for some complete residuated category **Sign**, and morphisms all residuated maps $\langle F, \alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$ between them that preserve the corresponding sentence actions. If the complete residuated category **Sign** is a 1-object category, corresponding to a complete residuated lattice **A**, as above, then the notion of a **Sign**-module system degenerates to the notion of an **A**-module of [10].

All results of Galatos and Tsinakis may be abstracted to the level of **Sign**-module systems. In fact most of the results of [10] that refer to complete lattices and closure operators, or consequence relations, without taking into account structurality, can be lifted to the case of complete lattice families and closure, or consequence, families on them in a straightforward signature-wise fashion. Many of these are presented here without proofs, referring to the corresponding results of [10] from which the proofs can be directly obtained. The difference occurs when instead of the complete residuated lattice **A** and an **A**-module **P**, a complete residuated category **Sign** and a **Sign**-module system $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ are under consideration. Even then, the results, albeit more general, follow closely the proofs of the corresponding results in [10].

The category $\mathcal{M}$ forms the basis for abstracting the results obtained in the category ${}_{\mathbf{A}}\mathcal{M}$ of **A**-modules. For example, it is the case in $\mathcal{M}$ also, that the consequence systems on an object SEN, i.e., the π-institutions, with sentence functor SEN, correspond to epimorphic images of SEN. Thus, closure systems of π-institutions may be identified with objects of the category $\mathcal{M}$. Two such closure systems are termed *equivalent* if there exists an equivalence between the module systems corresponding to them in a formal sense. Moreover, equivalence of closure systems may also be defined, as in [10], by stipulating the existence of a pair of mutually inverse structural residuated maps between the two sentence functors that preserve consequence in an appropriate sense. These definitions, not only generalize the ones given in [10], but they are also able to capture the notion of deductive equivalence of π-institutions of [20].

Roughly speaking, if two consequence systems on two module systems are equivalent via an equivalence defined by mutually inverse module system morphisms on the underlying module systems, then it is always the case that the module systems corresponding to these consequence systems are naturally equivalent. A categorical characterization is obtained of the module systems for which these two notions of equivalence coincide: they are precisely the projective objects (in a sense defined precisely in Subsection 6.1) in $\mathcal{M}$.

Let us now sketch how this result subsumes the one characterizing equivalence between two term π-institutions $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ and $\mathcal{I}' = \langle \mathbf{Sign}, \mathrm{SEN}', C'\rangle$, assuming, for simplicity, that their sentence functors are over the same signature category and the two signature categories are related by the identity functor between them. We define a category $\mathbf{Sign}^{\mathcal{P}}$, with the same objects as $\mathbf{Sign}$, whose morphisms are sets of morphisms of $\mathbf{Sign}$ between the corresponding objects. Then, we define $\mathcal{P}\mathrm{SEN}$ and $\mathcal{P}\mathrm{SEN}'$ as the powerset functors corresponding to SEN and SEN$'$, respectively, where action of a morphism $f^{\mathcal{P}}$ in $\mathbf{Sign}^{\mathcal{P}}$ on a subset X of sentences is defined by applying all morphisms in $f^{\mathcal{P}}$ to all elements in X. It is shown that, defined in this way, $\mathcal{P}\mathrm{SEN}$ and $\mathcal{P}\mathrm{SEN}'$ are projective $\mathbf{Sign}^{\mathcal{P}}$-module systems. They are in fact cyclic, i.e., generated by a source signature-variable pair in the sense of [20]. All projective cyclic $\mathbf{Sign}$-module systems are characterized in Theorem 32, which abstracts a corresponding characterization of projective cyclic $\mathbf{A}$-modules for a complete residuated lattice $\mathbf{A}$ obtained in Theorem 5.7 of [10].

On the other hand, if the π-institutions $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ and $\mathcal{I}' = \langle \mathbf{Sign}, \mathrm{SEN}', C'\rangle$ are multi-term [11] but not term, then, the corresponding $\mathbf{Sign}^{\mathcal{P}}$-module systems $\mathcal{P}\mathrm{SEN}$ and $\mathcal{P}\mathrm{SEN}'$ are not cyclic. But they are shown to be coproducts of cyclic projective $\mathbf{Sign}$-module systems and, therefore, based on a result abstracted from [10], they are also projective. Gil-Férez [11] has shown that these π-institutions encompass various Gentzen-style consequence systems over sequents (see, also, [18, 16]).

Following [10], we close our presentation by an exploration of conditions that ensure that the interpretations that define an equivalence of two finitary consequence systems over finitary module systems are also finitary, i.e., they send compact elements to compact elements.

Many of the proofs in this work as either direct adaptations of corresponding proofs in [10] or are easily generalized versions. Therefore, our intellectual debt to the work of [10] is considerable. Moreover, both the original work of [20], as well as its subsequence generalization in [11], have influenced the presentation of the systems studied here. Finally, for various basic standard categorical notions and notation, that will be left undefined, the reader is referred to any of the introductory references [14, 8, 1].

2 Equivalence of Various Logical Systems

In this section, we review some of the historic developments that paved the way for the general study of equivalences between consequence relations of various logical systems. We start by looking at the original framework of Blok and Pigozzi [5]. Then, we introduce equivalence between k-deductive systems [6], which is also due to Blok and Pigozzi [7]. Next, we revisit equivalence between consequence relations on sets of sequents [18, 15, 16]. Finally, switching to the categorical side of the theory, we review equivalence between π-institutions [20], in general, and, in particular, the characterization theorems of the equivalence between term π-institutions [20] and multi-term π-institutions [11]. All these cases form special examples for various aspects of the general theory that will be developed in later sections.

2.1 Algberaizability of Deductive Systems

Let $\mathcal{L}$ be a language type and V a fixed denumerable set of propositional variables. Denote by $\mathrm{Fm}_{\mathcal{L}}(V)$ the collection of all formulas (or terms) over the language $\mathcal{L}$ that are constructed in the usual recursive way using the variables in V. The associated absolutely free formula algebra will be denoted by $\mathbf{Fm}_{\mathcal{L}}(V)$. A substitution is a mapping $\sigma : V \to \mathrm{Fm}_{\mathcal{L}}(V)$, which can be extended to an endomorphism, also denoted by σ, on the formula algebra $\mathbf{Fm}_{\mathcal{L}}(V)$. A *consequence relation* $\vdash$ over $\mathrm{Fm}_{\mathcal{L}}(V)$ is a subset $\vdash \subseteq \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)) \times \mathrm{Fm}_{\mathcal{L}}(V)$, satisfying, for all $\Phi \cup \Psi \cup \{\phi, \psi, \chi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

1. $\Phi \vdash \phi$, for all $\phi \in \Phi$;
2. $\Phi \vdash \psi$, for all $\psi \in \Psi$, and $\Psi \vdash \chi$ imply $\Phi \vdash \chi$.

A consequence relation $\vdash$ is called *finitary*, if for all $\Phi \cup \{\phi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$, $\Phi \vdash \phi$ implies that there exists finite $\Psi \subseteq \Phi$, such that $\Psi \vdash \phi$. It is called *substitution invariant* or *structural*, if for every substitution σ, $\Phi \vdash \phi$ implies $\sigma(\Phi) \vdash \sigma(\phi)$, for all $\Phi \cup \{\phi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$.

By analogy with this case, one may also define substitution invariant and finitary consequence relations on $\mathrm{Eq}_{\mathcal{L}}(V) = \mathrm{Fm}^2_{\mathcal{L}}(V)$, the set of pairs of $\mathcal{L}$-formulas, also called $\mathcal{L}$-equations, and often written as $\phi \approx \psi$ instead of $\langle \phi, \psi \rangle$. In this case, an application of a substitution to an equation is performed pointwise, i.e., $\sigma(\phi \approx \psi) = \sigma(\phi) \approx \sigma(\psi)$. Given a class K of $\mathcal{L}$-algebras (in the usual universal algebraic sense), we denote by $\models_{\mathsf{K}} \subseteq \mathcal{P}(\mathrm{Eq}_{\mathcal{L}}(V)) \times \mathrm{Eq}_{\mathcal{L}}(V)$ the substitution invariant consequence relation on the set of $\mathcal{L}$-equations associated with K. This relation is finitary iff K is closed under ultraproducts, which is the case when K is a quasi-variety of $\mathcal{L}$-algebras.

A *deductive system* in the sense of Blok and Pigozzi [5] is a pair $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$, where $\mathcal{L}$ is a language type and $\vdash_{\mathcal{S}}$ is a substitution invariant, finitary consequence relation over $\mathrm{Fm}_{\mathcal{L}}(V)$. It is called *algebraizable* if there exist a class of $\mathcal{L}$–algebras K, a finite set of equations $\delta(p) \approx \epsilon(p) = \{\delta_i(p) \approx \epsilon_i(p) : i \in I\}$ on a single variable p and a finite set of formulas $\Delta(p, q) = \{\Delta_j(p, q) : j \in J\}$ in two variables p, q, such that for every $\Phi \cup \{\phi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$ and all $\phi \approx \psi \in \mathrm{Eq}_{\mathcal{L}}(V)$,

1. $\Phi \vdash_{\mathcal{S}} \phi$ iff $\delta(\Phi) \approx \epsilon(\Phi) \models_{\mathsf{K}} \delta(\phi) \approx \epsilon(\phi)$;
2. $\phi \approx \psi \mathrel{=\!\mid\models_{\mathsf{K}}} \delta(\Delta(\phi, \psi)) \approx \epsilon(\Delta(\phi, \psi))$.

Natural conventions have been used here, e.g., $\delta(\Delta(\phi, \psi)) \approx \epsilon(\Delta(\phi, \psi)) = \{\delta_i(\Delta_j(\phi, \psi)) \approx \epsilon_i(\Delta_j(\phi, \psi)) : i \in I, j \in J\}$. The class K is called an *equivalent algebraic semantics for* $\mathcal{S}$, the set $\delta \approx \epsilon$ a set of *defining equations* and the set Δ a set of *equivalence formulas*. The two conditions that define algebraizability are shown in [5] to be equivalent to the following two "symmetric" conditions: for all $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Eq}_{\mathcal{L}}(V)$ and all $\phi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

3. $E \models_{\mathsf{K}} \phi \approx \psi$ iff $\{\Delta(e_1, e_2) : e_1 \approx e_2 \in E\} \vdash_{\mathcal{S}} \Delta(\phi, \psi)$;

4. $\phi \dashv\vdash_{\mathcal{S}} \Delta(\delta(\phi), \epsilon(\phi))$.

A *theory of* $\mathcal{S}$, or of $\vdash_{\mathcal{S}}$, or an $\mathcal{S}$*-theory*, is a subset $T \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$, such that, for all $\phi \in \mathrm{Fm}_{\mathcal{L}}(V)$, $T \vdash_{\mathcal{S}} \phi$ implies $\phi \in T$, i.e., T is closed under consequence. The set of $\mathcal{S}$-theories forms a lattice $\mathbf{Th}(\vdash_{\mathcal{S}}) = \langle \mathrm{Th}(\vdash_{\mathcal{S}}), \subseteq \rangle$ under inclusion. Similarly, the lattice of theories $\mathbf{Th}(\models_{\mathsf{K}})$ of the equational consequence $\models_{\mathsf{K}}$ may be defined. The main result of [5], which has given impetus to all results discussed in this paper, characterizes algebraizability in terms of an isomorphism between these lattices.

Theorem 1 (Blok and Pigozzi [5]) *A deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ is algebraizable with equivalent algebraic semantics a quasivariety K iff there exists an isomorphism between $\mathbf{Th}(\vdash_{\mathcal{S}})$ and $\mathbf{Th}(\models_{\mathsf{K}})$ that commutes with inverse substitutions.*

2.2 Equivalence Between k-Deductive Systems

The next step in the study of equivalence of consequence relations was equivalence of k-deductive systems. Let k be a positive integer. A k*-formula* is an element of $\mathrm{Fm}^k_{\mathcal{L}}(V)$, the k-th direct power of $\mathrm{Fm}_{\mathcal{L}}(V)$. A finitary consequence relation $\vdash$ over $\mathrm{Fm}^k_{\mathcal{L}}(V)$ is defined by analogy to the consequence over $\mathrm{Fm}_{\mathcal{L}}(V)$ and $\mathrm{Fm}^2_{\mathcal{L}}(V)$ of the previous subsection. It is called *substitution invariant* or *structural*, if for every substitution σ, $\mathbf{\Phi} \vdash \boldsymbol{\phi}$ implies $\sigma(\mathbf{\Phi}) \vdash \sigma(\boldsymbol{\phi})$, for all $\mathbf{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Fm}^k_{\mathcal{L}}(V)$, where an application of a substitution to a k-tuple is performed point-wise. A k-deductive system in the sense of Blok and Pigozzi [6] is a pair $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$, where $\mathcal{L}$ is a language type and $\vdash_{\mathcal{S}}$ is a substitution invariant, finitary consequence relation over $\mathrm{Fm}^k_{\mathcal{L}}(V)$.

Given two positive integers k and l, a (k, l)*-translation* is a finite collection $\boldsymbol{\tau} = \{\boldsymbol{\tau}_i(\mathbf{p}) : i \in I\}$ of l-formulas in k variables $\mathbf{p} = \langle p_0, \ldots, p_{k-1} \rangle$. A k-deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ and an l-deductive system $\mathcal{S}' = \langle \mathcal{L}, \vdash_{\mathcal{S}'} \rangle$ over the same language type $\mathcal{L}$, are called *equivalent* if there exist a a (k, l)-translation $\boldsymbol{\tau}$ and an (l, k)-translation $\boldsymbol{\rho}$, such that for every $\mathbf{\Phi} \cup \{\boldsymbol{\phi}\} \subseteq \mathrm{Fm}^k_{\mathcal{L}}(V)$ and all $\boldsymbol{\psi} \in \mathrm{Fm}^l_{\mathcal{L}}(V)$,

1. $\mathbf{\Phi} \vdash_{\mathcal{S}} \boldsymbol{\phi}$ iff $\boldsymbol{\tau}(\mathbf{\Phi}) \vdash_{\mathcal{S}'} \boldsymbol{\tau}(\boldsymbol{\phi})$;

2. $\boldsymbol{\psi} \dashv\vdash_{\mathcal{S}'} \boldsymbol{\tau}(\boldsymbol{\rho}(\boldsymbol{\psi}))$.

These two conditions defining equivalence turn out to be equivalent to the following two "symmetric" conditions: for all $\mathbf{\Psi} \cup \{\boldsymbol{\psi}\} \subseteq \mathrm{Fm}^l_{\mathcal{L}}(V)$ and all $\boldsymbol{\phi} \in \mathrm{Fm}^k_{\mathcal{L}}(V)$,

3. $\mathbf{\Psi} \vdash_{\mathcal{S}'} \boldsymbol{\psi}$ iff $\boldsymbol{\rho}(\mathbf{\Psi}) \vdash_{\mathcal{S}} \boldsymbol{\rho}(\boldsymbol{\psi})$;

4. $\boldsymbol{\phi} \dashv\vdash_{\mathcal{S}} \boldsymbol{\rho}(\boldsymbol{\tau}(\boldsymbol{\phi}))$.

A *theory* of a k-deductive system $\mathcal{S}$, or of $\vdash_{\mathcal{S}}$, or an $\mathcal{S}$-*theory*, is a subset $T \subseteq \mathrm{Fm}^k_{\mathcal{L}}(V)$, that is closed under consequence. The set of $\mathcal{S}$-theories forms a lattice $\mathbf{Th}(\vdash_{\mathcal{S}}) = \langle \mathrm{Th}(\vdash_{\mathcal{S}}), \subseteq\rangle$ under inclusion. In Theorem 4.11 of [7], which generalizes Theorem 1, the following characterization of equivalence in terms of an isomorphism between the corresponding theory lattices is provided.

Theorem 2 (Blok and Pigozzi [7]) *Let $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}}\rangle$ be a k-deductive system and $\mathcal{S}' = \langle \mathcal{L}, \vdash_{\mathcal{S}'}\rangle$ be an l-deductive system. Then $\mathcal{S}$ and $\mathcal{S}'$ are equivalent iff there exists an isomorphism between $\mathbf{Th}(\vdash_{\mathcal{S}})$ and $\mathbf{Th}(\vdash_{\mathcal{S}'})$ that commutes with substitutions.*

2.3 Equivalence Between Term π-Institutions

A further step in the study of equivalence of consequence relations was the equivalence of two π-institutions. Recall that a *closure operator* C on a set X is a function $C : \mathcal{P}(X) \to \mathcal{P}(X)$, such that

1. $y \in C(Y)$, for all $y \in Y \subseteq X$;
2. $Y \subseteq Z$ implies $C(Y) \subseteq C(Z)$, for all $Y, Z \subseteq X$;
3. $C(C(Y)) = C(Y)$, for all $Y \subseteq X$.

A π-*institution* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ [9] consists of a category $\mathbf{Sign}$ of *signatures*, a set-valued functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, which gives for a given signature $\Sigma \in |\mathbf{Sign}|$, the set $\mathrm{SEN}(\Sigma)$ of all Σ-*sentences* and a closure system $C = \{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$, where $C_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \to \mathcal{P}(\mathrm{SEN}(\Sigma))$ is a closure operator on $\mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$, such that, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\phi \in C_\Sigma(\Phi) \quad \text{implies} \quad \mathrm{SEN}(f)(\phi) \in C_{\Sigma'}(\mathrm{SEN}(f)(\Phi)).$$

Recall that, given a consequence relation $\vdash \subseteq \mathcal{P}(X) \times X$ on a set X, one may define a closure operator $C_\vdash : \mathcal{P}(X) \to \mathcal{P}(X)$ on X by $C_\vdash(Y) = \{x \in X : Y \vdash x\}$, for all $Y \subseteq X$, and, given a closure operator $C : \mathcal{P}(X) \to \mathcal{P}(X)$ on X, one may define a consequence relation $\vdash_C \subseteq \mathcal{P}(X) \times X$ on X, by $Y \vdash_C x$ iff $x \in C(Y)$, for all $Y \cup \{x\} \subseteq X$. Moreover, we have $\vdash_{C_\vdash} = \vdash$ and $C_{\vdash_C} = C$, for every consequence relation $\vdash$ on X and every closure operator C on X. Therefore consequence relations and closure operators are interchangeable.

As a consequence of this observation, a π-institution may also be presented as a tuple $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \vdash\rangle$ where $\vdash$ is a consequence system $\vdash = \{\vdash_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$, where $\vdash_\Sigma \subseteq \mathcal{P}(\mathrm{SEN}(\Sigma)) \times \mathrm{SEN}(\Sigma)$ is a consequence relation on $\mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$, such that, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\Phi \vdash_\Sigma \phi \quad \text{implies} \quad \mathrm{SEN}(f)(\Phi) \vdash_{\Sigma'} \mathrm{SEN}(f)(\phi).$$

A sentence functor SEN : $\mathbf{Sign} \to \mathbf{Set}$ is called *term* if there exists a pair $\langle V, v\rangle$, with $V \in |\mathbf{Sign}|$ and $v \in \mathrm{SEN}(V)$, such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, there exists $f_{\langle\Sigma,\phi\rangle} \in \mathbf{Sign}(V,\Sigma)$, with $\mathrm{SEN}(f_{\langle\Sigma,\phi\rangle})(v) = \phi$, and such that, for all $\Sigma' \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma,\Sigma')$, $f \circ f_{\langle\Sigma,\phi\rangle} = f_{\langle\Sigma',\mathrm{SEN}(f)(\phi)\rangle}$. A π-institution $\mathcal{I} = \langle\mathbf{Sign}, \mathrm{SEN}, \vdash\rangle$ is *term* if its sentence functor SEN is term.

Let $\mathbf{Sign}$ and $\mathbf{Sign}'$ be two categories and SEN : $\mathbf{Sign} \to \mathbf{Set}$ and SEN' : $\mathbf{Sign}' \to \mathbf{Set}$ two set-valued functors. A *translation from* SEN *to* SEN' is a pair $\langle F,\alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$ consisting of a functor $F : \mathbf{Sign} \to \mathbf{Sign}'$ and a natural transformation $\alpha : \mathrm{SEN} \to \mathcal{P}\mathrm{SEN}' \circ F$. Two π-institutions $\mathcal{I} = \langle\mathbf{Sign}, \mathrm{SEN}, \vdash\rangle$ and $\mathcal{I}' = \langle\mathbf{Sign}', \mathrm{SEN}', \vdash'\rangle$ are called *equivalent* if there exist a translation $\langle F,\alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$, a translation $\langle G,\beta\rangle : \mathrm{SEN}' \to \mathrm{SEN}$ and an adjoint equivalence $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign} \to \mathbf{Sign}'$, such that, for all $\Sigma \in |\mathbf{Sign}|$, $\Sigma' \in |\mathbf{Sign}'|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$ and $\psi \in \mathrm{SEN}'(\Sigma')$,

1. $\Phi \vdash_\Sigma \phi$ iff $\alpha_\Sigma(\Phi) \vdash'_{F(\Sigma)} \alpha_\Sigma(\phi)$;
2. $\psi \dashv\vdash'_{\Sigma'} \mathrm{SEN}'(\epsilon_{\Sigma'})(\alpha_{G(\Sigma')}(\beta_{\Sigma'}(\psi)))$.

These two conditions defining equivalence turn out to be equivalent to the following two "symmetric" conditions: for all $\Sigma \in |\mathbf{Sign}|$, $\Sigma' \in |\mathbf{Sign}'|$, $\phi \in \mathrm{SEN}(\Sigma)$ and $\Psi \cup \{\psi\} \subseteq \mathrm{SEN}'(\Sigma')$,

3. $\Psi \vdash'_{\Sigma'} \psi$ iff $\beta_{\Sigma'}(\Psi) \vdash_{G(\Sigma')} \beta_{\Sigma'}(\psi)$;
4. $\mathrm{SEN}(\eta_\Sigma)(\phi) \dashv\vdash_{G(F(\Sigma))} \beta_{F(\Sigma)}(\alpha_\Sigma(\phi))$.

Let $\mathcal{I} = \langle\mathbf{Sign}, \mathrm{SEN}, \vdash\rangle$ be a π-institution. Given $\Sigma \in |\mathbf{Sign}|$, a Σ-*theory* of $\mathcal{I}$ is a closed subset of $\mathrm{SEN}(\Sigma)$. The set of all Σ-theories is denoted by $\mathrm{Th}_\Sigma(\mathcal{I})$. The *category of theories* $\mathbf{Th}(\mathcal{I})$ of $\mathcal{I}$ has as objects all pairs $\langle\Sigma, T\rangle$, with $\Sigma \in |\mathbf{Sign}|$ and $T \in \mathrm{Th}_\Sigma(\mathcal{I})$ and morphisms $f : \langle\Sigma,T\rangle \to \langle\Sigma',T'\rangle$ all $f \in \mathbf{Sign}(\Sigma,\Sigma')$, such that $\mathrm{SEN}(f)(T) \subseteq T'$. The *theory functor* of $\mathcal{I}$ is the functor $\mathrm{SEN}^\vdash : \mathbf{Sign} \to \mathbf{Set}$, defined, for all $\Sigma \in |\mathbf{Sign}|$, by

$$\mathrm{SEN}^\vdash(\Sigma) = \mathrm{Th}_\Sigma(\mathcal{I}),$$

and for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma,\Sigma')$ and all $T \in \mathrm{SEN}^\vdash(\Sigma)$, by

$$\mathrm{SEN}^\vdash(f)(T) = \{\phi' \in \mathrm{SEN}(\Sigma') : \mathrm{SEN}(f)(T) \vdash_{\Sigma'} \phi\},$$

or, using closure operators, $\mathrm{SEN}^\vdash(f)(T) = C_{\Sigma'}(\mathrm{SEN}(f)(T))$.

Let $\mathcal{I}$ and $\mathcal{I}'$ be two π-institutions. A functor $F : \mathbf{Th}(\mathcal{I}_1) \to \mathbf{Th}(\mathcal{I}_2)$ is called *signature-respecting* if there exists a functor $F^\dagger : \mathbf{Sign}_1 \to \mathbf{Sign}_2$, such that the following rectangle commutes

$$\begin{tikzcd}
\mathbf{Th}(\mathcal{I}_1) \arrow[r, "F"] \arrow[d, "\mathrm{SIG}_1"'] & \mathbf{Th}(\mathcal{I}_2) \arrow[d, "\mathrm{SIG}_2"] \\
\mathbf{Sign}_1 \arrow[r, "F^\dagger"'] & \mathbf{Sign}_2
\end{tikzcd}$$

where SIG_1, SIG_2 denote, respectively, the forgetful functors of $\mathcal{I}_1$ and $\mathcal{I}_2$ that map into the signature component. If this is the case, it is easy to verify that $F^\dagger$ is necessarily unique. A signature-respecting functor $F : \mathbf{Th}(\mathcal{I}_1) \to \mathbf{Th}(\mathcal{I}_2)$ is said to *commute with substitutions* if, for every $f : \Sigma_1 \to \Sigma_1' \in \mathrm{Mor}(\mathbf{Sign}_1)$,

$$\mathrm{SEN}'^{\vdash'}(F^\dagger(f))(F(\langle \Sigma_1, T_1 \rangle)) = F(\mathrm{SEN}^{\vdash}(f)(\langle \Sigma_1, T_1 \rangle)), \tag{1}$$

for every $\langle \Sigma_1, T_1 \rangle \in |\mathbf{Th}(\mathcal{I}_1)|$.

In the main theorem, Theorem 10.5 of [20], which generalizes Theorem 2, a characterization of the equivalence of two term π-institution in terms of an adjoint equivalence between their corresponding categories of theories is provided.

Theorem 3 (Voutsadakis [20]) *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \vdash \rangle$ and $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', \vdash' \rangle$ be two term π-institutions. Then $\mathcal{I}$ and $\mathcal{I}'$ are equivalent iff there exists a signature-respecting adjoint equivalence $\langle F, G, \eta, \epsilon \rangle : \mathbf{Th}(\mathcal{I}) \to \mathbf{Th}(\mathcal{I}')$, which commutes with substitutions.*

As an illustration of the concept of a π-institution and an aid to those readers that are more familiar with the universal algebraic side of the subject, let us briefly depict how a k-deductive system may be recast in the form of a π-institution. Let $\mathcal{L} = \langle \Lambda, \rho \rangle$ be a propositional language, i.e., Λ a set of connectives of finite arities and $\rho : \Lambda \to \omega$ the associated arity function, and V a countable set of variables. $\mathrm{Fm}_{\mathcal{L}}(V)$ denotes the set of formulas constructed by recursion using variables in V and connectives in $\mathcal{L}$ in the usual way. An assignment of formulas to variables is a mapping $f : V \to \mathrm{Fm}_{\mathcal{L}}(V)$. It will be denoted by $f : V \rightharpoondown V$. Such an assignment can be extended uniquely to a substitution, i.e., an endomorphism of the formula algebra $\mathbf{Fm}_{\mathcal{L}}(V)$, denoted by $f^* : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$.

Let $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ be a k-deductive system over $\mathcal{L}$ in the sense of [6]. We construct the π-institution $\mathcal{I}_{\mathcal{S}} = \langle \mathbf{Sign}_{\mathcal{S}}, \mathrm{SEN}_{\mathcal{S}}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}_{\mathcal{S}}|} \rangle$ as follows:

(i) $\mathbf{Sign}_{\mathcal{S}}$ is the one-object category with object V and morphisms all assignments $f : V \rightharpoondown V$. The identity morphism is the inclusion $i_V : V \to \mathrm{Fm}_{\mathcal{L}}(V)$. Composition $g \circ f$ of two assignments f and g is defined by $g \circ f = g^* f$.

(ii) $\mathrm{SEN}_{\mathcal{S}} : \mathbf{Sign}_{\mathcal{S}} \to \mathbf{Set}$ maps V to $\mathrm{Fm}^k_{\mathcal{L}}(V)$ and $f : V \rightharpoondown V$ to $(f^*)^k : \mathrm{Fm}^k_{\mathcal{L}}(V) \to \mathrm{Fm}^k_{\mathcal{L}}(V)$. It is easy to see that $\mathrm{SEN}_{\mathcal{S}}$ is a functor.

(iii) Finally, $C_V : \mathcal{P}(\mathrm{Fm}^k_{\mathcal{L}}(V)) \to \mathcal{P}(\mathrm{Fm}^k_{\mathcal{L}}(V))$ is the standard closure operator $C_{\mathcal{S}} : \mathcal{P}(\mathrm{Fm}^k_{\mathcal{L}}(V)) \to \mathcal{P}(\mathrm{Fm}^k_{\mathcal{L}}(V))$ associated with the k-deductive system $\mathcal{S}$, i.e.,

$$C_V(\mathbf{\Phi}) = \{\boldsymbol{\phi} \in \mathrm{Fm}^k_{\mathcal{L}}(V) : \mathbf{\Phi} \vdash_{\mathcal{S}} \boldsymbol{\phi}\}, \quad \text{for all} \quad \mathbf{\Phi} \subseteq \mathrm{Fm}^k_{\mathcal{L}}(V).$$

C_V, defined in this way, satisfies all properties required in the definition of a closure system of a π-institution. Therefore, the tuple $\mathcal{I}_{\mathcal{S}}$ constitutes a π-institution. It will be called the **π-institution associated with** the k-deductive system $\mathcal{S}$. Note that $\mathcal{I}_{\mathcal{S}}$ is a term π-institution for any k-deductive system $\mathcal{S}$. Indeed, the pair $\langle V, \mathbf{p}\rangle$, where $\mathbf{p} = \langle p_0, \ldots, p_{k-1}\rangle$ is a k-variable, is a source signature-variable pair for $\mathcal{I}_{\mathcal{S}}$.

2.4 Consequence Relations on Sets of Sequents

In this subsection, we review equivalence of consequence relations on sets of sequents.

Given two nonnegative integers m and n, an (m,n)-*sequent* is an expression of the form $\phi_0, \ldots, \phi_{m-1} \rhd \psi_0, \ldots, \psi_{n-1}$, where $\phi_i, \psi_j \in \mathrm{Fm}_{\mathcal{L}}(V)$, $i < m, j < n$. Sometimes it is abbreviated as $\vec{\phi} \rhd \vec{\psi}$. A substitution σ may be applied to an (m,n)-sequent point-wise. A *trace* is a nonempty subset tr of the Cartesian product $\omega \times \omega$. A sequent $\phi_0, \ldots, \phi_{m-1} \rhd \psi_0, \ldots, \psi_{n-1}$ is called a tr-*sequent* if $(m,n) \in \mathrm{tr}$. The set of all tr-sequents is denoted by $\mathrm{tr}\text{-}\mathrm{Seq}_{\mathcal{L}}(V)$.

Given a trace tr, a finitary consequence relation $\vdash$ over $\mathrm{tr}\text{-}\mathrm{Seq}_{\mathcal{L}}(V)$ is defined by analogy to the consequence over $\mathrm{Fm}_{\mathcal{L}}(V)$. It is called *substitution invariant* or *structural*, if for every substitution σ, $\mathfrak{P} \vdash \vec{\phi} \rhd \vec{\psi}$ implies $\sigma(\mathfrak{P}) \vdash \sigma(\vec{\phi} \rhd \vec{\psi})$, for all $\mathfrak{P} \cup \{\vec{\phi} \rhd \vec{\psi}\} \subseteq \mathrm{tr}\text{-}\mathrm{Seq}_{\mathcal{L}}(V)$. A *Gentzen system with trace* tr is a pair $\mathbf{G} = \langle \mathcal{L}, \vdash_{\mathbf{G}}\rangle$, where $\mathcal{L}$ is a language type and $\vdash_{\mathbf{G}}$ is a substitution invariant consequence relation over $\mathrm{tr}\text{-}\mathrm{Seq}_{\mathcal{L}}(V)$.

Let tr and tr$'$ be two traces. A $(\mathrm{tr}, \mathrm{tr}')$-*translation* is a tr-indexed family $\{\boldsymbol{\tau}_{m,n} : (m,n) \in \mathrm{tr}\}$, where $\boldsymbol{\tau}_{m,n}$ is a set of tr$'$-sequents $\vec{\delta}(\vec{p}, \vec{q}) \rhd \vec{\epsilon}(\vec{p}, \vec{q})$ in the variables $\vec{p} = p_0, \ldots, p_{m-1}, \vec{q} = q_0, \ldots, q_{n-1}$, for all $(m,n) \in \mathrm{tr}$. Given such a $(\mathrm{tr}, \mathrm{tr}')$-translation and an (m,n)-sequent $\vec{\phi} \rhd \vec{\psi}$, $\boldsymbol{\tau}(\vec{\phi} \rhd \vec{\psi}) := \boldsymbol{\tau}_{m,n}(\vec{\phi} \rhd \vec{\psi})$ denotes the set of tr$'$-sequents resulting from substituting ϕ_i for p_i and ψ_j for q_j, $i < m, j < n$, in every tr$'$-sequent $\vec{\delta}(\vec{p}, \vec{q}) \rhd \vec{\epsilon}(\vec{p}, \vec{q}) \in \boldsymbol{\tau}_{m,n}$.

Let $\mathbf{G} = \langle \mathcal{L}, \vdash_{\mathbf{G}}\rangle$ and $\mathbf{G}' = \langle \mathcal{L}, \vdash_{\mathbf{G}'}\rangle$ be two Gentzen systems. $\mathbf{G}$ and $\mathbf{G}'$ are called *equivalent* if there exist a $(\mathrm{tr}, \mathrm{tr}')$-translation $\boldsymbol{\tau}$ and a $(\mathrm{tr}', \mathrm{tr})$-translation $\boldsymbol{\rho}$, such that such that for every $\mathfrak{P} \cup \{\vec{\phi} \rhd \vec{\psi}\} \subseteq \mathrm{tr}\text{-}\mathrm{Seq}_{\mathcal{L}}(V)$ and all $\vec{\phi}' \rhd \vec{\psi}' \in \mathrm{tr}'\text{-}\mathrm{Seq}_{\mathcal{L}}(V)$,

1. $\mathfrak{P} \vdash_{\mathbf{G}} \vec{\phi} \rhd \vec{\psi}$ iff $\boldsymbol{\tau}(\mathfrak{P}) \vdash_{\mathbf{G}'} \boldsymbol{\tau}(\vec{\phi} \rhd \vec{\psi})$;
2. $\vec{\phi}' \rhd \vec{\psi}' \dashv\vdash_{\mathbf{G}'} \boldsymbol{\tau}(\boldsymbol{\rho}(\vec{\phi}' \rhd \vec{\psi}'))$.

These two conditions defining equivalence turn out to be equivalent to the following two "symmetric" conditions: for all $\mathfrak{P}' \cup \{\vec{\phi}' \rhd \vec{\psi}'\} \subseteq \mathrm{tr}'\text{-}\mathrm{Seq}_{\mathcal{L}}(V)$ and all $\vec{\phi} \rhd \vec{\psi} \in \mathrm{tr}\text{-}\mathrm{Seq}_{\mathcal{L}}(V)$,

3. $\mathfrak{P}' \vdash_{\mathbf{G}'} \vec{\phi}' \rhd \vec{\psi}'$ iff $\boldsymbol{\rho}(\mathfrak{P}') \vdash_{\mathbf{G}} \boldsymbol{\rho}(\vec{\phi}' \rhd \vec{\psi}')$;
4. $\vec{\phi} \rhd \vec{\psi} \dashv\vdash_{\mathbf{G}} \boldsymbol{\rho}(\boldsymbol{\tau}(\vec{\phi} \rhd \vec{\psi}))$.

Let $\mathbf{G} = \langle \mathcal{L}, \vdash_{\mathbf{G}}\rangle$ be a Gentzen system and let $\mathfrak{T}$ be a set of tr-sequents. The set $\mathfrak{T}$ is a *theory of* $\mathbf{G}$, or a $\mathbf{G}$-*theory* if it is closed under the consequence of $\mathbf{G}$. The set of all $\mathbf{G}$-theories is denoted by $\mathrm{Th}(\mathbf{G})$ and it becomes a complete lattice, $\mathbf{Th}(\mathbf{G}) = \langle \mathrm{Th}(\mathbf{G}), \subseteq\rangle$, when ordered by inclusion.

A Gentzen system $\mathbf{G} = \langle \mathcal{L}, \vdash_{\mathbf{G}} \rangle$ is said to be *standard* provided that it is over a trace not containing $(0,0)$ or $\mathcal{L}$ contains some constant symbols. In Theorem 6.8 of [16], which generalizes Theorem 2, the following characterization of equivalence of two standard Gentzen systems in terms of an isomorphism between the corresponding theory lattices is provided.

Theorem 4 (Raftery [16]) *Two standard Gentzen systems* $\mathbf{G}$ *and* $\mathbf{G}'$ *are equivalent iff there is a lattice isomorphism between* $\mathbf{Th}(\mathbf{G})$ *and* $\mathbf{Th}(\mathbf{G}')$ *that commutes with substitutions.*

Theorem 3 does not subsume Theorem 4 because a Gentzen system recast in a natural way as a π-institution (similar to the way used for a k-deductive system in the previous subsection), results, in general, in a π-institution that is not term. That is one of the reasons why Gil-Férez introduced in [11] the notion of a multi-term π-institution, that generalizes term π-institutions and is able to accommodate those π-institutions that naturally arise from Gentzen systems. These will be reviewed in the next subsection.

2.5 Equivalence Between Multi-Term π-Institutions

Since the class of term π-institutions does not encompass π-institutions that naturally arise from Gentzen systems, Gil-Férez [11] introduced a new wider class of π-institutions, called multi-term, that include these π-institutions. He then extended Theorem 3 to characterize the equivalence of multi-term π-institutions.

Let $\mathbf{Sign}$ be a category and $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a functor. The *category of elements of* SEN, denoted $\mathbf{Elt}(\mathrm{SEN})$ has as its objects all pairs $\langle \Sigma, \phi \rangle$, where $\Sigma \in |\mathbf{Sign}|$ and $\phi \in \mathrm{SEN}(\Sigma)$ and as morphisms $f : \langle \Sigma, \phi \rangle \to \langle \Sigma', \phi' \rangle$ all morphisms $f \in \mathbf{Sign}(\Sigma, \Sigma')$, such that $\mathrm{SEN}(f)(\phi) = \phi'$. In [11], a sentence functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is called *multi-term* if there exists an endomorphism $Y : \mathbf{Elt}(\mathrm{SEN}) \to \mathbf{Elt}(\mathrm{SEN})$, called a *multi-source signature-variable pair*, satisfying

- $Y(\langle \Sigma, \phi \rangle) = Y(\langle \Sigma', \phi' \rangle)$ and $Y(g) = i_{Y(\langle \Sigma, \phi \rangle)}$, for all $g : \langle \Sigma, \phi \rangle \to \langle \Sigma', \phi' \rangle$ in $\mathbf{Elt}(\mathrm{SEN})$;

and a natural transformation $f : Y \to \mathrm{I}_{\mathbf{Elt}(\mathrm{SEN})}$, where $\mathrm{I}_{\mathbf{Elt}(\mathrm{SEN})} : \mathbf{Elt}(\mathrm{SEN}) \to \mathbf{Elt}(\mathrm{SEN})$ denotes the identity functor.

Thus, for all $g : \langle \Sigma, \phi \rangle \to \langle \Sigma', \phi' \rangle$ in $\mathbf{Elt}(\mathrm{SEN})$, the following triangle commutes

$$\begin{tikzcd} & Y(\langle \Sigma, \phi \rangle) \arrow[dl, "f_{\langle \Sigma, \phi \rangle}"'] \arrow[dr, "f_{\langle \Sigma', \phi' \rangle}"] & \\ \langle \Sigma, \phi \rangle \arrow[rr, "g"'] & & \langle \Sigma', \phi' \rangle \end{tikzcd}$$

A π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ is called *multi-term* if SEN is a multi-term functor.

Note, first, that a term sentence functor SEN : $\mathbf{Sign} \to \mathbf{Set}$ is multi-term. For this, it suffices to set $Y : \mathbf{Elt}(\mathrm{SEN}) \to \mathbf{Elt}(\mathrm{SEN})$ to be the constant endofunctor with $Y(\langle \Sigma, \phi \rangle) = \langle V, v \rangle$, where $\langle V, v \rangle$ is a source signature-variable pair for SEN. Thus, every term π-institution is also multi-term.

Next, let us illustrate how a standard Gentzen system may be recast as a π-institution and show that, although the resulting π-institution is not in general term, it is multi-term.

Let tr be a trace and $\mathbf{G} = \langle \mathcal{L}, \vdash_{\mathbf{G}} \rangle$ be a standard Gentzen system with trace tr. We construct the π-institution $\mathcal{I}_{\mathbf{G}} = \langle \mathbf{Sign}_{\mathcal{L}}, \mathrm{SEN}_{\mathcal{L}}, C_{\mathbf{G}} \rangle$ as follows:

(i) $\mathbf{Sign}_{\mathcal{L}}$ is the one-object category with object V and morphisms $\sigma : V \to V$ all substitutions i.e., all endomorphisms $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$. Composition and identities are exactly as in the endomorphism monoid of $\mathbf{Fm}_{\mathcal{L}}(V)$.

(ii) $\mathrm{SEN}_{\mathcal{L}} : \mathbf{Sign}_{\mathcal{L}} \to \mathbf{Set}$ maps V to $\mathrm{SEN}_{\mathcal{L}}(V) = \text{tr-Seq}_{\mathcal{L}}(V)$ and $\sigma \in \mathbf{Sign}_{\mathcal{L}}(V, V)$ to $\mathrm{SEN}_{\mathcal{L}}(\sigma) : \mathrm{SEN}_{\mathcal{L}}(V) \to \mathrm{SEN}_{\mathcal{L}}(V)$, defined by $\mathrm{SEN}_{\mathcal{L}}(\sigma) = \sigma : \text{tr-Seq}_{\mathcal{L}}(V) \to \text{tr-Seq}_{\mathcal{L}}(V)$, the latter denoting the point-wise application of σ to every tr-sequent. It is easy to see that $\mathrm{SEN}_{\mathcal{L}}$ is a functor.

(iii) Finally, $C_{\mathbf{G}} : \mathcal{P}(\text{tr-Seq}_{\mathcal{L}}(V)) \to \mathcal{P}(\text{tr-Seq}_{\mathcal{L}}(V))$ is the standard closure operator associated with the consequence system $\vdash_{\mathbf{G}}$ of the Gentzen system $\mathbf{G}$, i.e., defined, for all $\mathfrak{P} \cup \{\vec{\phi} \rhd \vec{\psi}\} \subseteq \text{tr-Seq}_{\mathcal{L}}(V)$, by

$$\vec{\phi} \rhd \vec{\psi} \in C_{\mathbf{G}}(\mathfrak{P}) \quad \text{iff} \quad \mathfrak{P} \vdash_{\mathbf{G}} \vec{\phi} \rhd \vec{\psi}.$$

The triple $\mathcal{I}_{\mathbf{G}}$ determines a π-institution. It will be called the *π-institution associated with the Gentzen system* $\mathbf{G}$.

Unless tr is a singleton, $\mathrm{SEN}_{\mathcal{L}}$ is not term, whence $\mathcal{I}_{\mathbf{G}}$ is not a term π-institution. But, regardless of the form of tr, $\mathrm{SEN}_{\mathcal{L}}$ is multi-term. In fact, define the endofunctor $Y : \mathbf{Elt}(\mathrm{SEN}_{\mathcal{L}}) \to \mathbf{Elt}(\mathrm{SEN}_{\mathcal{L}})$ by setting, for all $\langle V, \phi_0, \ldots, \phi_{m-1} \rhd \psi_0, \ldots, \psi_{n-1} \rangle \in |\mathbf{Elt}(\mathrm{SEN}_{\mathcal{L}})|$,

$$Y(\langle V, \phi_0, \ldots, \phi_{m-1} \rhd \psi_0, \ldots, \psi_{n-1} \rangle) = \langle V, p_0, \ldots, p_{m-1} \rhd q_0, \ldots, q_{n-1} \rangle,$$

where p_i, q_j are distinct variables in V, $i < m, j < n$. Moreover, let $f : Y \to \mathrm{I}_{\mathbf{Elt}(\mathrm{SEN}_{\mathcal{L}})}$ be given by setting, for all $\langle V, \phi_0, \ldots, \phi_{m-1} \rhd \psi_0, \ldots, \psi_{n-1} \rangle \in |\mathbf{Elt}(\mathrm{SEN}_{\mathcal{L}})|$, $f_{\langle V, \vec{\phi} \rhd \vec{\psi} \rangle} : \langle V, \vec{p} \rhd \vec{q} \rangle \to \langle V, \vec{\phi} \rhd \vec{\psi} \rangle$ be the substitution sending p_i to ϕ_i and q_j to ψ_j, for all $i < m$ and $j < n$, and sending every other variable to ϕ_0 (or ψ_0 if $\vec{\phi}$ happens to be empty). With these definitions, Y becomes a multi-source signature-variable pair and f a natural transformation. Thus $\mathrm{SEN}_{\mathcal{L}}$ is multi-term and, as a consequence $\mathcal{I}_{\mathbf{G}}$ is a multi-term π-institution.

Gil-Férez, in the main theorem, Theorem 8.9, of [11] proves the following extension of Theorem 3, characterizing equivalence of two multi-term π-institutions, which by what has just been shown, includes also Theorem 4 as a special case.

Theorem 5 (Gil-Férez [11]) *If $\mathcal{I}$ and $\mathcal{I}'$ are two multi-term π-institutions, then $\mathcal{I}$ and $\mathcal{I}'$ are equivalent if and only if there exists an adjoint equivalence $\langle F, G, \eta, \epsilon\rangle : \mathbf{Th}(\mathcal{I}) \to \mathbf{Th}(\mathcal{I}')$ that commutes with substitutions.*

3 Consequence Systems over Power Sets

In this section, we show in more detail how one may define consequence systems over powersets of sentences associated with a given sentence functor. This is the main case that we are interested in and will motivate the introduction in the following sections of consequence families and consequence systems on arbitrary lattice families and module systems.

Let **Sign** be a category and SEN : **Sign** $\to$ **Set** be a set-valued functor. An **asymmetric consequence family over** SEN is a a collection of consequence relations $\vdash = \{\vdash_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$, where $\vdash_\Sigma \subseteq \mathcal{P}(\mathrm{SEN}(\Sigma)) \times \mathrm{SEN}(\Sigma)$ is such that, for all $X \cup Y \cup \{x, y, z\} \subseteq \mathrm{SEN}(\Sigma)$,

(1) if $x \in X$, then $X \vdash_\Sigma x$;

(2) if $X \vdash_\Sigma y$, for all $y \in Y$, and $Y \vdash_\Sigma z$, then $X \vdash_\Sigma z$.

The asymmetric consequence family $\vdash$ is said to be **Sign-invariant** or an **asymmetric consequence system over** SEN if, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $X \cup \{x\} \subseteq \mathrm{SEN}(\Sigma)$,

(3) if $X \vdash_\Sigma x$, then $\mathrm{SEN}(f)(X) \vdash_{\Sigma'} \mathrm{SEN}(f)(x)$.

The consequence family $\vdash$ is said to be **finitary** if, for all $\Sigma \in |\mathbf{Sign}|$, $\vdash_\Sigma$ is finitary in the ordinary sense, i.e., for all $X \cup \{x\} \subseteq \mathrm{SEN}(\Sigma)$, if $X \vdash_\Sigma x$, then, there exists a finite $X_0 \subseteq X$, such that $X_0 \vdash_\Sigma x$.

These definitions clearly generalize that of an *asymmetric* (*finitary*) *consequence relation over a set* S given in Section 2.3 of [10], by considering the case when **Sign** is the trivial one object category with single object $\bigstar$, in which SEN : $\bigstar \to$ **Set** reduces to a single set $S := \mathrm{SEN}(\bigstar)$. Note, also, that the present framework includes the notion of an *action* $\star : \Sigma \times S \to S$ of a monoid $\mathbf{\Sigma} = \langle \Sigma, \cdot, e\rangle$ on a set S, as follows: Consider the category $\mathbf{\Sigma}$ representing the monoid $\mathbf{\Sigma}$ in the well-known way, i.e., $\mathbf{\Sigma}$ has a single object Σ, its arrows correspond to the elements of the monoid, composition is the monoid composition and the identity arrow corresponds to the monoid identity. Let $\mathrm{SEN}(\Sigma) = S$ and $\mathrm{SEN}(m)(s) = m \star s$, for all $m \in \Sigma$ and all $s \in S$. It is easy to see that this setup exactly corresponds to a monoid action of $\mathbf{\Sigma}$ on S (see both [4] and [10]). The notion of a consequence system over SEN, given an action $\star$, corresponds exactly to a $\mathbf{\Sigma}$-*invariant consequence relation* on a set S.

Let **Sign** be a category and SEN : **Sign** $\to$ **Set** a set-valued functor. A **symmetric consequence family on** SEN is a family $\vdash = \{\vdash_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$, where $\vdash_\Sigma \subseteq \mathcal{P}(\mathrm{SEN}(\Sigma)) \times \mathcal{P}(\mathrm{SEN}(\Sigma))$, for all $\Sigma \in |\mathbf{Sign}|$, that satisfies, for all $\Sigma \in |\mathbf{Sign}|$, $X, Y, Z \subseteq \mathrm{SEN}(\Sigma)$,

(1) if $Y \subseteq X$, then $X \vdash_\Sigma Y$;

(2) if $X \vdash_\Sigma Y$ and $Y \vdash_\Sigma Z$, then $X \vdash_\Sigma Z$;

(3) $X \vdash_\Sigma \bigcup_{X \vdash_\Sigma Y} Y$;

The symmetric consequence family $\vdash$ is called **Sign-invariant** or a **symmetric consequence system on** SEN if, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $X, Y \subseteq \mathrm{SEN}(\Sigma)$ and $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

(4) $X \vdash_\Sigma Y$ implies $\mathrm{SEN}(f)(X) \vdash_{\Sigma'} \mathrm{SEN}(f)(Y)$.

A symmetric consequence family $\vdash$ on SEN is called **finitary** if, for all $\Sigma \in |\mathbf{Sign}|$ and all $X, Y \subseteq \mathrm{SEN}(\Sigma)$, if $X \vdash_\Sigma Y$ and Y is finite, then, there exists finite $X_0 \subseteq X$, such that $X_0 \vdash_\Sigma Y$.

Let $\vdash$ be an asymmetric consequence family on SEN. Its symmetric counterpart $\vdash^s$ may be defined by setting, for all $\Sigma \in |\mathbf{Sign}|$ and all $X, Y \subseteq \mathrm{SEN}(\Sigma)$,

$$X \vdash^s_\Sigma Y \quad \text{iff} \quad X \vdash_\Sigma y, \text{ for all } y \in Y.$$

Conversely, if $\vdash$ is a symmetric consequence family on SEN, its asymmetric counterpart $\vdash^a$ may be defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $X \cup \{x\} \subseteq \mathrm{SEN}(\Sigma)$, by

$$X \vdash^a_\Sigma x \quad \text{iff} \quad X \vdash_\Sigma \{x\}.$$

Lemma 6 *Symmetric consequence families on* SEN *are in bijective correspondence with asymmetric consequence families on* SEN *via the correspondence* $\vdash \mapsto \vdash^a$ *and* $\vdash \mapsto \vdash^s$. *Moreover, finitarity and* **Sign**-*invariance are preserved under these maps.*

In Section 4.2, starting from the prototypical example of symmetric consequence families, as defined here, we will define the encompassing notion of a consequence family on an arbitrary complete lattice family.

Given a category **Sign**, let us define, for all $\Sigma, \Sigma', \Sigma'' \in |\mathbf{Sign}|$ and all $A_1 \subseteq \mathbf{Sign}(\Sigma, \Sigma')$, $A_2 \subseteq \mathbf{Sign}(\Sigma', \Sigma'')$,

$$A_2 \circ^{\Sigma,\Sigma'}_{\Sigma',\Sigma''} A_1 = \{a_2 \circ a_1 : a_1 \in A_1, a_2 \in A_2\}.$$

Then, for all $\Sigma, \Sigma', \Sigma'' \in |\mathbf{Sign}|$, all $A_1 \subseteq \mathbf{Sign}(\Sigma, \Sigma')$, $A_2 \subseteq \mathbf{Sign}(\Sigma', \Sigma'')$ and $B \subseteq \mathbf{Sign}(\Sigma, \Sigma'')$, we have that

$$A_2 \circ^{\Sigma,\Sigma'}_{\Sigma',\Sigma''} A_1 \subseteq B \quad \text{iff} \quad A_2 \subseteq B /^{\Sigma,\Sigma'}_{\Sigma',\Sigma''} A_1 \quad \text{iff} \quad A_1 \subseteq A_2 \backslash^{\Sigma,\Sigma'}_{\Sigma',\Sigma''} B,$$

where

$$\begin{array}{rcl} B /^{\Sigma,\Sigma'}_{\Sigma',\Sigma''} A_1 & = & \{a \in \mathbf{Sign}(\Sigma', \Sigma'') : \{a\} \circ^{\Sigma,\Sigma'}_{\Sigma',\Sigma''} A_1 \subseteq B\}; \\ A_2 \backslash^{\Sigma,\Sigma'}_{\Sigma',\Sigma''} B & = & \{a \in \mathbf{Sign}(\Sigma, \Sigma') : A_2 \circ^{\Sigma,\Sigma'}_{\Sigma',\Sigma''} \{a\} \subseteq B\}. \end{array}$$

Note, also, the following facts: Given a category **Sign** and a functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, there exists, for every $\Sigma, \Sigma' \in |\mathbf{Sign}|$, a mapping

$$\star^{\Sigma,\Sigma'} : \mathcal{P}(\mathbf{Sign}(\Sigma, \Sigma')) \times \mathcal{P}(\mathrm{SEN}(\Sigma)) \to \mathcal{P}(\mathrm{SEN}(\Sigma')),$$

defined, for all $A \subseteq \mathbf{Sign}(\Sigma, \Sigma')$ and all $X \subseteq \mathrm{SEN}(\Sigma)$, by

$$A \star^{\Sigma,\Sigma'} X = \{\mathrm{SEN}(a)(x) : a \in A, x \in X\}.$$

This family of mappings satisfies, for all $\Sigma, \Sigma', \Sigma'' \in |\mathbf{Sign}|$, all $X \subseteq \mathrm{SEN}(\Sigma)$, all $A_2 \subseteq \mathbf{Sign}(\Sigma, \Sigma')$ and all $A_1 \subseteq \mathbf{Sign}(\Sigma', \Sigma'')$,

(1) $(A_1 \circ^{\Sigma,\Sigma'}_{\Sigma',\Sigma''} A_2) \star^{\Sigma,\Sigma''} X = A_1 \star^{\Sigma',\Sigma''} (A_2 \star^{\Sigma,\Sigma'} X)$;

(2) $\{\mathrm{id}_\Sigma\} \star^{\Sigma,\Sigma} X = X$.

Moreover, for all $A \subseteq \mathbf{Sign}(\Sigma, \Sigma')$, $X \subseteq \mathrm{SEN}(\Sigma)$ and $Y \subseteq \mathrm{SEN}(\Sigma')$, we have that

$$A \star^{\Sigma,\Sigma'} X \subseteq Y \quad \text{iff} \quad A \subseteq Y /^{\Sigma,\Sigma'} X \quad \text{iff} \quad X \subseteq A \backslash^{\Sigma,\Sigma'} Y,$$

where

$$\begin{array}{rcl} Y /^{\Sigma,\Sigma'} X &=& \{a \in \mathbf{Sign}(\Sigma, \Sigma') : \{a\} \star^{\Sigma,\Sigma'} X \subseteq Y\} \\ A \backslash^{\Sigma,\Sigma'} Y &=& \{x \in \mathrm{SEN}(\Sigma) : A \star^{\Sigma,\Sigma'} \{x\} \subseteq Y\}. \end{array}$$

If all conditions above are fulfilled, we say that $\mathcal{P}\mathrm{SEN}$ is a **$\mathbf{Sign}^{\mathcal{P}}$-module system**, where the superscript $^{\mathcal{P}}$ is intended to suggest the use of sets of morphisms, rather than single morphisms, acting on sets of sentences of the sentence functor SEN.

Let $\mathcal{P}\mathrm{SEN}_1 : \mathbf{Sign}_1 \to \mathbf{Set}$ be a $\mathbf{Sign}_1^{\mathcal{P}}$-module system and $\mathcal{P}\mathrm{SEN}_2 : \mathbf{Sign}_2 \to \mathbf{Set}$ be a $\mathbf{Sign}_2^{\mathcal{P}}$-module system. A pair $\langle F, \alpha \rangle : \mathcal{P}\mathrm{SEN}_1 \to \mathcal{P}\mathrm{SEN}_2$, such that $F : \mathbf{Sign}_1 \to \mathbf{Sign}_2$ is a functor and $\alpha = \{\alpha_\Sigma\}_{\Sigma \in |\mathbf{Sign}_1|}$ is a collection of mappings (not necessarily constituting a natural transformation) $\alpha_\Sigma : \mathcal{P}(\mathrm{SEN}_1(\Sigma)) \to \mathcal{P}(\mathrm{SEN}_2(F(\Sigma)))$ is said to be **$(\mathbf{Sign}_1^{\mathcal{P}}, \mathbf{Sign}_2^{\mathcal{P}})$-invariant** or **structural** if, for all $\Sigma, \Sigma' \in |\mathbf{Sign}_1|$, all $A \subseteq \mathbf{Sign}_1(\Sigma, \Sigma')$ and all $X \subseteq \mathrm{SEN}_1(\Sigma)$,

$$\alpha_{\Sigma'}(A \star^{\Sigma,\Sigma'} X) = F(A) \star^{F(\Sigma),F(\Sigma')} \alpha_\Sigma(X).$$

A $(\mathbf{Sign}^{\mathcal{P}}, \mathbf{Sign}^{\mathcal{P}})$-invariant collection of mappings

$$\alpha_\Sigma : \mathcal{P}(\mathrm{SEN}_1(\Sigma)) \to \mathcal{P}(\mathrm{SEN}_2(F(\Sigma))),$$

with $\mathrm{SEN}_1, \mathrm{SEN}_2 : \mathbf{Sign} \to \mathbf{Set}$ will simply be called **$\mathbf{Sign}^{\mathcal{P}}$-invariant**.

Note that $\alpha_\Sigma : \mathcal{P}(\mathrm{SEN}_1(\Sigma)) \to \mathcal{P}(\mathrm{SEN}_2(F(\Sigma)))$ is $(\mathbf{Sign}_1^{\mathcal{P}}, \mathbf{Sign}_2^{\mathcal{P}})$-invariant, then, for all $\Sigma, \Sigma' \in |\mathbf{Sign}_1|$, all $a \in \mathbf{Sign}_1(\Sigma, \Sigma')$ and all $x \in \mathrm{SEN}_1(\Sigma)$,

$$\alpha_{\Sigma'}(a \star^{\Sigma,\Sigma'} x) = F(a) \star^{F(\Sigma),F(\Sigma')} \alpha_\Sigma(x),$$

which, taking into account the definitions of $\star^{\Sigma,\Sigma'}$ and $\star^{F(\Sigma),F(\Sigma')}$, is equivalent to the naturality of $\alpha : \mathcal{P}\mathrm{SEN} \to \mathcal{P}\mathrm{SEN}' \circ F$.

The pair $\langle F, \alpha \rangle$, as above, is said to **preserve unions** if, for all $\Sigma \in |\mathbf{Sign}_1|$ and all $\mathcal{X} \subseteq \mathcal{P}(\mathrm{SEN}_1(\Sigma))$,

$$\alpha_\Sigma(\bigcup \mathcal{X}) = \bigcup_{X \in \mathcal{X}} \alpha_\Sigma(X).$$

Let $\mathbf{Sign}_1, \mathbf{Sign}_2$ be categories, $\mathrm{SEN}_1 : \mathbf{Sign}_1 \to \mathbf{Set}$ and $\mathrm{SEN}_2 : \mathbf{Sign}_2 \to \mathbf{Set}$ be set-valued functors and $\vdash^1, \vdash^2$ consequence systems over $\mathrm{SEN}_1, \mathrm{SEN}_2$, respectively. If there exists an adjoint equivalence $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$, $\langle F, \alpha\rangle : \mathcal{P}\mathrm{SEN}_1 \to \mathcal{P}\mathrm{SEN}_2$ and $\langle G, \beta\rangle : \mathcal{P}\mathrm{SEN}_2 \to \mathcal{P}\mathrm{SEN}_1$ preserve unions and are structural and, for all $\Sigma \in |\mathbf{Sign}_1|$, all $X \cup \{x\} \subseteq \mathrm{SEN}_1(\Sigma)$ and all $\Sigma' \in |\mathbf{Sign}_2|$, $y \in \mathrm{SEN}_2(\Sigma')$ we have

(1) $X \vdash^1_\Sigma x$ iff $\alpha_\Sigma(X) \vdash^2_{F(\Sigma)} \alpha_\Sigma(x)$

(2) $y \dashv\vdash^2_{\Sigma'} \epsilon_{\Sigma'} \star^{F(G(\Sigma')),\Sigma'} \alpha_{G(\Sigma')}(\beta_{\Sigma'}(y))$

then $\vdash^1$ and $\vdash^2$ will be called **equivalent via** $\langle F, \alpha\rangle$, $\langle G, \beta\rangle$ **and** $\langle F, G, \eta, \epsilon\rangle$.

This framework generalizes the corresponding one presented in [10] and, in addition, captures the deductive equivalence of π-institutions as presented in [20].

4 Module Systems and Consequence Systems

4.1 Module Systems

Let **Sign** be a category and $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ a set-valued functor. The functor SEN is said to be a **complete lattice family** if, for all $\Sigma \in |\mathbf{Sign}|$, $\mathrm{SEN}(\Sigma)$ has the structure of a complete lattice, with the order relation denoted by $\leq^\Sigma$. We will use a superscript to keep track of the signature when referring to the order $\leq^\Sigma$, e.g., to the meet $\wedge^\Sigma$, the join $\vee^\Sigma$, etc., of the complete lattice $\mathrm{SEN}(\Sigma)$.

Let $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ be complete lattices (in the universal algebraic sense). A map $\star : A \times B \to C$ is called **residuated** if there exist maps $\backslash_\star : A \times C \to B$ and $/_\star : C \times B \to A$, called the **residuals** of $\star$, such that, for all $x \in A, y \in B$ and $z \in C$,

$$x \star y \leq z \quad \text{iff} \quad x \leq z/_\star y \quad \text{iff} \quad y \leq x\backslash_\star z.$$

A category **Sign** will be said to be a **complete residuated category** if, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $\mathbf{Sign}(\Sigma, \Sigma')$ has the structure of a complete lattice, with order, meet and join denoted, respectively, by $\leq^{\Sigma,\Sigma'}$, $\wedge^{\Sigma,\Sigma'}$ and $\vee^{\Sigma,\Sigma'}$, and, for every $\Sigma, \Sigma', \Sigma'' \in |\mathbf{Sign}|$, the composition operation

$$\circ^{\Sigma,\Sigma'}_{\Sigma',\Sigma''} : \mathbf{Sign}(\Sigma', \Sigma'') \times \mathbf{Sign}(\Sigma, \Sigma') \to \mathbf{Sign}(\Sigma, \Sigma'')$$

is residuated, with residuals $\backslash^{\Sigma,\Sigma'}_{\Sigma',\Sigma''} : \mathbf{Sign}(\Sigma', \Sigma'') \times \mathbf{Sign}(\Sigma, \Sigma'') \to \mathbf{Sign}(\Sigma, \Sigma')$ and $/^{\Sigma,\Sigma'}_{\Sigma',\Sigma''} : \mathbf{Sign}(\Sigma, \Sigma'') \times \mathbf{Sign}(\Sigma, \Sigma') \to \mathbf{Sign}(\Sigma', \Sigma'')$.

Consider an arbitrary category **Sign**. Define the **complexification** $\mathbf{Sign}^{\mathcal{P}}$ of **Sign** as follows:

- $|\mathbf{Sign}^{\mathcal{P}}| = |\mathbf{Sign}|$;
- $\mathbf{Sign}^{\mathcal{P}}(\Sigma, \Sigma') = \mathcal{P}(\mathbf{Sign}(\Sigma, \Sigma'))$, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$;

- $g^{\mathcal{P}} \circ^{\Sigma,\Sigma'}_{\Sigma',\Sigma''} f^{\mathcal{P}} = \{g \circ f : g \in g^{\mathcal{P}}, f \in f^{\mathcal{P}}\}$, for all $f^{\mathcal{P}} \in \mathbf{Sign}^{\mathcal{P}}(\Sigma,\Sigma')$, $g^{\mathcal{P}} \in \mathbf{Sign}^{\mathcal{P}}(\Sigma',\Sigma'')$;
- $i^{\mathcal{P}}_{\Sigma} = \{i_\Sigma\}$.

Proposition 7 *For every category* **Sign**, $\mathbf{Sign}^{\mathcal{P}}$ *is a complete residuated category, where* $f^{\mathcal{P}} \leq^{\Sigma,\Sigma'} g^{\mathcal{P}}$ *iff* $f^{\mathcal{P}} \subseteq g^{\mathcal{P}}$, *for all* $\Sigma,\Sigma' \in |\mathbf{Sign}|$ *and all* $f^{\mathcal{P}}, g^{\mathcal{P}} \in \mathbf{Sign}^{\mathcal{P}}(\Sigma,\Sigma')$.

Proof:
For all $\Sigma,\Sigma',\Sigma'' \in |\mathbf{Sign}|$, all $f^{\mathcal{P}} \in \mathbf{Sign}^{\mathcal{P}}(\Sigma,\Sigma')$, $g^{\mathcal{P}} \in \mathbf{Sign}^{\mathcal{P}}(\Sigma',\Sigma'')$ and all $h^{\mathcal{P}} \in \mathbf{Sign}^{\mathcal{P}}(\Sigma,\Sigma'')$, define the following operations:

$$\begin{array}{rcl} h^{\mathcal{P}} /^{\Sigma,\Sigma'}_{\Sigma',\Sigma''} f^{\mathcal{P}} & = & \{g \in \mathbf{Sign}(\Sigma',\Sigma'') : \{g\} \circ^{\Sigma,\Sigma'}_{\Sigma',\Sigma''} f^{\mathcal{P}} \subseteq h^{\mathcal{P}}\} \\ g^{\mathcal{P}} \backslash^{\Sigma,\Sigma'}_{\Sigma',\Sigma''} h^{\mathcal{P}} & = & \{f \in \mathbf{Sign}(\Sigma,\Sigma') : g^{\mathcal{P}} \circ^{\Sigma,\Sigma'}_{\Sigma',\Sigma''} \{f\} \subseteq h^{\mathcal{P}}\}. \end{array}$$

It is then easy to see that

$$\begin{array}{rl} g^{\mathcal{P}} \circ^{\Sigma,\Sigma'}_{\Sigma',\Sigma''} f^{\mathcal{P}} \leq^{\Sigma,\Sigma''} h^{\mathcal{P}} & \text{iff} \quad g^{\mathcal{P}} \leq^{\Sigma',\Sigma''} h^{\mathcal{P}} /^{\Sigma,\Sigma'}_{\Sigma',\Sigma''} f^{\mathcal{P}} \\ & \text{iff} \quad f^{\mathcal{P}} \leq^{\Sigma,\Sigma'} g^{\mathcal{P}} \backslash^{\Sigma,\Sigma'}_{\Sigma',\Sigma''} h^{\mathcal{P}}. \end{array}$$

□

Let **Sign** be a complete residuated category and SEN : **Sign** → **Set** a complete lattice family. The family SEN will be said to be a **complete lattice system** or (by analogy with [10]) a **Sign-module system** if the operation $\star^{\Sigma,\Sigma'} : \mathbf{Sign}(\Sigma,\Sigma') \times \mathrm{SEN}(\Sigma) \to \mathrm{SEN}(\Sigma')$ defined, for all $f \in \mathbf{Sign}(\Sigma,\Sigma')$ and all $\phi \in \mathrm{SEN}(\Sigma)$ by $f \star^{\Sigma,\Sigma'} \phi = \mathrm{SEN}(f)(\phi)$ is residuated, with residuals $\backslash^{\Sigma,\Sigma'} : \mathbf{Sign}(\Sigma,\Sigma') \times \mathrm{SEN}(\Sigma') \to \mathrm{SEN}(\Sigma)$ and $/^{\Sigma,\Sigma'} : \mathrm{SEN}(\Sigma') \times \mathrm{SEN}(\Sigma) \to \mathbf{Sign}(\Sigma,\Sigma')$.

Consider an arbitrary category **Sign** and an arbitrary functor SEN : **Sign** → **Set**. Define the **complexification** $\mathcal{P}$SEN : $\mathbf{Sign}^{\mathcal{P}} \to$ **Set** of SEN as follows:

- $\mathcal{P}\mathrm{SEN}(\Sigma) = \mathcal{P}(\mathrm{SEN}(\Sigma))$, for all $\Sigma \in |\mathbf{Sign}|$;
- $\mathcal{P}\mathrm{SEN}(f^{\mathcal{P}})(X) = \{\mathrm{SEN}(f)(x) : f \in f^{\mathcal{P}}, x \in X\}$, for all $\Sigma,\Sigma' \in |\mathbf{Sign}|$, all $f^{\mathcal{P}} \in \mathbf{Sign}^{\mathcal{P}}(\Sigma,\Sigma')$ and all $X \subseteq \mathrm{SEN}(\Sigma)$.

This defines a functor, since, for all $\Sigma,\Sigma',\Sigma'' \in |\mathbf{Sign}|$, all $f^{\mathcal{P}} \in \mathbf{Sign}^{\mathcal{P}}(\Sigma,\Sigma')$, all $g^{\mathcal{P}} \in \mathbf{Sign}^{\mathcal{P}}(\Sigma',\Sigma'')$ and all $X \subseteq \mathrm{SEN}(\Sigma)$,

$$\begin{array}{l} \mathcal{P}\mathrm{SEN}(g^{\mathcal{P}})(\mathcal{P}\mathrm{SEN}(f^{\mathcal{P}})(X)) \\ \qquad = \mathcal{P}\mathrm{SEN}(g^{\mathcal{P}})(\{\mathrm{SEN}(f)(X) : f \in f^{\mathcal{P}}, x \in X\}) \\ \qquad = \{\mathrm{SEN}(g)(\mathrm{SEN}(f)(x)) : g \in g^{\mathcal{P}}, f \in f^{\mathcal{P}}, x \in X\} \\ \qquad = \{\mathrm{SEN}(g \circ f)(x) : g \in g^{\mathcal{P}}, f \in f^{\mathcal{P}}, x \in X\} \\ \qquad = \{\mathrm{SEN}(h)(x) : h \in g^{\mathcal{P}} \circ^{\Sigma,\Sigma'}_{\Sigma',\Sigma''} f^{\mathcal{P}}, x \in X\} \\ \qquad = \mathcal{P}\mathrm{SEN}(g^{\mathcal{P}} \circ^{\Sigma,\Sigma'}_{\Sigma',\Sigma''} f^{\mathcal{P}})(X). \end{array}$$

Moreover, we have:

Proposition 8 *Let* **Sign** *be a category and* SEN : **Sign** $\rightarrow$ **Set** *be a functor. Then* $\mathcal{P}$SEN : $\mathbf{Sign}^{\mathcal{P}} \rightarrow \mathbf{Set}$ *is a* $\mathbf{Sign}^{\mathcal{P}}$*-module system.*

Proof: For all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, all $f^{\mathcal{P}} \in \mathbf{Sign}^{\mathcal{P}}(\Sigma, \Sigma')$, all $X \subseteq \mathrm{SEN}(\Sigma)$ and all $Y \subseteq \mathrm{SEN}(\Sigma')$, define the following operations:

$$\begin{array}{rcl} Y/^{\Sigma,\Sigma'}X & = & \{f \in \mathbf{Sign}(\Sigma, \Sigma') : \mathrm{SEN}(f)(X) \subseteq Y\} \\ f^{\mathcal{P}}\backslash^{\Sigma,\Sigma'}Y & = & \{x \in \mathrm{SEN}(\Sigma) : \mathcal{P}\mathrm{SEN}(f^{\mathcal{P}})(\{x\}) \subseteq Y\}. \end{array}$$

It is then easy to see that

$$f^{\mathcal{P}} \star^{\Sigma,\Sigma'} X \leq^{\Sigma'} Y \quad \text{iff} \quad f^{\mathcal{P}} \leq^{\Sigma,\Sigma'} Y/^{\Sigma,\Sigma'}X \quad \text{iff} \quad X \leq^{\Sigma} f^{\mathcal{P}}\backslash^{\Sigma,\Sigma'}Y.$$

$\square$

In Lemma 9, which is an analog of Lemma 3.7 of [10], we list several properties of the operations involved in the definitions of **Sign**-module systems, which are inherited from corresponding well-known properties from the theory of residuated lattices. We provide, however, a few of the proofs to give a feeling to the reader not familiar with the lattice-theoretic results.

Lemma 9 *Let* **Sign** *be a complete residuated category and* SEN : **Sign** $\rightarrow$ **Set** *be a* **Sign***-module system. Then the following hold, for all* $\Sigma, \Sigma' \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma), \psi \in \mathrm{SEN}(\Sigma')$ *and* $a \in \mathbf{Sign}(\Sigma, \Sigma')$*:*

(1) $\star^{\Sigma,\Sigma'} : \mathbf{Sign}(\Sigma, \Sigma') \times \mathrm{SEN}(\Sigma) \rightarrow \mathrm{SEN}(\Sigma')$ *preserves arbitrary joins in both coordinates. In particular, it is order-preserving in both coordinates.*

(2) *The operations* $\backslash^{\Sigma,\Sigma'}$ *and* $/^{\Sigma,\Sigma'}$ *preserve arbitrary meets in the numerator and they convert arbitrary joins in the denominator to arbitrary meets. In particular, they are both order-preserving in the numerator and order-reversing in the denominator.*

(3) $(\psi/^{\Sigma,\Sigma'}\phi) \star^{\Sigma,\Sigma'} \phi \leq^{\Sigma'} \psi$*;*

(4) $a \star^{\Sigma,\Sigma'} (a\backslash^{\Sigma,\Sigma'}\psi) \leq^{\Sigma'} \psi$*;*

(5) $\phi \leq^{\Sigma} a\backslash^{\Sigma,\Sigma'}(a \star^{\Sigma,\Sigma'} \phi)$ *and* $a \leq^{\Sigma,\Sigma'} (a \star^{\Sigma,\Sigma'} \phi)/^{\Sigma,\Sigma'}\phi$*;*

(6) $(a\backslash^{\Sigma,\Sigma'}\psi)/^{\Sigma,\Sigma'}\phi = a\backslash^{\Sigma,\Sigma'}(\psi/^{\Sigma,\Sigma'}\phi)$*;*

(7) $[(\psi/^{\Sigma,\Sigma'}\phi) \star^{\Sigma,\Sigma'} \phi]/^{\Sigma,\Sigma'}\phi = \psi/^{\Sigma,\Sigma'}\phi$*;*

(8) $i_{\Sigma} \leq^{\Sigma,\Sigma} \phi/^{\Sigma,\Sigma}\phi$*;*

(9) $(\phi/^{\Sigma,\Sigma}\phi) \star^{\Sigma,\Sigma} \phi = \phi$.

Proof:

(1) Let $A \subseteq \mathbf{Sign}(\Sigma,\Sigma')$ and $\phi \in \mathrm{SEN}(\Sigma)$. Then, clearly, $(\bigvee^{\Sigma,\Sigma'} A) \star^{\Sigma,\Sigma'} \phi \leq^{\Sigma'} (\bigvee^{\Sigma,\Sigma'} A) \star^{\Sigma,\Sigma'} \phi$, which implies that $\bigvee^{\Sigma,\Sigma'} A \leq^{\Sigma,\Sigma'} ((\bigvee^{\Sigma,\Sigma'} A) \star^{\Sigma,\Sigma'} \phi)/^{\Sigma,\Sigma'}\phi$. Therefore, for all $a \in A$, $a \leq^{\Sigma,\Sigma'} ((\bigvee^{\Sigma,\Sigma'} A) \star^{\Sigma,\Sigma'} \phi)/^{\Sigma,\Sigma'}\phi$. This shows that $a \star^{\Sigma,\Sigma'} \phi \leq^{\Sigma'} (\bigvee^{\Sigma,\Sigma'} A) \star^{\Sigma,\Sigma'} \phi$, whence $\bigvee^{\Sigma'}_{a\in A}(a \star^{\Sigma,\Sigma'} \phi) \leq^{\Sigma'} (\bigvee^{\Sigma,\Sigma'} A) \star^{\Sigma,\Sigma'} \phi$. For the reverse inequality, we have, for all $a \in A$, $a \star^{\Sigma,\Sigma'} \phi \leq^{\Sigma'} \bigvee^{\Sigma'}_{a\in A}(a \star^{\Sigma,\Sigma'} \phi)$, whence $a \leq^{\Sigma,\Sigma'} \bigvee^{\Sigma'}_{a\in A}(a \star^{\Sigma,\Sigma'} \phi)/^{\Sigma,\Sigma'}\phi$. Therefore, $\bigvee^{\Sigma,\Sigma'} A \leq^{\Sigma,\Sigma'} \bigvee^{\Sigma'}_{a\in A}(a \star^{\Sigma,\Sigma'} \phi)/^{\Sigma,\Sigma'}\phi$, showing that $(\bigvee^{\Sigma,\Sigma'} A) \star^{\Sigma,\Sigma'} \phi \leq^{\Sigma'} \bigvee^{\Sigma'}_{a\in A}(a \star^{\Sigma,\Sigma'} \phi)$.

Preservation of joins in the second coordinate may be shown similarly.

(2) Let $\Psi \subseteq \mathrm{SEN}(\Sigma')$ and $\phi \in \mathrm{SEN}(\Sigma)$. Then, we have $(\bigwedge^{\Sigma'} \Psi)/^{\Sigma,\Sigma'}\phi \leq^{\Sigma,\Sigma'} (\bigwedge^{\Sigma'} \Psi)/^{\Sigma,\Sigma'}\phi$ iff $((\bigwedge^{\Sigma'} \Psi)/^{\Sigma,\Sigma'}\phi) \star^{\Sigma,\Sigma'} \phi \leq^{\Sigma'} \bigwedge^{\Sigma'} \Psi \leq^{\Sigma'} \psi$, for all $\psi \in \Psi$. Therefore, $(\bigwedge^{\Sigma'} \Psi)/^{\Sigma,\Sigma'}\phi \leq^{\Sigma,\Sigma'} \psi/^{\Sigma,\Sigma'}\phi$, for all $\psi \in \Psi$, showing that $(\bigwedge^{\Sigma'} \Psi)/^{\Sigma,\Sigma'}\phi \leq^{\Sigma,\Sigma'} \bigwedge_{\psi\in\Psi}(\psi/^{\Sigma,\Sigma'}\phi)$. For the converse inequality, for all $\psi \in \Psi$, $\bigwedge^{\Sigma,\Sigma'}_{\psi\in\Psi}(\psi/^{\Sigma,\Sigma'}\phi) \leq^{\Sigma,\Sigma'} \psi/^{\Sigma,\Sigma'}\phi$, whence $(\bigwedge^{\Sigma,\Sigma'}_{\psi\in\Psi}(\psi/^{\Sigma,\Sigma'}\phi)) \star^{\Sigma,\Sigma'} \phi \leq^{\Sigma'} \psi$. Thus, $(\bigwedge^{\Sigma,\Sigma'}_{\psi\in\Psi}(\psi/^{\Sigma,\Sigma'}\phi)) \star^{\Sigma,\Sigma'} \phi \leq^{\Sigma'} \bigwedge^{\Sigma'} \Psi$, showing that $\bigwedge^{\Sigma,\Sigma'}_{\psi\in\Psi}(\psi/^{\Sigma,\Sigma'}\phi) \leq^{\Sigma,\Sigma'} (\bigwedge^{\Sigma'} \Psi)/^{\Sigma,\Sigma'}\phi$.

Conversion of arbitrary joins in the denominator to arbitrary meets may be shown similarly. Moreover, both parts for $\backslash^{\Sigma,\Sigma'}$ also follow along the same lines.

(3)-(9) May be proven using similar arguments. □

4.2 Consequence Systems on Module Systems

Let SEN be a complete lattice family. Motivated by symmetric consequence families on the power sets of sentence functors, as defined in Section 3, we define a **symmetric consequence family on** SEN to be a collection $\vdash = \{\vdash_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ of binary relations $\vdash_\Sigma \subseteq \mathrm{SEN}(\Sigma)\times\mathrm{SEN}(\Sigma)$, such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $x, y, z \in \mathrm{SEN}(\Sigma)$,

(1) if $y \leq^\Sigma x$, then $x \vdash_\Sigma y$;

(2) if $x \vdash_\Sigma y$ and $y \vdash_\Sigma z$, then $x \vdash_\Sigma z$;

(3) $x \vdash_\Sigma \bigvee^\Sigma_{x\vdash_\Sigma y} y$;

For every $\Sigma \in |\mathbf{Sign}|$, $\vdash_\Sigma$ satisfies conditions (1) and (2), above, iff it is a pre-order on $\mathrm{SEN}(\Sigma)$ containing the binary relation $\geq^\Sigma$.

If **Sign** is a complete residuated category and SEN is a **Sign**-module system, then a symmetric consequence family on SEN is called a **symmetric consequence system** if it is **structural**, i.e., for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, all $x, y \in \mathrm{SEN}(\Sigma)$ and all $a \in \mathbf{Sign}(\Sigma,\Sigma')$, we have

$$x \vdash_\Sigma y \quad \text{implies} \quad a \star^{\Sigma,\Sigma'} x \vdash_{\Sigma'} a \star^{\Sigma,\Sigma'} y.$$

4.3 Module System Morphisms

In all examples of translations between various syntactic entities that were presented in Section 2, the translations applied to sets of sentences, as was done in Section 3, are union-preserving. This notion is captured by the concept of a residuated map.

Let $\mathbf{Sign}_1, \mathbf{Sign}_2$ be two categories and $\mathrm{SEN}_1 : \mathbf{Sign}_1 \to \mathbf{Set}$ and $\mathrm{SEN}_2 : \mathbf{Sign}_2 \to \mathbf{Set}$ be two complete lattice families. By a **map** $\langle F, \alpha\rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2$ we will understand a functor $F : \mathbf{Sign}_1 \to \mathbf{Sign}_2$ and a family $\alpha = \{\alpha_\Sigma\}_{\Sigma \in |\mathbf{Sign}_1|}$ of mappings $\alpha_\Sigma : \mathrm{SEN}_1(\Sigma) \to \mathrm{SEN}_2(F(\Sigma))$, $\Sigma \in |\mathbf{Sign}_1|$. A map $\langle F, \alpha\rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2$ is called **residuated** if there exists a collection $\alpha^* = \{\alpha^*_\Sigma\}_{\Sigma \in |\mathbf{Sign}_1|}$, called the **residual** of $\langle F, \alpha\rangle$, with $\alpha^*_\Sigma : \mathrm{SEN}_2(F(\Sigma)) \to \mathrm{SEN}_1(\Sigma)$, satisfying, for all $\Sigma \in |\mathbf{Sign}_1|$, $x \in \mathrm{SEN}_1(\Sigma)$ and $y \in \mathrm{SEN}_2(F(\Sigma))$,

$$\alpha_\Sigma(x) \leq^{F(\Sigma)} y \quad \text{iff} \quad x \leq^\Sigma \alpha^*_\Sigma(y).$$

It turns out that the residual of a residuated map $\langle F, \alpha\rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2$ is uniquely determined by setting, for all $\Sigma \in |\mathbf{Sign}_1|$ and all $y \in \mathrm{SEN}_2(F(\Sigma))$,

$$\alpha^*_\Sigma(y) = \max\{x \in \mathrm{SEN}_1(\Sigma) : \alpha_\Sigma(x) \leq^{F(\Sigma)} y\}.$$

The following lemma adapts well-known results on residuated maps to the current context. It is an analog of Lemma 3.1 of [10].

Lemma 10 *Let* $\mathbf{Sign}_1, \mathbf{Sign}_2$ *and* $\mathbf{Sign}_3$ *be categories,* $\mathrm{SEN}_1 : \mathbf{Sign}_1 \to \mathbf{Set}$, $\mathrm{SEN}_2 : \mathbf{Sign}_2 \to \mathbf{Set}$ *and* $\mathrm{SEN}_3 : \mathbf{Sign}_3 \to \mathbf{Set}$ *complete lattice families and* $\langle F, \alpha\rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2$ *and* $\langle G, \beta\rangle : \mathrm{SEN}_2 \to \mathrm{SEN}_3$ *residuated maps.*

(1) *The map* $\langle F, \alpha\rangle$ *preserves arbitrary joins and the map* α^* *preserves arbitrary meets;*

(2) *For every* $\Sigma \in |\mathbf{Sign}_1|$, *all* $\phi \in \mathrm{SEN}_1(\Sigma)$ *and all* $\psi \in \mathrm{SEN}_2(F(\Sigma))$, *we have* $x \leq^\Sigma \alpha^*_\Sigma(\alpha_\Sigma(x))$ *and* $\alpha_\Sigma(\alpha^*_\Sigma(y))) \leq^{F(\Sigma)} y$;

(3) $\langle G, \beta\rangle \circ \langle F, \alpha\rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_3$ *is residuated with residual* $(\beta\alpha)^* = \alpha^*\beta^*$.

Let $\mathbf{Sign}_1, \mathbf{Sign}_2$ be complete residuated categories and $\mathrm{SEN}_1 : \mathbf{Sign}_1 \to \mathbf{Set}$, $\mathrm{SEN}_2 : \mathbf{Sign}_2 \to \mathbf{Set}$ be module systems. A map $\langle F, \alpha\rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2$ is called **structural** if, for all $\Sigma, \Sigma' \in |\mathbf{Sign}_1|$, $a \in \mathbf{Sign}_1(\Sigma, \Sigma')$ and $x \in \mathrm{SEN}_1(\Sigma)$,

$$\alpha_{\Sigma'}(a \star^{\Sigma,\Sigma'} x) = F(a) \star^{F(\Sigma),F(\Sigma')} \alpha_\Sigma(x). \tag{2}$$

In Equation (2), as elsewhere in the paper, $\star$ has been used to refer to both the action of $\mathbf{Sign}_1$ on SEN_1 and the action of $\mathbf{Sign}_2$ on SEN_2. This overloading of notation will sometimes be used in what follows to avoid multiple superscripts and/or subscripts. Hopefully it will not cause any confusion, since the meaning can be disambiguated from the context. As noted in Section 3, $\langle F, \alpha\rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2$ is structural iff $\alpha : \mathrm{SEN}_1 \to \mathrm{SEN}_2 \circ F$ is a natural transformation.

A **module system morphism** $\langle F, \alpha\rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2$ is a structural residuated map. Module system morphisms will be referred to also as **translations** by analogy with [10]. We use $\mathcal{M}$ to denote the category of all module systems and module system morphisms between them.

4.4 Closure Families over Complete Lattice Families

A **closure family** $\gamma : \mathrm{SEN} \to \mathrm{SEN}$ on a complete lattice family $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is a map $\langle \mathrm{I}_{\mathbf{Sign}}, \gamma \rangle : \mathrm{SEN} \to \mathrm{SEN}$ that satisfies, for all $\Sigma \in |\mathbf{Sign}|$ and all $x, y \in \mathrm{SEN}(\Sigma)$,

Expanding: $x \leq^\Sigma \gamma_\Sigma(x)$;

Monotone: $x \leq^\Sigma y$ implies $\gamma_\Sigma(x) \leq^\Sigma \gamma_\Sigma(y)$;

Idempotent: $\gamma_\Sigma(\gamma_\Sigma(x)) = \gamma_\Sigma(x)$;

An **interior family** $\gamma : \mathrm{SEN} \to \mathrm{SEN}$ on a complete lattice family $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, on the other hand, is a map $\langle \mathrm{I}_{\mathbf{Sign}}, \gamma \rangle : \mathrm{SEN} \to \mathrm{SEN}$ that satisfies, for all $\Sigma \in |\mathbf{Sign}|$ and all $x, y \in \mathrm{SEN}(\Sigma)$, monotonicity and idempotency together with

Contracting: $\gamma_\Sigma(x) \leq^\Sigma x$.

If $\gamma : \mathrm{SEN} \to \mathrm{SEN}$ is a closure family on a complete lattice family $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, we denote, for all $\Sigma \in |\mathbf{Sign}|$, by $\mathrm{SEN}^\gamma(\Sigma)$ the image of $\mathrm{SEN}(\Sigma)$ under γ_Σ, i.e.,

$$\mathrm{SEN}^\gamma(\Sigma) = \gamma_\Sigma(\mathrm{SEN}(\Sigma)), \text{ for all } \Sigma \in |\mathbf{Sign}|.$$

This defines a functor $\mathrm{SEN}^\gamma : \overline{\mathbf{Sign}} \to \mathbf{Set}$ from the discretization $\overline{\mathbf{Sign}}$ of $\mathbf{Sign}$ into $\mathbf{Set}$, which forms a complete lattice family with the order of $\mathrm{SEN}^\gamma(\Sigma)$ inherited from $\mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$.

Let $\mathbf{Sign}$ be a category and $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ a complete lattice family. A simple subfunctor (i.e., one with the same domain) $\mathrm{SEN}' : \overline{\mathbf{Sign}} \to \mathbf{Set}$ of the discretized functor $\overline{\mathrm{SEN}} : \overline{\mathbf{Sign}} \to \mathbf{Set}$ is called **completely meet closed** if, for all $\Sigma \in |\mathbf{Sign}|$ and all $X \subseteq \mathrm{SEN}'(\Sigma)$, $\bigwedge^\Sigma X \in \mathrm{SEN}'(\Sigma)$. Given a completely meet closed subfunctor $\mathrm{SEN}' : \overline{\mathbf{Sign}} \to \mathbf{Set}$ of the discretized functor $\overline{\mathrm{SEN}} : \overline{\mathbf{Sign}} \to \mathbf{Set}$, we define the map $\gamma^{\mathrm{SEN}'} : \overline{\mathrm{SEN}} \to \overline{\mathrm{SEN}}$ by setting, for all $\Sigma \in |\mathbf{Sign}|$ and all $x \in \mathrm{SEN}(\Sigma)$, $\gamma^{\mathrm{SEN}'}_\Sigma(x) = \bigwedge^\Sigma(\uparrow x \cap \mathrm{SEN}'(\Sigma))$, where the meet is the one of the complete lattice $\mathrm{SEN}(\Sigma)$. The following lemma adapts Lemma 3.3 of [10], which is a standard lattice theoretic result [2], to the context of closure families over complete lattice families. It can be proven by applying Lemma 3.3 of [10] to the present context signature-wise.

Lemma 11 *Let* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *be a complete lattice family,* $\gamma : \mathrm{SEN} \to \mathrm{SEN}$ *a closure family on* SEN *and* $\mathrm{SEN}' : \overline{\mathbf{Sign}} \to \mathbf{Set}$ *a completely meet closed subfunctor of the discretized functor* $\overline{\mathrm{SEN}} : \overline{\mathbf{Sign}} \to \mathbf{Set}$.

1. $\mathrm{SEN}^\gamma : \overline{\mathbf{Sign}} \to \mathbf{Set}$ *is a completely meet closed subfunctor of* $\overline{\mathrm{SEN}}$*;*
2. $\gamma^{\mathrm{SEN}'}$ *is a closure family on* $\overline{\mathrm{SEN}}$*;*
3. $\gamma^{\mathrm{SEN}^\gamma} = \gamma$ *and* $\mathrm{SEN}^{\gamma^{\mathrm{SEN}'}} = \mathrm{SEN}'$*;*

4. $\mathrm{SEN}^\gamma : \overline{\mathbf{Sign}} \to \mathbf{Set}$ *is a complete lattice family, such that, for all* $\Sigma \in |\mathbf{Sign}|$, $\langle \mathrm{SEN}^\gamma(\Sigma), \leq_\gamma^\Sigma\rangle$ *is a complete meet sub-semilattice of* $\mathrm{SEN}(\Sigma)$ *with join* $\bigvee_\gamma^\Sigma \gamma_\Sigma(X) = \gamma_\Sigma(\bigvee^\Sigma \gamma_\Sigma(X)) = \gamma_\Sigma(\bigvee^\Sigma X)$ *and meet* $\bigwedge_\gamma^\Sigma \gamma_\Sigma(X) = \bigwedge^\Sigma \gamma_\Sigma(X)$.

It is not difficult to see that $\gamma : \mathrm{SEN} \to \mathrm{SEN}$ is a closure family on the complete lattice family SEN iff the map $\langle \mathrm{I}_{\overline{\mathbf{Sign}}}, \gamma'\rangle : \overline{\mathrm{SEN}} \to \mathrm{SEN}^\gamma$, given, for all $\Sigma \in |\mathbf{Sign}|$ and all $x \in \mathrm{SEN}(\Sigma)$, by $\gamma'_\Sigma(x) = \gamma_\Sigma(x)$ is residuated and the "inclusion" family $\{\iota^\gamma\} = \{\iota^\gamma_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$, with $\iota^\gamma_\Sigma(x) = x$, for all $\Sigma \in |\mathbf{Sign}|, x \in \mathrm{SEN}^\gamma(\Sigma)$, is its residual. Sometimes, abusing notation slightly, we will write simply γ for the map $\langle \mathrm{I}_{\overline{\mathbf{Sign}}}, \gamma'\rangle : \overline{\mathrm{SEN}} \to \mathrm{SEN}^\gamma$.

The following lemma is a version of Lemma 3.4 of [10] applicable to residuated maps between complete lattice families. It can be proven by applying Lemma 3.4 of [10] signature-wise to the relevant mappings.

Lemma 12 *Let* $\mathbf{Sign}_1$, $\mathbf{Sign}_2$ *be categories,* $\mathrm{SEN}_1 : \mathbf{Sign}_1 \to \mathbf{Set}$ *and* $\mathrm{SEN}_2 : \mathbf{Sign}_2 \to \mathbf{Set}$ *be complete lattice families and* $\langle F, \alpha\rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2$ *a residuated map.*

(1) $\alpha^*\alpha$ *is a closure family on* SEN_1 *and, for all* $\Sigma \in |\mathbf{Sign}_1|$, $\alpha_\Sigma\alpha^*_\Sigma : \mathrm{SEN}_2(F(\Sigma)) \to \mathrm{SEN}_2(F(\Sigma))$ *is contracting, monotone and idempotent;*

(2) $\alpha\alpha^*\alpha = \alpha$ *and, for all* $\Sigma \in |\mathbf{Sign}_1|$, $\alpha^*_\Sigma\alpha_\Sigma\alpha^*_\Sigma = \alpha^*_\Sigma$*;*

(2) *For all* $\Sigma \in |\mathbf{Sign}_1|$, $\alpha^*_\Sigma(\alpha_\Sigma(\mathrm{SEN}_1(\Sigma)))$ *and* $\alpha_\Sigma(\alpha^*_\Sigma(\mathrm{SEN}_2(F(\Sigma))))$ *are isomorphic ordered sets.*

Let **Sign** be a complete residuated category and $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ a **Sign**-module system. A closure family $\gamma : \mathrm{SEN} \to \mathrm{SEN}$ is called **structural** or a **closure system** if, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, all $a \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $x \in \mathrm{SEN}(\Sigma)$,

$$a \star^{\Sigma,\Sigma'} \gamma_\Sigma(x) \leq^{\Sigma'} \gamma_{\Sigma'}(a \star^{\Sigma,\Sigma'} x).$$

Note that, if γ is structural, then SEN^γ can be extended to morphisms by defining $\mathrm{SEN}^\gamma(f)(x) = \gamma_{\Sigma'}(\mathrm{SEN}(f)(x))$, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $x \in \mathrm{SEN}(\Sigma)$, in such a way that $\mathrm{SEN}^\gamma : \mathbf{Sign} \to \mathbf{Set}$ becomes a functor, i.e., it is a complete lattice family (see also Lemma 16). The overloading of notation (using $\mathrm{SEN}^\gamma : \overline{\mathbf{Sign}} \to \mathbf{Set}$ and $\mathrm{SEN}^\gamma : \mathbf{Sign} \to \mathbf{Set}$) is unambiguous on objects and we will use it in the latter sense only when γ is structural.

Given a consequence family $\vdash$ on a complete lattice family $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, define the map $\gamma^\vdash : \mathrm{SEN} \to \mathrm{SEN}$ by setting, for all $\Sigma \in |\mathbf{Sign}|$ and all $x \in \mathrm{SEN}(\Sigma)$, $\gamma^\vdash_\Sigma(x) = \bigvee^\Sigma_{x\vdash_\Sigma y} y$. On the other hand, given a closure family $\gamma : \mathrm{SEN} \to \mathrm{SEN}$, define the consequence family $\vdash^\gamma$ on SEN by setting, for all $\Sigma \in |\mathbf{Sign}|$ and all $x, y \in \mathrm{SEN}(\Sigma)$, $x \vdash^\gamma_\Sigma y$ iff $y \leq^\Sigma \gamma_\Sigma(x)$. Then, we get the following (see Lemma 3.5 of [10])

Lemma 13 *Consequence families on a complete lattice family* SEN : **Sign** $\rightarrow$ **Set** *are in bijective correspondence with closure families on* SEN *via the maps* $\vdash \mapsto \gamma^{\vdash}$ *and* $\gamma \mapsto \vdash^{\gamma}$. *If* **Sign** *is a complete residuated category and* SEN *is a* **Sign**-*module system, then* $\vdash$ *is structural, i.e., a consequence system, iff* $\gamma^{\vdash}$ *is structural, i.e., a closure system.*

We start the study of closure systems on module systems in earnest in Subsection 4.6.

4.5 Theories

Let **Sign** be a category, SEN : **Sign** $\rightarrow$ **Set** a complete lattice family and $\vdash$ a consequence family on SEN. For $\Sigma \in |\mathbf{Sign}|$, a **$\Sigma$-theory of** $\vdash$ is an element $t \in \mathrm{SEN}(\Sigma)$, such that, for all $x \in \mathrm{SEN}(\Sigma)$,

$$t \vdash_\Sigma x \quad \text{implies} \quad x \leq^\Sigma t.$$

If t is a Σ-theory of $\vdash$, then $x \leq^\Sigma t$ and $x \vdash_\Sigma y$ imply $y \leq^\Sigma t$, for all $x, y \in \mathrm{SEN}(\Sigma)$. We use $\mathrm{Th}_\Sigma(\vdash)$ to denote the collection of all Σ-theories of $\vdash$.

The following result, which characterizes the collection $\mathrm{Th}_\Sigma(\vdash)$ in terms of the closure family $\gamma^{\vdash}$ is well-known. The proof is also presented as the proof of Lemma 3.6 of [10]. The proof of Lemma 14 uses the same arguments signature-wise and is, therefore, omitted.

Lemma 14 *If* $\vdash$ *is a consequence family on the complete lattice family* SEN : **Sign** $\rightarrow$ **Set**, *then* $\mathrm{Th}_\Sigma(\vdash) = \mathrm{SEN}^{\gamma^{\vdash}}(\Sigma)$, *for all* $\Sigma \in |\mathbf{Sign}|$.

Based on Lemma 14, we let $\mathrm{Th}(\vdash) = \mathrm{SEN}^{\gamma^{\vdash}} : \overline{\mathbf{Sign}} \rightarrow \mathbf{Set}$ be the **complete lattice family of theories of** $\vdash$.

4.6 Closure Systems over Module Systems

The following lemma provides alternative characterizations for closure systems over **Sign**-module systems, abstracting the ones given for structural closure operators in Lemma 3.8 of [10].

Lemma 15 *Let* **Sign** *be a complete residuated category,* SEN : **Sign** $\rightarrow$ **Set** *a* **Sign**-*module system and* $\gamma : \mathrm{SEN} \rightarrow \mathrm{SEN}$ *a closure family on* SEN. *Then the following statements are equivalent:*

(1) γ *is structural;*

(2) $\gamma_{\Sigma'}(a \star^{\Sigma,\Sigma'} \gamma_\Sigma(x)) = \gamma_{\Sigma'}(a \star^{\Sigma,\Sigma'} x)$, *for all* $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $x \in \mathrm{SEN}(\Sigma)$ *and* $a \in \mathbf{Sign}(\Sigma, \Sigma')$*;*

(3) $\gamma_{\Sigma'}(y) /^{\Sigma,\Sigma'} x = \gamma_{\Sigma'}(y) /^{\Sigma,\Sigma'} \gamma_\Sigma(x)$, *for all* $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $x \in \mathrm{SEN}(\Sigma)$ *and* $y \in \mathrm{SEN}(\Sigma')$*;*

(4) $\gamma_\Sigma(a\backslash^{\Sigma,\Sigma'}y) \leq^\Sigma a\backslash^{\Sigma,\Sigma'}\gamma_{\Sigma'}(y)$, *for all* $\Sigma,\Sigma' \in |\mathbf{Sign}|$, $y \in \mathrm{SEN}(\Sigma')$ *and* $a \in \mathbf{Sign}(\Sigma,\Sigma')$*;*

(5) $a\backslash^{\Sigma,\Sigma'}\gamma_{\Sigma'}(y) \in \mathrm{SEN}^\gamma(\Sigma)$, *for all* $\Sigma,\Sigma' \in |\mathbf{Sign}|$, $y \in \mathrm{SEN}(\Sigma')$ *and* $a \in \mathbf{Sign}(\Sigma,\Sigma')$.

Proof:

(1)↔(2) The right-to-left direction is obvious and for the left-to-right, we have, for all $\Sigma,\Sigma' \in |\mathbf{Sign}|$, all $a \in \mathbf{Sign}(\Sigma,\Sigma')$ and all $x \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{rcl}\gamma_{\Sigma'}(a\star^{\Sigma,\Sigma'}x) & \leq^{\Sigma'} & \gamma_{\Sigma'}(a\star^{\Sigma,\Sigma'}\gamma_\Sigma(x))\\ & \leq^{\Sigma'} & \gamma_{\Sigma'}(\gamma_{\Sigma'}(a\star^{\Sigma,\Sigma'}x))\\ & = & \gamma_{\Sigma'}(a\star^{\Sigma,\Sigma'}x).\end{array}$$

(1)→(3) Since $x \leq^\Sigma \gamma_\Sigma(x)$, by Lemma 9, $\gamma_{\Sigma'}(y)/^{\Sigma,\Sigma'}\gamma_\Sigma(x) \leq^{\Sigma,\Sigma'} \gamma_{\Sigma'}(y)/^{\Sigma,\Sigma'}x$.

For the reverse inequality, we have

$$\begin{array}{rcl}[\gamma_{\Sigma'}(y)/^{\Sigma,\Sigma'}x]\star^{\Sigma,\Sigma'}\gamma_\Sigma(x) & \leq^{\Sigma'} & \gamma_{\Sigma'}([\gamma_{\Sigma'}(y)/^{\Sigma,\Sigma'}x]\star^{\Sigma,\Sigma'}x)\\ & \leq^{\Sigma'} & \gamma_{\Sigma'}(\gamma_{\Sigma'}(y)) \quad \text{(by Lemma 9)}\\ & = & \gamma_{\Sigma'}(y),\end{array}$$

whence $\gamma_{\Sigma'}(y)/^{\Sigma,\Sigma'}x \leq^{\Sigma,\Sigma'} \gamma_{\Sigma'}(y)/^{\Sigma,\Sigma'}\gamma_\Sigma(x)$.

(3)→(1) Since $a\star^{\Sigma,\Sigma'}x \leq^{\Sigma'} \gamma_{\Sigma'}(a\star^{\Sigma,\Sigma'}x)$, we get that $a \leq^{\Sigma,\Sigma'} \gamma_{\Sigma'}(a\star^{\Sigma,\Sigma'}x)/^{\Sigma,\Sigma'}x = \gamma_{\Sigma'}(a\star^{\Sigma,\Sigma'}x)/^{\Sigma,\Sigma'}\gamma_\Sigma(x)$, showing that $a\star^{\Sigma,\Sigma'}\gamma_\Sigma(x) \leq^{\Sigma'} \gamma_{\Sigma'}(a\star^{\Sigma,\Sigma'}x)$.

(1)↔(4) We have $a\star^{\Sigma,\Sigma'}\gamma_\Sigma(a\backslash^{\Sigma,\Sigma'}y) \leq^{\Sigma'} \gamma_{\Sigma'}(a\star^{\Sigma,\Sigma'}(a\backslash^{\Sigma,\Sigma'}y)) \leq^{\Sigma'} \gamma_{\Sigma'}(y)$, whence $\gamma_\Sigma(a\backslash^{\Sigma,\Sigma'}y) \leq^\Sigma a\backslash^{\Sigma,\Sigma'}\gamma_{\Sigma'}(y)$. Conversely,

$$\begin{array}{rcl}a\star^{\Sigma,\Sigma'}\gamma_\Sigma(x) & \leq^{\Sigma'} & a\star^{\Sigma,\Sigma'}\gamma_\Sigma(a\backslash^{\Sigma,\Sigma'}(a\star^{\Sigma,\Sigma'}x))\\ & \leq^{\Sigma'} & a\star^{\Sigma,\Sigma'}[a\backslash^{\Sigma,\Sigma'}\gamma_{\Sigma'}(a\star^{\Sigma,\Sigma'}x)]\\ & \leq^{\Sigma'} & \gamma_{\Sigma'}(a\star^{\Sigma,\Sigma'}x).\end{array}$$

(1)→(5) We have

$$\begin{array}{rcl}a\star^{\Sigma,\Sigma'}\gamma_\Sigma(a\backslash^{\Sigma,\Sigma'}\gamma_{\Sigma'}(y)) & \leq^{\Sigma'} & \gamma_{\Sigma'}(a\star^{\Sigma,\Sigma'}(a\backslash^{\Sigma,\Sigma'}\gamma_{\Sigma'}(y)))\\ & \leq^{\Sigma'} & \gamma_{\Sigma'}(\gamma_{\Sigma'}(y))\\ & = & \gamma_{\Sigma'}(y),\end{array}$$

whence $\gamma_\Sigma(a\backslash^{\Sigma,\Sigma'}\gamma_{\Sigma'}(y)) \leq^\Sigma a\backslash^{\Sigma,\Sigma'}\gamma_{\Sigma'}(y)$. This is enough to show that $a\backslash^{\Sigma,\Sigma'}\gamma_{\Sigma'}(y) \in \mathrm{SEN}^\gamma(\Sigma)$.

(5)→(1) Since $a\star^{\Sigma,\Sigma'}x \leq^{\Sigma,\Sigma'} \gamma_{\Sigma'}(a\star^{\Sigma,\Sigma'}x)$, we get $x \leq^\Sigma a\backslash^{\Sigma,\Sigma'}\gamma_{\Sigma'}(a\star^{\Sigma,\Sigma'}x)$. Therefore, $\gamma_\Sigma(x) \leq^\Sigma a\backslash^{\Sigma,\Sigma'}\gamma_{\Sigma'}(a\star^{\Sigma,\Sigma'}x)$, yielding $a\star^{\Sigma,\Sigma'}\gamma_\Sigma(x) \leq^{\Sigma,\Sigma'} \gamma_{\Sigma'}(a\star^{\Sigma,\Sigma'}x)$.

□

Lemma 16 *Let* **Sign** *be a complete residuated category and* SEN : **Sign** → **Set** *a* **Sign***-module system. If* $\gamma : \mathrm{SEN} \to \mathrm{SEN}$ *is a closure system on* SEN*, then* $\mathrm{SEN}^\gamma : \mathbf{Sign} \to \mathbf{Set}$ *is a* **Sign***-module system, where, for all* $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $\star_\gamma^{\Sigma,\Sigma'} : \mathbf{Sign}(\Sigma, \Sigma') \times \mathrm{SEN}^\gamma(\Sigma) \to \mathrm{SEN}^\gamma(\Sigma')$ *is defined, for all* $a \in \mathbf{Sign}(\Sigma, \Sigma')$ *and all* $x \in \mathrm{SEN}^\gamma(\Sigma)$*, by* $a \star_\gamma^{\Sigma,\Sigma'} x = \gamma_{\Sigma'}(a \star^{\Sigma,\Sigma'} x)$*. Moreover,* $\gamma : \mathrm{SEN} \to \mathrm{SEN}^\gamma$ *is a* **Sign***-module system morphism.*

Proof:

Although this has been pointed out before, we will prove, first, that SEN^γ : **Sign** → **Set** is a functor. In fact, if $\Sigma, \Sigma', \Sigma'' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $g \in \mathbf{Sign}(\Sigma', \Sigma'')$ and $x \in \mathrm{SEN}^\gamma(\Sigma)$, we get, using the structurality of γ,

$$\begin{array}{rcl} \mathrm{SEN}^\gamma(g)(\mathrm{SEN}^\gamma(f)(x)) & = & \mathrm{SEN}^\gamma(g)(\gamma_{\Sigma'}(\mathrm{SEN}(f)(x))) \\ & = & \gamma_{\Sigma''}(\mathrm{SEN}(g)(\gamma_{\Sigma'}(\mathrm{SEN}(f)(x)))) \\ & = & \gamma_{\Sigma''}(\mathrm{SEN}(g)(\mathrm{SEN}(f)(x))) \\ & = & \gamma_{\Sigma''}(\mathrm{SEN}(gf)(x)) \\ & = & \mathrm{SEN}^\gamma(x). \end{array}$$

It is clear that $\mathrm{SEN}^\gamma(\Sigma)$ has the structure of a complete lattice, as a sublattice of $\mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$. Thus, for the first statement of the lemma, it suffices to show that $\star_\gamma$ is residuated. Let $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $a \in \mathbf{Sign}(\Sigma, \Sigma')$, $x \in \mathrm{SEN}^\gamma(\Sigma)$ and $y \in \mathrm{SEN}^\gamma(\Sigma')$. Then

$$\begin{array}{rll} a \star_\gamma^{\Sigma,\Sigma'} x \leq^{\Sigma'} y & \text{iff} & \gamma_{\Sigma'}(a \star^{\Sigma,\Sigma'} x) \leq^{\Sigma'} y \\ & \text{iff} & a \star^{\Sigma,\Sigma'} x \leq^{\Sigma'} y \\ & \text{iff} & x \leq^{\Sigma} a \backslash^{\Sigma,\Sigma'} y. \end{array}$$

Since, by Lemma 15, $a\backslash^{\Sigma,\Sigma'} y \in \mathrm{SEN}^\gamma(\Sigma)$, this string of equivalences proves that $\star_\gamma^{\Sigma,\Sigma'}$ is left residuated with residual $\backslash_\gamma^{\Sigma,\Sigma'}$, the restriction of $\backslash^{\Sigma,\Sigma'}$ to $\mathbf{Sign}(\Sigma, \Sigma') \times \mathrm{SEN}^\gamma(\Sigma')$. Similarly, we get

$$\begin{array}{rll} a \star_\gamma^{\Sigma,\Sigma'} x \leq^{\Sigma'} y & \text{iff} & \gamma_{\Sigma'}(a \star^{\Sigma,\Sigma'} x) \leq^{\Sigma'} y \\ & \text{iff} & a \star^{\Sigma,\Sigma'} x \leq^{\Sigma'} y \\ & \text{iff} & a \leq^{\Sigma,\Sigma'} y /^{\Sigma,\Sigma'} x, \end{array}$$

showing that $\star_\gamma^{\Sigma,\Sigma'}$ is also right residuated with residual $/_\gamma^{\Sigma,\Sigma'}$, the restriction of $/^{\Sigma,\Sigma'}$ to $\mathrm{SEN}^\gamma(\Sigma') \times \mathrm{SEN}^\gamma(\Sigma)$.

The last statement is easy to see, since $\gamma : \mathrm{SEN} \to \mathrm{SEN}^\gamma$ is residuated with residual $\iota^\gamma : \mathrm{SEN}^\gamma \to \mathrm{SEN}$ and is structural by Lemma 15. □

5 Representation and Equivalence

5.1 Representation

Let $\mathbf{Sign}_1$, $\mathbf{Sign}_2$ be categories. $\mathrm{SEN}_1 : \mathbf{Sign}_1 \to \mathbf{Set}$, $\mathrm{SEN}_2 : \mathbf{Sign}_2 \to \mathbf{Set}$ be two complete lattice families and $\gamma : \mathrm{SEN}_1 \to \mathrm{SEN}_1$, $\delta : \mathrm{SEN}_2 \to \mathrm{SEN}_2$

closure families on $\mathrm{SEN}_1, \mathrm{SEN}_2$, respectively. A **representation of** γ **in** δ is a residuated map $\langle F, f\rangle : \mathrm{SEN}_1^\gamma \to \mathrm{SEN}_2^\delta$, such that, for all $\Sigma \in |\mathbf{Sign}_1|$ and all $x, y \in \mathrm{SEN}_1^\gamma(\Sigma)$,

$$x \leq_\gamma^\Sigma y \quad \text{iff} \quad f_\Sigma(x) \leq_\delta^{F(\Sigma)} f_\Sigma(y).$$

A representation $\langle F, f\rangle : \mathrm{SEN}_1^\gamma \to \mathrm{SEN}_2^\delta$ of γ in δ is **induced** by a residuated map $\langle F, \alpha\rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2$ if, for all $\Sigma \in |\mathbf{Sign}|$ and all $x \in \mathrm{SEN}(\Sigma)$,

$$\begin{tikzcd}
\mathrm{SEN}(\Sigma) \arrow[r, "\alpha_\Sigma"] \arrow[d, "\gamma_\Sigma"'] & \mathrm{SEN}'(F(\Sigma)) \arrow[d, "\delta_{F(\Sigma)}"] \\
\mathrm{SEN}^\gamma(\Sigma) \arrow[r, "f_\Sigma"'] & \mathrm{SEN}'^\delta(F(\Sigma))
\end{tikzcd}$$

$$f_\Sigma(\gamma_\Sigma(x)) = \delta_{F(\Sigma)}(\alpha_\Sigma(x)).$$

Denote by $\vdash^\gamma$ an arbitrary consequence family on a complete lattice family $\mathrm{SEN}_1 : \mathbf{Sign}_1 \to \mathbf{Set}$ to underscore the fact that $\gamma : \mathrm{SEN}_1 \to \mathrm{SEN}_1$ is the canonically associated closure family. Then, a consequence family $\vdash^\gamma$ on SEN_1 is **represented** in the consequence family $\vdash^\delta$ on SEN_2 if the closure family γ is represented in δ. Similarly, a representation of $\vdash^\gamma$ in $\vdash^\delta$ is **induced** by a residuated map $\langle F, \alpha\rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2$ if the representation of the closure family γ in the closure family δ is induced by $\langle F, \alpha\rangle$. Corollary 20 shows that $\vdash^\gamma$ is represented in $\vdash^\delta$ via $\langle F, \alpha\rangle$ iff, for all $\Sigma \in |\mathbf{Sign}_1|$ and all $x, y \in \mathrm{SEN}_1(\Sigma)$,

$$x \vdash_\Sigma^\gamma y \quad \text{iff} \quad \alpha_\Sigma(x) \vdash_{F(\Sigma)}^\delta \alpha_\Sigma(y).$$

The following lemma is an analog of Lemma 4.1 of [10]. Informally speaking, given a residuated map between two complete lattice families and a closure family on the codomain of the residuated map, it provides a way of endowing the domain of the residuated map with a closure family. More precisely, the given closure family is pulled back using the residuated map and its residual.

Lemma 17 *Let* $\mathbf{Sign}_1, \mathbf{Sign}_2$ *be categories,* $\mathrm{SEN}_1 : \mathbf{Sign}_1 \to \mathbf{Set}$*,* $\mathrm{SEN}_2 : \mathbf{Sign}_2 \to \mathbf{Set}$ *be complete lattice families and* $\langle F, \alpha\rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2$ *a residuated map.*

(1) *If* $\delta : \mathrm{SEN}_2 \to \mathrm{SEN}_2$ *is a closure family on* SEN_2*, then the map* $\delta^\alpha = \alpha^*\delta\alpha : \mathrm{SEN}_1 \to \mathrm{SEN}_1$ *is a closure family on* SEN_1*.*

(2) *If* $\mathbf{Sign}_1, \mathbf{Sign}_2$ *are complete residuated categories,* $\mathrm{SEN}_1, \mathrm{SEN}_2$ *are module systems,* $\langle F, \alpha\rangle$ *is a module system morphism and* δ *is structural, then* δ^α *is also structural.*

Proof:

The proof of Part (1) will be omitted, since it follows by applying signature-wise the proof of the statement of Lemma 4.1.1 of [10]. Suppose that $\mathbf{Sign}_1$, $\mathbf{Sign}_2$ are complete residuated categories, $\mathrm{SEN}_1 : \mathbf{Sign}_1 \to \mathbf{Set}$ and $\mathrm{SEN}_2 :$

$\mathbf{Sign}_2 \to \mathbf{Set}$ are module systems, $\langle F, \alpha \rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2$ is a module system morphism and $\delta : \mathrm{SEN}_2 \to \mathrm{SEN}_2$ is structural. To show that δ^α is structural, we have, for all $\Sigma, \Sigma' \in |\mathbf{Sign}_1|$, all $a \in \mathbf{Sign}_1(\Sigma, \Sigma')$ and all $x \in \mathrm{SEN}_1(\Sigma)$,

$$\begin{array}{rcl}
\alpha_\Sigma(a \star^{\Sigma,\Sigma'} \delta^\alpha_\Sigma(x)) & = & \alpha_\Sigma(a \star^{\Sigma,\Sigma'} \alpha^*_\Sigma(\delta_{F(\Sigma)}(\alpha_\Sigma(x)))) \\
& = & F(a) \star^{F(\Sigma),F(\Sigma')} \alpha_\Sigma(\alpha^*_\Sigma(\delta_{F(\Sigma)}(\alpha_\Sigma(x)))) \\
& \leq^{F(\Sigma')} & F(a) \star^{F(\Sigma),F(\Sigma')} \delta_{F(\Sigma)}(\alpha_\Sigma(x)) \\
& \leq^{F(\Sigma')} & \delta_{F(\Sigma')}(F(a) \star^{F(\Sigma),F(\Sigma')} \alpha_\Sigma(x)) \\
& = & \delta_{F(\Sigma')}(\alpha_{\Sigma'}(a \star^{\Sigma,\Sigma'} x)).
\end{array}$$

Therefore, we obtain $a \star^{\Sigma,\Sigma'} \delta^\alpha_\Sigma(x) \leq^{\Sigma'} \alpha^*_{\Sigma'}(\delta_{F(\Sigma')}(\alpha_{\Sigma'}(a \star^{\Sigma,\Sigma'} x)))$, i.e., $a \star^{\Sigma,\Sigma'} \delta^\alpha_\Sigma(x) \leq^{\Sigma'} \delta^\alpha_{\Sigma'}(a \star^{\Sigma,\Sigma'} x)$, showing that δ^α is also structural. □

The closure family $\delta^\alpha : \mathrm{SEN}_1 \to \mathrm{SEN}_1$ is called the $\langle F, \alpha \rangle$**-transform** of the closure family $\delta : \mathrm{SEN}_2 \to \mathrm{SEN}_2$. Similarly, the $\langle F, \alpha \rangle$**-transform** of a consequence family $\vdash$ on SEN_2 is the consequence family $\vdash^\alpha$ on SEN defined, for all $\Sigma \in |\mathbf{Sign}_1|$ and all $x, y \in \mathrm{SEN}(\Sigma)$, by

$$x \vdash^\alpha_\Sigma y \quad \text{iff} \quad \alpha_\Sigma(x) \vdash_{F(\Sigma)} \alpha_\Sigma(y).$$

Finally, given a completely meet closed subfunctor $\mathrm{SEN}^\bullet_2 : \overline{\mathbf{Sign}}_2 \to \mathbf{Set}$ of $\overline{\mathrm{SEN}}_2 : \overline{\mathbf{Sign}}_2 \to \mathbf{Set}$, define the $\overline{\langle F, \alpha \rangle}$**-transform** of $\mathrm{SEN}^\bullet_2$ as the subfunctor $\mathrm{SEN}^{\bullet,\alpha}_1 : \overline{\mathbf{Sign}}_1 \to \mathbf{Set}$ of $\overline{\mathrm{SEN}}_1 : \overline{\mathbf{Sign}}_1 \to \mathbf{Set}$, given by

$$\mathrm{SEN}^{\bullet,\alpha}_1(\Sigma) = \alpha^*_\Sigma(\mathrm{SEN}^\bullet_2(F(\Sigma))), \text{ for all } \Sigma \in |\mathbf{Sign}_1|.$$

The following analog of Lemma 4.2 of [10] details the relations between these notions of τ-transforms.

Lemma 18 *Let* $\mathbf{Sign}_1, \mathbf{Sign}_2$ *be categories,* $\mathrm{SEN}_1 : \mathbf{Sign}_1 \to \mathbf{Set}$, $\mathrm{SEN}_2 : \mathbf{Sign}_2 \to \mathbf{Set}$ *be complete lattice families,* $\langle F, \alpha \rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2$ *a residuated map and* $\delta : \mathrm{SEN}_2 \to \mathrm{SEN}_2$ *a closure family on* SEN_2*. Then, the following are equivalent:*

(1) $\gamma = \delta^\alpha$*;*

(2) $x \vdash^\gamma_\Sigma y$ *iff* $\alpha_\Sigma(x) \vdash^\delta_{F(\Sigma)} \alpha_\Sigma(y)$*, for all* $\Sigma \in |\mathbf{Sign}_1|$ *and all* $x, y \in \mathrm{SEN}(\Sigma)$*;*

(3) $\mathrm{SEN}^\gamma_1 = \mathrm{SEN}^{\delta,\alpha}_2$.

Proof:

(1)→(2) Let $\Sigma \in |\mathbf{Sign}_1|$ and $x, y \in \mathrm{SEN}_1(\Sigma)$. Then $x \vdash^{\delta^\alpha}_\Sigma y$ iff $y \leq^\Sigma \delta^\alpha_\Sigma(x)$ iff $y \leq^\Sigma \alpha^*_\Sigma(\delta_{F(\Sigma)}(\alpha_\Sigma(x)))$ iff $\alpha_\Sigma(y) \leq^{F(\Sigma)} \delta_{F(\Sigma)}(\alpha_\Sigma(x))$ iff $\alpha_\Sigma(x) \vdash^\delta_{F(\Sigma)} \alpha_\Sigma(y)$.

(2)→(1) Again, let $\Sigma \in |\mathbf{Sign}_1|$ and $x, y \in \mathrm{SEN}_1(\Sigma)$. We have $y \leq^\Sigma \gamma_\Sigma(x)$ iff $x \vdash^\gamma_\Sigma y$ iff $\alpha_\Sigma(x) \vdash^\delta_{F(\Sigma)} \alpha_\Sigma(y)$ iff $\alpha_\Sigma(y) \leq^{F(\Sigma)} \delta_{F(\Sigma)}(\alpha_\Sigma(x))$ iff $y \leq^\Sigma \alpha^*_\Sigma(\delta_{F(\Sigma)}(\alpha_\Sigma(x)))$ iff $y \leq^\Sigma \delta^\alpha_\Sigma(x)$. Thus, $\gamma = \delta^\alpha$.

(1)↔(3) We have that $\mathrm{SEN}_1^\gamma = \mathrm{SEN}_2^{\delta,\alpha}$ iff, for all $\Sigma \in |\mathbf{Sign}_1|$ and all $x \in \mathrm{SEN}_1(\Sigma)$, $x = \gamma_\Sigma(x)$ iff $x = \alpha^*_\Sigma(\delta_{F(\Sigma)}(z))$, for some $z \in \mathrm{SEN}_2(F(\Sigma))$. The latter holds iff $\delta^\alpha_\Sigma(x) = x$. (Right-to-left: Take $z = \alpha_\Sigma(x)$; Left-to-right: $\delta^\alpha_\Sigma(x) = \delta^\alpha_\Sigma(\alpha^*_\Sigma(\delta_{F(\Sigma)}(z))) = \alpha^*_\Sigma(\delta_{F(\Sigma)}(\alpha_\Sigma(\alpha^*_\Sigma(\delta_{F(\Sigma)}(z))))) \leq^\Sigma \alpha^*_\Sigma(\delta_{F(\Sigma)}(\delta_{F(\Sigma)}(z))) = \alpha^*_\Sigma(\delta_{F(\Sigma)}(z)) = x$.) Hence, $\mathrm{SEN}_1^\gamma = \mathrm{SEN}_2^{\delta,\alpha}$ iff γ and δ^α have exactly the same fixed points iff $\gamma = \delta^\alpha$.

□

Lemma 19, which is an analog of Lemma 4.3 of [10], asserts that, given a residuated map $\langle F,\alpha\rangle$ between two complete lattice families SEN_1 and SEN_2 and a closure family δ on SEN_2, the closure family δ^α on SEN_1, that is induced by δ and $\langle F,\alpha\rangle$, as in Lemma 17, admits a natural representation $\langle F,f\rangle$ into δ. Moreover, this representation is induced by $\langle F,\alpha\rangle$ and δ^α is shown to be the only closure family on SEN_1 that admits a representation in δ induced by $\langle F,\alpha\rangle$.

Lemma 19 *Let* $\mathbf{Sign}_1, \mathbf{Sign}_2$ *be categories,* $\mathrm{SEN}_1 : \mathbf{Sign}_1 \to \mathbf{Set}$*,* $\mathrm{SEN}_2 : \mathbf{Sign}_2 \to \mathbf{Set}$ *be complete lattice families,* $\langle F,\alpha\rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2$ *a residuated map and* $\delta : \mathrm{SEN}_2 \to \mathrm{SEN}_2$ *a closure family on* SEN_2*.*

1. $\langle F,f\rangle : \mathrm{SEN}_1^{\delta^\alpha} \to \mathrm{SEN}_2^\delta$*, with* $f = \delta\alpha \restriction_{\mathrm{SEN}^{\delta^\alpha}}$ *is residuated with residual the map* $f^* = \alpha^* \restriction_{\mathrm{SEN}_2^\delta} = \delta^\alpha\alpha^* \restriction_{\mathrm{SEN}_2^\delta} : \mathrm{SEN}_2^\delta \to \mathrm{SEN}_1^{\delta^\alpha}$.
2. $\langle F,f\rangle$ *is a representation of* δ^α *in* δ *induced by* $\langle F,\alpha\rangle$*.*
3. $\delta^\alpha : \mathrm{SEN}_1 \to \mathrm{SEN}_1$ *is the only closure family on* SEN_1 *that is represented in* $\delta : \mathrm{SEN}_2 \to \mathrm{SEN}_2$ *under a representation induced by* $\langle F,\alpha\rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2$*.*
4. *If* $\mathbf{Sign}_1, \mathbf{Sign}_2$ *are complete residuated categories,* $\mathrm{SEN}_1, \mathrm{SEN}_2$ *are module systems,* $\langle F,\alpha\rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2$ *is a module system morphism and* δ *is structural, then* $\langle F,f\rangle$ *is also structural.*

Proof:

Parts (1) and (2) are proven using the same techniques as the ones used in the proofs of the corresponding statements of Lemma 4.3.1-2 of [10]. We present only the proof of Parts (3) and (4).

(3) Let $\gamma : \mathrm{SEN}_1 \to \mathrm{SEN}_1$ be a closure family on SEN_1, represented in δ by a representation $\langle F,f\rangle$ induced by $\langle F,\alpha\rangle$. We must show that $\gamma = \delta^\alpha$. Note, first, that, for all $\Sigma \in |\mathbf{Sign}_1|$ and all $x, y \in \mathrm{SEN}_1(\Sigma)$,

$$\begin{array}{rll} x \leq^\Sigma (f\gamma)^*_\Sigma(f_\Sigma(\gamma_\Sigma(y))) & \text{iff} & f_\Sigma(\gamma_\Sigma(x)) \leq^{F(\Sigma)} f_\Sigma(\gamma_\Sigma(y)) \\ & \text{iff} & \gamma_\Sigma(x) \leq^\Sigma \gamma_\Sigma(y) \\ & \text{iff} & x \leq^\Sigma \gamma^*_\Sigma(\gamma_\Sigma(y)). \end{array} \tag{3}$$

Now we have:

$$\begin{array}{rcl} \delta^\alpha & = & \alpha^*\delta\alpha \\ & = & \alpha^*\iota^\delta\delta\alpha \\ & = & \alpha^*\delta^*\delta\alpha \\ & = & (\delta\alpha)^*\delta\alpha \\ & = & (f\gamma)^*f\gamma \\ & = & \gamma^*\gamma \quad \text{(by (3))} \\ & = & \iota^\gamma\gamma \\ & = & \gamma. \end{array}$$

(4) Let $\Sigma, \Sigma' \in |\mathbf{Sign_1}|$, $a \in \mathbf{Sign_1}(\Sigma, \Sigma')$, $x \in \mathrm{SEN}_1^{\delta^\alpha}(\Sigma)$. We have

$$\begin{array}{rcl} f_{\Sigma'}(a \star_{\delta^\alpha}^{\Sigma,\Sigma'} x) & = & f_{\Sigma'}(\delta^\alpha_{\Sigma'}(a \star^{\Sigma,\Sigma'} x)) \\ & = & \delta_{F(\Sigma')}(\alpha_{\Sigma'}(a \star^{\Sigma,\Sigma'} x)) \\ & = & \delta_{F(\Sigma')}(F(a) \star^{F(\Sigma),F(\Sigma')} \alpha_\Sigma(x)) \\ & = & \delta_{F(\Sigma')}(F(a) \star^{F(\Sigma),F(\Sigma')} \delta_{F(\Sigma)}(\alpha_\Sigma(x))) \\ & = & F(a) \star_\delta^{F(\Sigma),F(\Sigma')} \delta_{F(\Sigma)}(\alpha_\Sigma(x)) \\ & = & F(a) \star_\delta^{F(\Sigma),F(\Sigma')} f_\Sigma(x). \end{array}$$

$\square$

Corollary 20 *Let* $\mathbf{Sign_1}, \mathbf{Sign_2}$ *be categories,* $\mathrm{SEN}_1 : \mathbf{Sign_1} \to \mathbf{Set}$, $\mathrm{SEN}_2 : \mathbf{Sign_2} \to \mathbf{Set}$ *two complete lattice families and* $\vdash^\gamma$ *and* $\vdash^\delta$ *be consequence families on* SEN_1 *and* SEN_2*, respectively. Then* $\vdash^\gamma$ *is represented in* $\vdash^\delta$ *via a residuated map* $\langle F, \alpha\rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2$ *iff, for all* $\Sigma \in |\mathbf{Sign_1}|$ *and all* $x, y \in \mathrm{SEN}_1(\Sigma)$, $x \vdash^\gamma_\Sigma y$ *iff* $\alpha_\Sigma(x) \vdash^\delta_{F(\Sigma)} \alpha_\Sigma(y)$.

The consequence family $\vdash^\gamma$ being represented in the consequence family $\vdash^\delta$ by $\langle F, f\rangle : \mathrm{SEN}_1^\gamma \to \mathrm{SEN}_2^\delta$ means that $\langle F, f\rangle$ is residuated and, for all $\Sigma \in |\mathbf{Sign_1}|$ and all $x, y \in \mathrm{SEN}_1(\Sigma)$,

$$x \vdash^\gamma_\Sigma y \quad \text{iff} \quad f_\Sigma(\gamma_\Sigma(x)) \vdash^\delta_{F(\Sigma)} f_\Sigma(\gamma_\Sigma(y)).$$

Indeed, if $\vdash^\gamma$ is represented in $\vdash^\delta$ by $\langle F, f\rangle$, then

$$\begin{array}{rll} x \vdash^\gamma_\Sigma y & \text{iff} & y \leq^\Sigma \gamma_\Sigma(x) \\ & \text{iff} & \gamma_\Sigma(y) \leq^\Sigma_\gamma \gamma_\Sigma(x) \\ & \text{iff} & f_\Sigma(\gamma_\Sigma(y)) \leq^{F(\Sigma)}_\delta f_\Sigma(\gamma_\Sigma(x)) \\ & \text{iff} & f_\Sigma(\gamma_\Sigma(y)) \leq^{F(\Sigma)} \delta_\Sigma(f_\Sigma(\gamma_\Sigma(x))) \\ & \text{iff} & f_\Sigma(\gamma_\Sigma(x)) \vdash^\delta_{F(\Sigma)} f_\Sigma(\gamma_\Sigma(y)). \end{array}$$

On the other hand, if $f_\Sigma(\gamma_\Sigma(y)) \leq^{F(\Sigma)}_\delta f_\Sigma(\gamma_\Sigma(x))$, we have $f_\Sigma(\gamma_\Sigma(y)) \leq^{F(\Sigma)} \delta_\Sigma(f_\Sigma(\gamma_\Sigma(x)))$, whence $f_\Sigma(\gamma_\Sigma(x)) \vdash^\delta_{F(\Sigma)} f_\Sigma(\gamma_\Sigma(y))$, i.e., $x \vdash^\gamma_\Sigma y$, showing that $\gamma_\Sigma(y) \leq^\Sigma_\gamma \gamma_\Sigma(x)$.

5.2 Equivalence

Let $\mathbf{Sign}_1, \mathbf{Sign}_2$ be complete residuated categories, $\mathrm{SEN}_1 : \mathbf{Sign}_1 \to \mathbf{Set}$, $\mathrm{SEN}_2 : \mathbf{Sign}_2 \to \mathbf{Set}$ be two module systems and $\gamma : \mathrm{SEN}_1 \to \mathrm{SEN}_1$ and $\delta : \mathrm{SEN}_2 \to \mathrm{SEN}_2$ closure systems on SEN_1 and SEN_2, respectively. An **equivalence** between γ and δ consists of a pair of module system morphisms $\langle F, f\rangle : \mathrm{SEN}_1^\gamma \to \mathrm{SEN}_2^\delta$ and $\langle G, g\rangle : \mathrm{SEN}_2^\delta \to \mathrm{SEN}_1^\gamma$, together with an adjoint equivalence $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$ with the property that

- $\eta^{-1}_{\Sigma_1} \star_\gamma^{G(F(\Sigma_1)),\Sigma_1} g_{F(\Sigma_1)} f_{\Sigma_1} = i_{\mathrm{SEN}_1^\gamma(\Sigma_1)}$, for all $\Sigma_1 \in |\mathbf{Sign}_1|$, and
- $\epsilon_{\Sigma_2} \star_\delta^{F(G(\Sigma_2)),\Sigma_2} f_{G(\Sigma_2)} g_{\Sigma_2} = i_{\mathrm{SEN}_2^\delta(\Sigma_2)}$, for all $\Sigma_2 \in |\mathbf{Sign}_2|$.

The closure systems γ and δ are said to be **equivalent** if there exists an equivalence between γ and δ.

An equivalence consisting of $\langle F, f\rangle, \langle G, g\rangle$ and $\langle F, G, \eta, \epsilon\rangle$ between γ and δ is **induced** by the module morphisms $\langle F, \alpha\rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2$ and $\langle G, \beta\rangle : \mathrm{SEN}_2 \to \mathrm{SEN}_1$ and the adjoint equivalence $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$ if,

$$\begin{tikzcd}
\mathrm{SEN}_1 \arrow[r, "\langle F,\alpha\rangle"] \arrow[d, "\gamma"'] & \mathrm{SEN}_2 \arrow[d, "\delta"] \\
\mathrm{SEN}_1^\gamma \arrow[r, "\langle F,f\rangle"'] & \mathrm{SEN}_2^\delta
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{SEN}_1 \arrow[d, "\gamma"'] & \mathrm{SEN}_2 \arrow[l, "\langle G,\beta\rangle"'] \arrow[d, "\delta"] \\
\mathrm{SEN}_1^\gamma & \mathrm{SEN}_2^\delta \arrow[l, "\langle G,g\rangle"]
\end{tikzcd}$$

$\langle F, f\rangle \circ \gamma = \delta \circ \langle F, \alpha\rangle$ and $\langle G, g\rangle \circ \delta = \gamma \circ \langle G, \beta\rangle$. If this is the case, γ and δ are said to be **equivalent via** $\langle F, \alpha\rangle$, $\langle G, \beta\rangle$ and $\langle F, G, \eta, \epsilon\rangle$.

For consequence systems, we say that $\vdash^\gamma$ is **equivalent** to $\vdash^\delta$ via $\langle F, \alpha\rangle$, $\langle G, \beta\rangle$ and $\langle F, G, \eta, \epsilon\rangle$, if γ is equivalent to δ via $\langle F, \alpha\rangle$, $\langle G, \beta\rangle$ and $\langle F, G, \eta, \epsilon\rangle$.

Lemma 21 is an analog of Theorem 4.7 of [10] and provides a characterization of the equivalence between two closure systems that is induced by given module system morphisms and a given natural equivalence.

Theorem 21 *Let* $\mathbf{Sign}_1$ *and* $\mathbf{Sign}_2$ *be complete residuated categories,* $\mathrm{SEN}_1 : \mathbf{Sign}_1 \to \mathbf{Set}$, $\mathrm{SEN}_2 : \mathbf{Sign}_2 \to \mathbf{Set}$ *two module systems,* $\gamma : \mathrm{SEN}_1 \to \mathrm{SEN}_1, \delta : \mathrm{SEN}_2 \to \mathrm{SEN}_2$ *closure systems on* SEN_1, SEN_2*, respectively, and* $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$ *an adjoint equivalence. Then, the following statements are equivalent:*

(1) γ *and* δ *are equivalent via* $\langle F, \alpha\rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2$, $\langle G, \beta\rangle : \mathrm{SEN}_2 \to \mathrm{SEN}_1$ *and the adjoint equivalence* $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$*;*

(2) $\gamma = \delta^\alpha$, $\epsilon \star_\delta \delta\alpha\beta = \delta$ *and* ϵ *consists of order isomorphisms on* SEN_2^δ*;*

(3) $\delta = \gamma^\beta$, $\eta^{-1} \star_\gamma \gamma\beta\alpha = \gamma$ *and* η *consists of order-isomorphisms on* SEN_1^γ.

Proof:
(1) implies (2) follows by Lemma 19 and the following:

$$\begin{array}{rcl}\epsilon \star_\delta \delta\alpha\beta &=& \epsilon \star_\delta f\gamma\beta\\ &=& \epsilon \star_\delta fg\delta\\ &=& \delta.\end{array}$$

For the converse, assume that $\gamma = \delta^\alpha$, $\epsilon \star_\delta \delta\alpha\beta = \delta$ and ϵ consists of order-isomorphisms on SEN_2^δ. Note that, if $\Sigma \in |\mathbf{Sign}_2|$ and $y \in \mathrm{SEN}_2^\delta(F(G(\Sigma)))$, then

$$\beta^*_\Sigma(\alpha^*_{G(\Sigma)}(y)) = \epsilon_\Sigma \star_\delta^{F(G(\Sigma)),\Sigma} \delta_{F(G(\Sigma))}(y).$$

Indeed, we have, for all $\Sigma \in |\mathbf{Sign}_2|$, $x \in \mathrm{SEN}_2(\Sigma)$ and all $y \in \mathrm{SEN}_2^\delta(G(F(\Sigma)))$,

$$\begin{array}{lll}\alpha_{G(\Sigma)}(\beta_\Sigma(x)) \leq^{F(G(\Sigma))} y & \text{iff} & \delta_{F(G(\Sigma))}(\alpha_{G(\Sigma)}(\beta_\Sigma(x))) \leq^{F(G(\Sigma))} y\\ & \text{iff} & \epsilon_\Sigma \star_\delta^{F(G(\Sigma)),\Sigma} \delta_{F(G(\Sigma))}(\alpha_{G(\Sigma)}(\beta_\Sigma(x)))\\ & & \qquad\qquad \leq^\Sigma \epsilon_\Sigma \star_\delta^{F(G(\Sigma)),\Sigma} y\\ & \text{iff} & \delta_\Sigma(x) \leq^\Sigma \epsilon_\Sigma \star_\delta^{F(G(\Sigma)),\Sigma} y\\ & \text{iff} & x \leq^\Sigma \epsilon_\Sigma \star_\delta^{F(G(\Sigma)),\Sigma} y.\end{array}$$

This allows us to conclude that $\delta = \beta^*\gamma\beta$. In fact, for all $\Sigma \in |\mathbf{Sign}_2|$ and all $y \in \mathrm{SEN}_2(\Sigma)$, we have

$$\begin{array}{rcl}\beta^*_\Sigma(\gamma_{G(\Sigma)}(\beta_\Sigma(y))) &=& \beta^*_\Sigma(\alpha^*_{G(\Sigma)}(\delta_{F(G(\Sigma))}(\alpha_{G(\Sigma)}(\beta_\Sigma(y)))))\\ &=& \epsilon_\Sigma \star_\delta^{F(G(\Sigma)),\Sigma} \delta_{F(G(\Sigma))}(\alpha_{G(\Sigma)}(\beta_\Sigma(y)))\\ &=& \delta_\Sigma(y).\end{array}$$

Moreover, we have, for all $\Sigma \in |\mathbf{Sign}_1|$ and all $x \in \mathrm{SEN}_1(\Sigma)$,

$$\begin{array}{l}\eta^{-1}_\Sigma \star_\gamma^{G(F(\Sigma)),\Sigma} \gamma_{G(F(\Sigma))}(\beta_{F(\Sigma)}(\alpha_\Sigma(x)))\\ \quad = \gamma_\Sigma(\eta^{-1}_\Sigma \star^{G(F(\Sigma)),\Sigma} \gamma_{G(F(\Sigma))}(\beta_{F(\Sigma)}(\alpha_\Sigma(x))))\\ \quad = \gamma_\Sigma(\eta^{-1}_\Sigma \star^{G(F(\Sigma)),\Sigma} \beta_{F(\Sigma)}(\alpha_\Sigma(x)))\\ \quad = \alpha^*_\Sigma(\delta_{F(\Sigma)}(\alpha_\Sigma(\eta^{-1}_\Sigma \star^{G(F(\Sigma)),\Sigma} \beta_{F(\Sigma)}(\alpha_\Sigma(x)))))\\ \quad = \alpha^*_\Sigma(\delta_{F(\Sigma)}(F(\eta^{-1}_\Sigma) \star^{F(G(F(\Sigma))),F(\Sigma)} \alpha_{G(F(\Sigma))}(\beta_{F(\Sigma)}(\alpha_\Sigma(x)))))\\ \quad = \alpha^*_\Sigma(\delta_{F(\Sigma)}(\epsilon_{F(\Sigma)} \star^{F(G(F(\Sigma))),F(\Sigma)} \alpha_{G(F(\Sigma))}(\beta_{F(\Sigma)}(\alpha_\Sigma(x)))))\\ \quad = \alpha^*_\Sigma(\epsilon_{F(\Sigma)} \star_\delta^{F(G(F(\Sigma))),F(\Sigma)} \delta_{F(G(F(\Sigma)))}(\alpha_{G(F(\Sigma))}(\beta_{F(\Sigma)}(\alpha_\Sigma(x)))))\\ \quad = \alpha^*_\Sigma(\delta_{F(\Sigma)}(\alpha_\Sigma(x)))\\ \quad = \gamma_\Sigma(x).\end{array}$$

Thus, all conditions in (3) hold. We use the conditions included in (2) and (3) to prove (1): Let $\langle F, f\rangle : \mathrm{SEN}_1^\gamma \to \mathrm{SEN}_2^\delta$ and $\langle G, g\rangle : \mathrm{SEN}_2^\delta \to \mathrm{SEN}_1^\gamma$ be the representations of $\gamma = \delta^\alpha$ in δ and of $\delta = \gamma^\beta$ in γ, as given in Lemma 19. Then $\langle F, f\rangle \circ \gamma = \delta \circ \langle F, \alpha\rangle$ and $\langle G, g\rangle \circ \delta = \gamma \circ \langle G, \beta\rangle$ It suffices, thus, to show that the adjoint equivalence satisfies the additional properties stipulated in the definition of an equivalence and that both $\langle F, f\rangle$ and $\langle G, g\rangle$ are structural. Let

$\Sigma \in |\mathbf{Sign}_1|$ and $x \in \mathrm{SEN}_1^\gamma(\Sigma)$. Then

$$\begin{aligned}\eta_\Sigma^{-1} \star_\gamma^{G(F(\Sigma)),\Sigma} & g_{F(\Sigma)}(f_\Sigma(x)) \\ &= \eta_\Sigma^{-1} \star_\gamma^{G(F(\Sigma)),\Sigma} \gamma_{G(F(\Sigma))}(\beta_{F(\Sigma)}(\delta_{F(\Sigma)}(\alpha_\Sigma(x)))) \\ &= \eta_\Sigma^{-1} \star_\gamma^{G(F(\Sigma)),\Sigma} g_{F(\Sigma)}(\delta_{F(\Sigma)}(\delta_{F(\Sigma)}(\alpha_\Sigma(x)))) \\ &= \eta_\Sigma^{-1} \star_\gamma^{G(F(\Sigma)),\Sigma} g_{F(\Sigma)}(\delta_{F(\Sigma)}(\alpha_\Sigma(x))) \\ &= \eta_\Sigma^{-1} \star_\gamma^{G(F(\Sigma)),\Sigma} \gamma_{G(F(\Sigma))}(\beta_{F(\Sigma)}(\alpha_\Sigma(x))) \\ &= \gamma_\Sigma(x) \\ &= x.\end{aligned}$$

That $\epsilon_\Sigma \star_\delta^{F(G(\Sigma)),\Sigma} f_{G(\Sigma)}(g_\Sigma(x)) = x$, for all $\Sigma \in |\mathbf{Sign}_2|$ and all $x \in \mathrm{SEN}_2^\delta(\Sigma)$, may be shown similarly. Finally, we must show that $\langle F, f\rangle : \mathrm{SEN}_1^\gamma \to \mathrm{SEN}_2^\delta$ and $\langle G, g\rangle : \mathrm{SEN}_2^\delta \to \mathrm{SEN}_1^\gamma$ are structural. We show that $\langle F, f\rangle$ is structural, since a similar proof applies to $\langle G, g\rangle$. For all $\Sigma, \Sigma' \in |\mathbf{Sign}_1|$, $a \in \mathbf{Sign}_1(\Sigma, \Sigma')$, and all $x \in \mathrm{SEN}_1^\gamma(\Sigma)$, we have

$$\begin{array}{rcl} f_{\Sigma'}(a \star_\gamma^{\Sigma,\Sigma'} x) &=& f_{\Sigma'}(\gamma_{\Sigma'}(a \star^{\Sigma,\Sigma'} x)) \\ &=& \delta_{\Sigma'}(\alpha_{\Sigma'}(a \star^{\Sigma,\Sigma'} x)) \\ &=& \delta_{\Sigma'}(F(a) \star^{F(\Sigma),F(\Sigma')} \alpha_\Sigma(x)) \\ &=& \delta_{\Sigma'}(F(a) \star^{F(\Sigma),F(\Sigma')} \delta_\Sigma(\alpha_\Sigma(x))) \\ &=& F(a) \star_\delta^{F(\Sigma),F(\Sigma')} \delta_\Sigma(\alpha_\Sigma(x)) \\ &=& F(a) \star_\delta^{F(\Sigma),F(\Sigma')} f_\Sigma(\gamma_\Sigma(x)) \\ &=& F(a) \star_\delta^{F(\Sigma),F(\Sigma')} f_\Sigma(x) \quad (\gamma_\Sigma(x) = x). \end{array}$$

The equivalence of (1) and (3) follows by a symmetric argument. □

As a corollary, we provide a similar characterization of the equivalence between two consequence families.

Corollary 22 *Let* $\mathbf{Sign}_1$ *and* $\mathbf{Sign}_2$ *be complete residuated categories,* $\mathrm{SEN}_1 : \mathbf{Sign}_1 \to \mathbf{Set}$, $\mathrm{SEN}_2 : \mathbf{Sign}_2 \to \mathbf{Set}$ *be two module systems and* $\vdash^\gamma, \vdash^\delta$ *consequence systems on* SEN_1, SEN_2, *respectively. Then* $\vdash^\gamma$ *is equivalent to* $\vdash^\delta$ *via the module system morphisms* $\langle F, \alpha\rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2$, $\langle G, \beta\rangle : \mathrm{SEN}_2 \to \mathrm{SEN}_1$ *and the adjoint equivalence* $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$, *where* η *and* ϵ *consist of order-isomorphisms on* SEN_1^γ *and* SEN_2^δ, *respectively, iff the following hold:*

(1) *For all* $\Sigma \in |\mathbf{Sign}_1|$, $x, y \in \mathrm{SEN}_1(\Sigma)$, $x \vdash_\Sigma^\gamma y$ *iff* $\alpha_\Sigma(x) \vdash_{F(\Sigma)}^\delta \alpha_\Sigma(y)$;

(2) *For all* $\Sigma \in |\mathbf{Sign}_2|$, $z \in \mathrm{SEN}_2(\Sigma)$, $z \dashv\vdash_\Sigma^\delta \mathrm{SEN}_2(\epsilon_\Sigma)(\alpha_{G(\Sigma)}(\beta_\Sigma(z)))$.

Proof:

Since Condition (2) is equivalent to $\epsilon\star_\delta\delta\alpha\beta = \delta$, we get the result by applying Theorem 21 and Corollary 20. □

6 Equivalences Induced by Translators

It is well known that not every equivalence of consequence relations in the sense of [10] is induced by translators. This result extends of course to the case of consequence systems studied in the present paper. However, Galatos and Tsinakis show in [10] that this is true for consequence relations on powersets of formulas, equations and sequents. Moreover, they exactly pinpoint those modules over complete residuated lattices for which equivalences are induced by translators. They show that these are exactly the projective modules in the category of modules. Their results are extended here to cover the case of consequence systems over module systems. More specifically, it will be shown that if $\mathbf{Sign}_1, \mathbf{Sign}_2$ are complete residuated categories, $\mathrm{SEN}_1 : \mathbf{Sign}_1 \to \mathbf{Set}$ is a module system, satisfying certain conditions, $\mathrm{SEN}_2 : \mathbf{Sign}_2 \to \mathbf{Set}$ is also a module system and γ, δ are closure systems on $\mathrm{SEN}_1, \mathrm{SEN}_2$, respectively, then every structural representation $\langle F, f\rangle : \mathrm{SEN}_1^\gamma \to \mathrm{SEN}_2^\delta$ of γ in δ is induced by a translator. Note that, here, since γ and δ are assumed to be closure systems, SEN_1^γ and SEN_2^δ are functors on $\mathbf{Sign}_2, \mathbf{Sign}_2$, respectively. Recall, also, that $\mathcal{M}$ denotes the category with objects module systems and morphisms module system morphisms (translators) between them.

6.1 Projective Objects

Consider complete residuated categories $\mathbf{Sign}_1, \mathbf{Sign}_2$, module systems $\mathrm{SEN}_1 : \mathbf{Sign}_1 \to \mathbf{Set}$, $\mathrm{SEN}_2 : \mathbf{Sign}_2 \to \mathbf{Set}$ and closure systems γ and δ on SEN_1 and SEN_2, respectively. Let $\langle F, f\rangle : \mathrm{SEN}_1^\gamma \to \mathrm{SEN}_2^\delta$ be a structural representation of γ in δ. The goal is to find a translator $\langle F, \alpha\rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2$ that induces $\langle F, f\rangle$, i.e., such that $\langle F, f\rangle \circ \gamma = \delta \circ \langle F, \alpha\rangle$. This is tantamount to finding a morphism $\langle F, \alpha\rangle$ in $\mathcal{M}$ that completes the square

$$\begin{array}{ccc} \mathrm{SEN}_1 & \xrightarrow{\langle F,\alpha\rangle} & \mathrm{SEN}_2 \\ \gamma \big\downarrow & & \big\downarrow \delta \\ \mathrm{SEN}_1^\gamma & \xrightarrow[\langle F,f\rangle]{} & \mathrm{SEN}_2^\delta \end{array} \tag{4}$$

As in [10], it will be shown that the objects $\mathrm{SEN}_1 \in |\mathcal{M}|$, for which the square can be completed are precisely the projective objects of $\mathcal{M}$, where an object $\mathrm{SEN} : \mathbf{Sign}_1 \to \mathbf{Set}$ of $\mathcal{M}$ is called **projective** if whenever there exist module systems $\mathrm{SEN}_2 : \mathbf{Sign}_2 \to \mathbf{Set}$, $\mathrm{SEN}_2' : \mathbf{Sign}_2 \to \mathbf{Set}$, over the same complete residuated category $\mathbf{Sign}_2$, and module system morphisms $g := \langle \mathrm{I}_{\mathbf{Sign}_2}, g\rangle : \mathrm{SEN}_2 \to \mathrm{SEN}_2'$ and $\langle K, k\rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2'$, with g surjective, then, there exists a module system morphism $\langle H, h\rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2$, such that $\langle K, k\rangle = g \circ \langle H, h\rangle$.

$$\begin{tikzcd}
\mathrm{SEN}_1 \arrow[r, "\langle H,h\rangle"] \arrow[dr, swap, "\langle K,k\rangle"] & \mathrm{SEN}_2 \arrow[d, "g"] \\
& \mathrm{SEN}_2'
\end{tikzcd} \tag{5}$$

Theorem 23 *Let* $\mathbf{Sign}_1$ *be a complete residuated category. The objects* $\mathrm{SEN}_1 : \mathbf{Sign}_1 \to \mathbf{Set}$ *of* $\mathcal{M}$ *for which all squares of type* (4) *can be completed are the projective objects of* $\mathcal{M}$.

Proof:

If SEN_1 is projective, then Square (4) can be completed by choosing $\mathrm{SEN}_2' = \mathrm{SEN}_2^\delta$, $\langle K, k\rangle = \langle F, f\rangle \circ \gamma$ and $g = \delta$ in Triangle (5).

Conversely, assume that SEN is such that every Square (4) can be completed and consider Triangle (5), with $\langle K, k\rangle, g$ given and $\langle H, h\rangle$ to be determined. (To avoid clattering the diagram below, we omit functor components.)

$$\begin{tikzcd}
\mathrm{SEN}_1 \arrow[rr, "h = \alpha"] \arrow[dr, "k"] \arrow[dd, swap, "k^*k"] & & \mathrm{SEN}_2 \arrow[dl, swap, "g"] \arrow[dd, "g^*g"] \\
& \mathrm{SEN}_2' & \\
\mathrm{SEN}_1^{k^*k} \arrow[ur, "k'"] \arrow[rr, swap, "f"] & & \mathrm{SEN}_2^{g^*g} \arrow[ul, swap, "g'"]
\end{tikzcd}$$

By Lemma 12, $k^*k : \mathrm{SEN}_1 \to \mathrm{SEN}_1$ is a closure system on SEN_1 and $\mathrm{SEN}_1^{k^*k}$ is isomorphic to $k(\mathrm{SEN}_1)$ via $\langle K, k'\rangle = \langle K, k \restriction_{\mathrm{SEN}_1^{k^*k}}\rangle$. Therefore, $\langle K, k\rangle$ factors as $\langle K, k\rangle = \langle K, k'\rangle \circ (k^*k)$. Similarly, $g = g' \circ (g^*g)$, where $g' = g \restriction_{\mathrm{SEN}_2^{g^*g}}$. But $\langle K, k'\rangle$ is an embedding and g' is an isomorphism, whence $\langle F, f\rangle = (g')^{-1} \circ \langle K, k'\rangle$ is an embedding. Let $\langle F, \alpha\rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2$ be the completion of the outer square. Then $fk^*k = g^*gh$ implying $g'fk^*k = g'g^*gh$, whence $k'k^*k = gh$. Thus, $k = gh$, whence the upper triangle commutes. □

6.2 Cyclic Module Systems

It will now be shown that the module systems on which all term π-institutions, as introduced in [20], and all multi-term π-institutions, as introduced in [11], are based are projective. Therefore, Theorem 23 asserts that all equivalences between consequence systems on such module systems are induced by translators. These results were established in [20, 11]. Moreover, we generalize the notion of a cyclic projective module of [10] to obtain the notion of a cyclic projective module system. We show that term π-institutions are based on cyclic module systems, whereas multi-term π-institutions are not based on cyclic module systems, but they are coproducts of projective cyclic module systems and are, as a result, also projective.

Let **Sign** be a complete residuated category. A **Sign**-module system SEN : **Sign** $\to$ **Set** is called **cyclic** if there exists $V \in |\mathbf{Sign}|$ and $v \in \mathrm{SEN}(V)$, such that, for all $\Sigma \in |\mathbf{Sign}|$, and all $x \in \mathrm{SEN}(\Sigma)$, there exists $a_{\langle\Sigma,x\rangle} \in \mathbf{Sign}(V,\Sigma)$, such that $a_{\langle\Sigma,x\rangle} \star^{V,\Sigma} v = x$. The pair $\langle V,v\rangle$ is called a **generator** for SEN.

Recall, from [20], that, given a category **Sign** and a set-valued functor SEN : **Sign** $\to$ **Set**, SEN is called **term** if there exists $V \in |\mathbf{Sign}|$ and $v \in \mathrm{SEN}(V)$, such that, for all $\Sigma,\Sigma' \in |\mathbf{Sign}|$, all $x \in \mathrm{SEN}(\Sigma)$ and all $f \in \mathbf{Sign}(\Sigma,\Sigma')$,

- there exists $f_{\langle\Sigma,x\rangle} \in \mathbf{Sign}(V,\Sigma)$, such that $\mathrm{SEN}(f_{\langle\Sigma,x\rangle})(v) = x$;
- $f \circ f_{\langle\Sigma,x\rangle} = f_{\langle\Sigma',\mathrm{SEN}(f)(x)\rangle}$.

Proposition 24 *Let* **Sign** *be a category and* SEN : **Sign** $\to$ **Set** *a functor. If* SEN *is term, then* $\mathcal{P}$SEN : $\mathbf{Sign}^{\mathcal{P}} \to$ **Set** *is a cyclic* $\mathbf{Sign}^{\mathcal{P}}$*-module system.*

Proof:

It has already been shown in Proposition 8 that $\mathcal{P}$SEN : $\mathbf{Sign}^{\mathcal{P}} \to$ **Set** is a $\mathbf{Sign}^{\mathcal{P}}$-module system. To show that it is cyclic, it suffices to show that the pair $\langle V,\{v\}\rangle$ is in fact a generator. To see this, let $\Sigma \in |\mathbf{Sign}|$ and $X \in \mathcal{P}\mathrm{SEN}(\Sigma)$. Then, there exists $f_{\langle\Sigma,X\rangle} = \{f_{\langle\Sigma,x\rangle} : x \in X\} \in \mathbf{Sign}^{\mathcal{P}}(V,\Sigma)$, such that

$$\begin{array}{rcl}\mathcal{P}\mathrm{SEN}(f_{\langle\Sigma,X\rangle})(\{v\}) & = & \{\mathrm{SEN}(f_{\langle\Sigma,x\rangle})(v) : x \in X\} \\ & = & \{x : x \in X\} \\ & = & X.\end{array}$$

□

The following lemma provides a characterization of cyclicity similar to the one provided by Lemma 5.2 of [10] for cyclic modules over complete residuated lattices.

Lemma 25 *Given a complete residuated category* **Sign***, a* **Sign***-module system* SEN *is cyclic with generator* $\langle V,v\rangle$ *iff, for all* $\Sigma \in |\mathbf{Sign}|$*, and all* $x \in \mathrm{SEN}(\Sigma)$*,* $(x/^{V,\Sigma}v) \star^{V,\Sigma} v = x$.

Proof:

Suppose that the condition in the statement of the lemma holds. Given $\Sigma \in |\mathbf{Sign}|$ and $x \in \mathrm{SEN}(\Sigma)$, let $a_{\langle\Sigma,x\rangle} \in \mathbf{Sign}(V,\Sigma)$ be defined by $a_{\langle\Sigma,x\rangle} := x/^{V,\Sigma}v$. Then, the condition in the definition of a cyclic **Sign**-module system with generator $\langle V,v\rangle$ holds.

Conversely, assume that SEN is cyclic with $\langle V,v\rangle$ a generator. Then, given $\Sigma \in |\mathbf{Sign}|$ and $x \in \mathrm{SEN}(\Sigma)$, there exists $a_{\langle\Sigma,x\rangle} \in \mathbf{Sign}(V,\Sigma)$, such that $a_{\langle\Sigma,x\rangle} \star^{V,\Sigma} v = x$, whence $a_{\langle\Sigma,x\rangle} \leq^{V,\Sigma} x/^{V,\Sigma}v$. Thus, by Lemma 9, $x = a_{\langle\Sigma,x\rangle} \star^{V,\Sigma} v \leq^{\Sigma} (x/^{V,\Sigma}v) \star^{V,\Sigma} v \leq^{\Sigma} x$, yielding $(x/^{V,\Sigma}v) \star^{V,\Sigma} v = x$. □

Next, we define a special cyclic module system, that will play an important role in what follows. First, note that the slice functor $\mathbf{Sign}(V, -)$ of any complete residuated category $\mathbf{Sign}$ over a given object $V \in |\mathbf{Sign}|$ forms a $\mathbf{Sign}$-module system with $\mathbf{Sign}$-module operation arrow composition.

Lemma 26 *Let* $\mathbf{Sign}$ *be a complete residuated category and* $V \in |\mathbf{Sign}|$*. Then* $\mathbf{Sign}(V, -) : \mathbf{Sign} \to \mathbf{Set}$ *is a* $\mathbf{Sign}$*-module system, if one defines, for all* $\Sigma, \Sigma' \in |\mathbf{Sign}|$*, all* $x \in \mathbf{Sign}(V, \Sigma)$ *and all* $a \in \mathbf{Sign}(\Sigma, \Sigma')$*,*

$$a \star^{\Sigma,\Sigma'} x = a \circ^{V,\Sigma}_{\Sigma,\Sigma'} x.$$

The special $\mathbf{Sign}$-module system, that was alluded to above, is the one associated with a closure system on the slice functor $\mathbf{Sign}(V, -)$, for some $V \in |\mathbf{Sign}|$. The following lemma parallels Lemma 5.3 of [10].

Lemma 27 *Let* $\mathbf{Sign}$ *be a complete residuated category,* $V \in |\mathbf{Sign}|$ *and* $\gamma : \mathbf{Sign}(V, -) \to \mathbf{Sign}(V, -)$ *a closure system on* $\mathbf{Sign}(V, -)$*. Then the* $\mathbf{Sign}$*-module system* $\mathbf{Sign}^\gamma(V, -)$ *is cyclic, with generator* $\langle V, \gamma_V(i_V)\rangle$*, where, for all* $\Sigma, \Sigma' \in |\mathbf{Sign}|$*,* $a \in \mathbf{Sign}(\Sigma, \Sigma')$ *and* $x \in \mathbf{Sign}(V, \Sigma)$*,* $a \star^{\Sigma,\Sigma'} x = a \circ^{V,\Sigma}_{\Sigma,\Sigma'} x$*.*

Proof:
Suppose, $\Sigma \in |\mathbf{Sign}|$ and $\gamma_\Sigma(a) \in \mathbf{Sign}^\gamma(V, \Sigma)$, for some $a \in \mathbf{Sign}(V, \Sigma)$. Then, $a \star^{V,\Sigma}_\gamma \gamma_V(i_V) = \gamma_\Sigma(a \circ^{V,V}_{V,\Sigma} i_V) = \gamma_\Sigma(a)$. Thus, $\mathbf{Sign}^\gamma(V, -)$ is cyclic with generator $\langle V, \gamma_V(i_V)\rangle$. □

Now consider a complete residuated category $\mathbf{Sign}$, $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ a $\mathbf{Sign}$-module system, $V \in |\mathbf{Sign}|$ and $v \in \mathrm{SEN}(V)$. Define $\mathrm{SEN}^v : \mathbf{Sign} \to \mathbf{Set}$ by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\mathrm{SEN}^v(\Sigma) = \{x \in \mathrm{SEN}(\Sigma) : (\exists f \in \mathbf{Sign}(V, \Sigma))(x = \mathrm{SEN}(f)(v))\},$$

and, for all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\mathrm{SEN}^v(f) = \mathrm{SEN}(f) \restriction_{\mathrm{SEN}^v(\Sigma)} .$$

The following representation lemma for cyclic $\mathbf{Sign}$-module systems abstracts a similar representation result, Lemma 5.4, in [10].

Lemma 28 *Let* $\mathbf{Sign}$ *be a complete residuated category,* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *a* $\mathbf{Sign}$*-module system,* $V \in |\mathbf{Sign}|$*,* $v \in \mathrm{SEN}(V)$*.*

(1) $\mathrm{SEN}^v : \mathbf{Sign} \to \mathbf{Set}$ *is a* $\mathbf{Sign}$*-module system in which joins coincide with those in* SEN*. The residual operation of* $\star^{\Sigma,\Sigma'}_v$ *in* SEN^v *is given by* $a \backslash^{\Sigma,\Sigma'}_v y = [(a \backslash^{\Sigma,\Sigma'} y)/^{V,\Sigma} v] \star^{V,\Sigma'} v$*.*

(2) *The map* $\gamma^v : \mathbf{Sign}(V, -) \to \mathbf{Sign}(V, -)$*, given, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $a \in \mathbf{Sign}(V, \Sigma)$ *by* $\gamma^v_\Sigma(a) = (a \star^{V,\Sigma} v)/^{V,\Sigma} v$ *is a closure system on* $\mathbf{Sign}(V, -)$*.*

(3) SEN^v *is isomorphic to* $\mathbf{Sign}^{\gamma^v}(V,-)$.

Thus, a **Sign**-*module system is cyclic iff it is isomorphic to a* **Sign**-*module system* $\mathbf{Sign}^\gamma(V,-)$, *for some closure system* $\gamma : \mathbf{Sign}(V,-) \to \mathbf{Sign}(V,-)$.

Proof:

(1) Let $\Sigma \in |\mathbf{Sign}|, q \in \mathrm{SEN}^v(\Sigma)$ and $a \in \mathbf{Sign}(\Sigma,\Sigma')$. Then $q = b \star^{V,\Sigma} v$, for some $b \in \mathbf{Sign}(V,\Sigma)$, whence

$$\begin{array}{rcl} a \star^{\Sigma,\Sigma'} q & = & a \star^{\Sigma,\Sigma'} (b \star^{V,\Sigma} v) \\ & = & (a \circ^{V,\Sigma}_{\Sigma,\Sigma'} b) \star^{V,\Sigma'} v \\ & \in & \mathrm{SEN}^v(\Sigma'). \end{array}$$

Furthermore, for $r \in \mathrm{SEN}^v(\Sigma)$, i.e., $r = c \star^{V,\Sigma} v$, for some $c \in \mathbf{Sign}(V,\Sigma)$, and all $q \in \mathrm{SEN}^v(\Sigma')$, we get that

$$\begin{array}{rll} a \star^{\Sigma,\Sigma'} r \leq^{\Sigma'} q & \text{iff} & a \star^{\Sigma,\Sigma'} (c \star^{V,\Sigma} v) \leq^{\Sigma'} q \\ & \text{iff} & c \star^{V,\Sigma} v \leq^{\Sigma} a \backslash^{\Sigma,\Sigma'} q \\ & \text{iff} & c \leq^{V,\Sigma} (a \backslash^{\Sigma,\Sigma'} q)/^{V,\Sigma} v \\ & \text{iff} & c \star^{V,\Sigma'} v \leq^{\Sigma} [(a \backslash^{\Sigma,\Sigma'} q)/^{V,\Sigma} v] \star^{V,\Sigma'} v \\ & & \text{(by Lemma 9).} \end{array}$$

Finally, note that, for all $\Sigma \in |\mathbf{Sign}|$ and all $a_i \in \mathbf{Sign}(V,\Sigma), i \in I$, $\bigvee^{\Sigma}_{i \in I}(a_i \star^{V,\Sigma} v) = (\bigvee^{V,\Sigma}_{i \in I} a_i) \star^{V,\Sigma} v \in \mathrm{SEN}^v(\Sigma)$. Thus, SEN^v is closed under arbitrary joins in SEN and is therefore a complete lattice system.

(2) Since $a \star^{V,\Sigma} v = a \star^{V,\Sigma} v$, we get that $a \leq^{V,\Sigma} (a \star^{V,\Sigma} v)/^{V,\Sigma} v = \gamma^v_\Sigma(a)$. Moreover, if $a \leq^{V,\Sigma} b$, then, by Lemma 9, we get that $\gamma^v_\Sigma(a) \leq^{V,\Sigma} \gamma^v_\Sigma(b)$ and, also by Lemma 9, $\gamma^v_\Sigma(\gamma^v_\Sigma(a)) = \gamma^v_\Sigma(a)$. Finally, to establish structurality, suppose that $\Sigma, \Sigma' \in |\mathbf{Sign}|$ and $a \in \mathbf{Sign}(\Sigma,\Sigma')$, $b \in \mathbf{Sign}(V,\Sigma)$. Then

$$\begin{array}{rcl} (a \circ^{V,\Sigma}_{\Sigma,\Sigma'} \gamma^v_\Sigma(b)) \star^{V,\Sigma'} v & = & (a \circ^{V,\Sigma}_{\Sigma,\Sigma'} [(b \star^{V,\Sigma} v)/^{V,\Sigma} v]) \star^{V,\Sigma'} v \\ & \leq^{\Sigma'} & (a \circ^{V,\Sigma}_{\Sigma,\Sigma'} b) \star^{V,\Sigma'} v, \end{array}$$

whence $a \circ^{V,\Sigma}_{\Sigma,\Sigma'} \gamma^v_\Sigma(b) \leq^{V,\Sigma'} \gamma^v_{\Sigma'}(a \circ^{V,\Sigma}_{\Sigma,\Sigma'} b)$.

(3) Define $f : \mathbf{Sign}^{\gamma^v}(V,-) \to \mathrm{SEN}^v$ by $f_\Sigma(a) = a \star^{V,\Sigma} v$, for all $\Sigma \in |\mathbf{Sign}|$ and all $a \in \mathbf{Sign}^{\gamma^v}(V,\Sigma)$, and $g : \mathrm{SEN}^v \to \mathbf{Sign}^{\gamma^v}(V,-)$ by $g_\Sigma(x) = x/^{V,\Sigma} v$, for all $\Sigma \in |\mathbf{Sign}|$ and $x \in \mathrm{SEN}^v(\Sigma)$. Note that both f and g are well-defined maps. Moreover, by Lemma 25,

$$f_\Sigma(g_\Sigma(x)) = (x/^{V,\Sigma} v) \star^{V,\Sigma} v = x,$$

and

$$g_\Sigma(f_\Sigma(a)) = (a \star^{V,\Sigma} v)/^{V,\Sigma} v = \gamma^v_\Sigma(a) = a.$$

Finally, it is easy to see that both f and g are order-preserving and, therefore, also order-reflecting.

□

Corollary 29 *Let* **Sign** *be a complete residuated category and* SEN : **Sign** → **Set** *a cyclic* **Sign***-module system, with generator* $\langle V, v\rangle$*. Then* SEN *is isomorphic to* $\mathbf{Sign}^{\gamma^v}(V,-)$*.*

Lemma 28, Part (3) yields the following corollary, if we take as SEN : **Sign** → **Set** the functor $\mathbf{Sign}(V,-) : \mathbf{Sign} \to \mathbf{Set}$ and as $v \in V$ a fixed morphism $u \in \mathbf{Sign}(V,V)$.

Corollary 30 *Let* **Sign** *be a complete residuated category,* $V \in |\mathbf{Sign}|$ *and* $u \in \mathbf{Sign}(V,V)$*. Then the* **Sign***-module system* $\mathbf{Sign}^u(V,-)$ *is isomorphic to* $\mathbf{Sign}^{\gamma^u}(V,-)$*.*

Note that the isomorphisms involved in Corollary 30 are given by $a \mapsto a/^{V,V}_{V,\Sigma}u$, for all $a \in \mathbf{Sign}^u(V,\Sigma)$, and $a \mapsto a \circ^{V,V}_{V,\Sigma} u$, for all $a \in \mathbf{Sign}^{\gamma^u}(V,\Sigma)$.

Lemma 31 *Let* **Sign** *be a complete residuated category,* $V \in |\mathbf{Sign}|$*,* $u \in \mathbf{Sign}(V,V)$ *and and* $\gamma : \mathbf{Sign}(V,-) \to \mathbf{Sign}(V,-)$ *a closure system on the complete lattice system* $\mathbf{Sign}(V,-)$*. Then, the following are equivalent:*

(1) $\gamma_V(u) = \gamma_V(i_V)$ *and* $\gamma_\Sigma(a) \circ^{V,V}_{V,\Sigma} u = a \circ^{V,V}_{V,\Sigma} u$*, for all* $a \in \mathbf{Sign}(V,\Sigma)$*;*

(2) $\gamma = \gamma^u$ *and* $u \circ^{V,V}_{V,V} u = u$*.*

Proof:

(2)→(1) We have

$$\begin{aligned}\gamma_V(u) &= \gamma^u_V(u)\\ &= (u \circ^{V,V}_{V,V} u)/^{V,V}_{V,V}u\\ &= u/^{V,V}_{V,V}u\\ &= (i_V \circ^{V,V}_{V,V} u)/^{V,V}_{V,V}u\\ &= \gamma^u_V(i_V)\\ &= \gamma_V(i_V)\end{aligned}$$

and, also,

$$\begin{aligned}\gamma_\Sigma(a) \circ^{V,V}_{V,\Sigma} u &= \gamma^u_\Sigma(a) \circ^{V,V}_{V,\Sigma} u\\ &= [(a \circ^{V,V}_{V,\Sigma} u)/^{V,V}_{V,\Sigma}u] \circ^{V,V}_{V,\Sigma} u\\ &= a \circ^{V,V}_{V,\Sigma} u.\end{aligned}$$

(1)→(2) For all $a \in \mathbf{Sign}(V,\Sigma)$, $\gamma_\Sigma(a) \circ^{V,V}_{V,\Sigma} u = a \circ^{V,V}_{V,\Sigma} u$ implies $\gamma_\Sigma(a) \leq^{V,\Sigma} (a \circ^{V,V}_{V,\Sigma} u)/^{V,V}_{V,\Sigma}u = \gamma^u_\Sigma(a)$. To show the reverse inequality, note that $\gamma_V(u) = \gamma_V(i_V)$ implies, for all $b \in \mathbf{Sign}(V,\Sigma)$,

$$\begin{aligned}\gamma_\Sigma(b \circ^{V,V}_{V,\Sigma} u) &= b \circ^{V,V}_{V,\Sigma} \gamma_V(u)\\ &= b \circ^{V,V}_{V,\Sigma} \gamma_V(i_V)\\ &= \gamma_\Sigma(b \circ^{V,V}_{V,\Sigma} i_V)\\ &= \gamma_\Sigma(b),\end{aligned}$$

whence, for all $a \in \mathbf{Sign}(V,\Sigma)$,

$$\begin{array}{ll}\gamma^u_\Sigma(a) \circ^{V,V}_{V,\Sigma} u \leq^{V,\Sigma} a \circ^{V,V}_{V,\Sigma} u & \\ \quad\text{implies} & \gamma_\Sigma(\gamma^u_\Sigma(a) \circ^{V,V}_{V,\Sigma} u) \leq^{V,\Sigma} \gamma_\Sigma(a \circ^{V,V}_{V,\Sigma} u) \\ \quad\text{implies} & \gamma_\Sigma(\gamma^u_\Sigma(a)) \leq^{V,\Sigma} \gamma_\Sigma(a) \\ \quad\text{implies} & \gamma^u_\Sigma(a) \leq^{V,\Sigma} \gamma_\Sigma(a).\end{array}$$

To show that $u \circ^{V,V}_{V,V} u = u$, notice, first, that

$$u = i_V \circ^{V,V}_{V,V} u \leq^{V,V} (u/^{V,V}_{V,V}u) \circ^{V,V}_{V,V} u \leq^{V,V} u,$$

whence $(u/^{V,V}_{V,V}u) \circ^{V,V}_{V,V} u = u$ and, also,

$$u \circ^{V,V}_{V,V} u \leq^{V,V} [(u \circ^{V,V}_{V,V} u)/^{V,V}_{V,V}u] \circ^{V,V}_{V,V} u \leq^{V,V} u \circ^{V,V}_{V,V} u,$$

whence $[(u \circ^{V,V}_{V,V} u)/^{V,V}_{V,V}u] \circ^{V,V}_{V,V} u = u \circ^{V,V}_{V,V} u$. Taking these two equalities into account we have the following:

$$\begin{array}{lll}\gamma = \gamma^u & \text{implies} & \gamma^u_V(u) = \gamma^u_V(i_V) \\ & \text{implies} & (u \circ^{V,V}_{V,V} u)/^{V,V}_{V,V}u = u/^{V,V}_{V,V}u \\ & \text{implies} & [(u \circ^{V,V}_{V,V} u)/^{V,V}_{V,V}u] \circ^{V,V}_{V,V} u = (u/^{V,V}_{V,V}u) \circ^{V,V}_{V,V} u \\ & \text{implies} & u \circ^{V,V}_{V,V} u = u.\end{array}$$

□

The following theorem provides a characterization of projective cyclic **Sign**-module systems, for a complete residuated category **Sign**. It abstracts in an obvious way Theorem 5.7 of [10]. We apply this theorem to the specific context of term π-institutions in Corollary 33, that follows.

Theorem 32 *Let* **Sign** *be a complete residuated category and* SEN : **Sign** $\rightarrow$ **Set** *a* **Sign**-*module system. Then, the following conditions are equivalent:*

(1) *For some* $V \in |\mathbf{Sign}|$, $v \in \mathrm{SEN}(V)$ *and* $u \in \mathbf{Sign}(V,V)$, *we have* $u \star^{V,V} v = v$, $[(a \star^{V,\Sigma} v)/^{V,\Sigma} v] \circ^{V,V}_{V,\Sigma} u = a \circ^{V,V}_{V,\Sigma} u$, *for all* $\Sigma \in |\mathbf{Sign}|$, $a \in \mathbf{Sign}(V,\Sigma)$, *and* $\mathrm{SEN} = \mathrm{SEN}^v$;

(2) *For some* $V \in |\mathbf{Sign}|$, $v \in \mathrm{SEN}(V)$ *and* $u \in \mathbf{Sign}(V,V)$, $\gamma^v_\Sigma(a) \circ^{V,V}_{V,\Sigma} u = a \circ^{V,V}_{V,\Sigma} u$, *for all* $\Sigma \in |\mathbf{Sign}|$, $a \in \mathbf{Sign}(V,\Sigma)$, $\gamma^v_V(u) = \gamma^v_\Sigma(i_V)$ *and* $\mathrm{SEN} = \mathrm{SEN}^v$;

(3) *For some* $V \in |\mathbf{Sign}|$, $v \in \mathrm{SEN}(V)$ *and* $u \in \mathbf{Sign}(V,V)$, *we have* $\gamma^u = \gamma^v$, $u \circ^{V,V}_{V,V} u = u$ *and* $\mathrm{SEN} = \mathrm{SEN}^v$;

(4) *For some* $V \in |\mathbf{Sign}|$ *and* $u \in \mathbf{Sign}(V,V)$, *the module system* SEN *is isomorphic to* $\mathbf{Sign}^u(V,-)$ *and* $u \circ^{V,V}_{V,V} u = u$;

(5) SEN *is cyclic and projective.*

Proof:

(1)↔(2) Since $\gamma^v_\Sigma(a) = (a \star^{V,\Sigma} v)/^{V,\Sigma}v$, the equivalence follows from

$$\gamma^v_V(u) = \gamma^v_V(i_V) \text{ iff } (u \star^{V,V} v)/^{V,V}v = v/^{V,V}v \text{ iff } u \star^{V,V} v = v.$$

(2)→(3) By Lemma 31.

(3)→(4) We have

$$\begin{array}{rcll} \mathrm{SEN} & \cong & \mathbf{Sign}^{\gamma^v}(V,-) & \text{(by Lemma 28)} \\ & \cong & \mathbf{Sign}^{\gamma^u}(V,-) & \text{(since } \gamma^v = \gamma^u\text{)} \\ & \cong & \mathbf{Sign}^{u}(V,-). & \text{(by Corollary 29)} \end{array}$$

(4)→(1) Take into account the isomorphism identifying v with u and replace in (1) $\star^{V,V}$, $\star^{V,\Sigma}$ and $/^{V,\Sigma}$ by $\circ^{V,V}_{V,V}$, $\circ^{V,V}_{V,\Sigma}$ and $/^{V,V}_{V,\Sigma}$, respectively.

(5)→(4) By Corollary 29, every cyclic **Sign**-module system is of the form $\mathbf{Sign}^\gamma(V,$ $-)$ for some closure system $\gamma : \mathbf{Sign}(V,-) \to \mathbf{Sign}(V,-)$.

Suppose, also, that $\mathbf{Sign}^\gamma(V,-)$ is projective. By projectivity, there exists a **Sign**-module system morphism $f : \mathbf{Sign}^\gamma(V,-) \to \mathbf{Sign}(V,-)$, such that $\gamma f = \iota^{\mathbf{Sign}^\gamma(V,-)}$.

$$\begin{tikzcd} \mathbf{Sign}^\gamma(V,-) \arrow[r, "f"] \arrow[dr, "\iota^{\mathbf{Sign}^\gamma(V,-)}"'] & \mathbf{Sign}(V,-) \arrow[d, "\gamma"] \\ & \mathbf{Sign}^\gamma(V,-) \end{tikzcd}$$

Set $u = f_V(\gamma_V(i_V))$. Then, for all $a \in \mathbf{Sign}(V,\Sigma)$,

$$\begin{array}{rcl} \gamma_\Sigma(a) & = & \gamma_\Sigma(a \circ^{V,V}_{V,\Sigma} i_V) \\ & = & \gamma_\Sigma(a \circ^{V,V}_{V,\Sigma} \gamma_V(i_V)) \\ & = & a(\circ^{V,V}_{V,\Sigma})_\gamma \gamma_V(i_V), \end{array}$$

whence $f_\Sigma(\gamma_\Sigma(a)) = a \circ^{V,V}_{V,\Sigma} f_V(\gamma_V(i_V)) = a \circ^{V,V}_{V,\Sigma} u$. Hence, the map $f : \mathbf{Sign}^\gamma(V,-) \to \mathbf{Sign}^u(V,-)$ is surjective. Since f is also injective by definition, we get that $\mathbf{Sign}^\gamma(V,-) \cong \mathbf{Sign}^u(V,-)$. Finally,

$$\begin{array}{rcl} u \circ^{V,V}_{V,V} u & = & f_V(\gamma_V(i_V)) \circ^{V,V}_{V,V} f_V(\gamma_V(i_V)) \\ & = & f_V(f_V(\gamma_V(i_V))(\circ^{V,V}_{V,V})_\gamma \gamma_V(i_V)) \\ & = & f_V(\gamma_V(f_V(\gamma_V(i_V)))) \\ & = & f_V(\gamma_V(i_V)) \\ & = & u. \end{array}$$

(4)→(5) Suppose that SEN is cyclic, such that $\mathrm{SEN} \cong \mathbf{Sign}^u(V,-)$ and $u \circ^{V,V}_{V,V} u = u$, for some $V \in |\mathbf{Sign}|$ and $u \in \mathbf{Sign}(V,V)$. Let $\mathbf{Sign}'$ be a complete residuated category and $\mathrm{SEN}', \mathrm{SEN}'' : \mathbf{Sign}' \to \mathbf{Set}$ be $\mathbf{Sign}'$-module systems, $g : \mathrm{SEN}' \to \mathrm{SEN}''$ a surjective module system morphism and $\langle K,k\rangle : \mathbf{Sign}^u(V,-) \to \mathrm{SEN}''$ a module system morphism. We must define a $\mathbf{Sign}$-module system morphism $\langle H,h\rangle : \mathbf{Sign}^u(V,-) \to \mathrm{SEN}'$ making the following triangle commute:

$$\begin{tikzcd}
\mathbf{Sign}^u(V,-) \arrow[r, "h"] \arrow[dr, "k"'] & \mathrm{SEN}' \arrow[d, "g"] \\
& \mathrm{SEN}''
\end{tikzcd}$$

Let $H = K$, set $w = k_V(u)$ and let $z \in \mathrm{SEN}'(K(V))$ be such that $g_{K(V)}(z) = w = k_V(u)$. Then define, for all $\Sigma \in |\mathbf{Sign}|$ and all $a \in \mathbf{Sign}(V,\Sigma)$,

$$h_\Sigma(a \circ^{V,V}_{V,\Sigma} u) = K(a) \star^{K(V),K(\Sigma)} z.$$

This defines a module system morphism and we have, for all $\Sigma \in |\mathbf{Sign}|$, $a \in \mathbf{Sign}(V,\Sigma)$,

$$\begin{aligned}
g_{K(\Sigma)}(h_\Sigma(a \circ^{V,V}_{V,\Sigma} u)) &= g_{K(\Sigma)}(K(a) \star^{K(V),K(\Sigma)} z) \\
&= K(a) \star^{K(V),K(\Sigma)} g_{K(V)}(z) \\
&= K(a) \star^{K(V),K(\Sigma)} w \\
&= K(a) \star^{K(V),K(\Sigma)} k_V(u) \\
&= k_\Sigma(a \circ^{V,V}_{V,\Sigma} u).
\end{aligned}$$

□

Corollary 33 *Let* $\mathbf{Sign}$ *be a category and* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *a functor. If* SEN *is term, then* $\mathcal{P}\mathrm{SEN} : \mathbf{Sign}^{\mathcal{P}} \to \mathbf{Set}$ *is a projective cyclic* $\mathbf{Sign}^{\mathcal{P}}$*-module system.*

Proof:
Using Theorem 32, we define $u \in \mathbf{Sign}^{\mathcal{P}}(V,V)$ by $u = f_{\langle V,\{v\}\rangle}$. Then we have $f_{\langle V,\{v\}\rangle} \star^{V,V} \{v\} = \{v\}$ and, for all $\Sigma \in |\mathbf{Sign}|$, and $f^{\mathcal{P}} \in \mathbf{Sign}^{\mathcal{P}}(V,\Sigma)$, $[(f^{\mathcal{P}} \star^{V,\Sigma} \{v\})/^{V,\Sigma}\{v\}] \circ^{V,V}_{V,\Sigma} f_{\langle V,\{v\}\rangle} = f^{\mathcal{P}} \circ^{V,V}_{V,\Sigma} f_{\langle V,\{v\}\rangle}$. □

In fact, using Theorem 32, we can see the following:

Corollary 34 *Let* $\mathbf{Sign}$ *be a category and* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *a functor. Then* $\mathcal{P}\mathrm{SEN} : \mathbf{Sign}^{\mathcal{P}} \to \mathbf{Set}$ *is a projective cyclic* $\mathbf{Sign}^{\mathcal{P}}$*-module system iff, there exists* $V \in |\mathbf{Sign}|$ *and* $v \in \mathrm{SEN}(V)$*, such that,*

- *for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $x \in \mathrm{SEN}(\Sigma)$*, there exists* $f_{\langle\Sigma,x\rangle} \in \mathbf{Sign}(V,\Sigma)$*, such that* $\mathrm{SEN}(f_{\langle\Sigma,x\rangle})(v) = x$*;*
- *for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $f,g \in \mathbf{Sign}(V,\Sigma)$*, if* $\mathrm{SEN}(f)(v) = \mathrm{SEN}(g)(v)$*, then* $f \circ f_{\langle V,v\rangle} = g \circ f_{\langle V,v\rangle}$*.*

6.3 Coproducts

Multi-term sentence functors (see [11]), in general, are not cyclic. The reason is that they can accommodate all sequent sentence functors and these are not cyclic, as is shown in Proposition 5.10 of [10]. Thus, the projectivity of these functors cannot be established using Theorem 32. In this section we study coproducts of module systems and show, by analogy with coproducts of modules over complete residuated lattices, that coproducts of projective module systems are also projective. This result can then be applied to the case of multi-term sentence functors, which, in fact, turn out to be coproducts of projective cyclic modules.

Let **Sign** be a complete residuated category and $\mathrm{SEN}^i : \mathbf{Sign} \to \mathbf{Set}$, $i \in I$, a family of **Sign**-module systems. The **coproduct** of this family is a **Sign**-module system $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, denoted by $\coprod_{i\in I} \mathrm{SEN}^i$, together with a family of injective module system morphisms $\sigma^i : \mathrm{SEN}^i \to \mathrm{SEN}$, $i \in I$, such that, for every **Sign**′-module system $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, and every family of module system morhisms $\langle F, \alpha^i\rangle : \mathrm{SEN}^i \to \mathrm{SEN}'$, $i \in I$, there exists a unique map $\langle F, \alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$, such that $\langle F, \alpha\rangle \circ \sigma^i = \langle F, \alpha^i\rangle$.

$$\begin{tikzcd} \mathrm{SEN}^i \arrow[r, "\sigma^i"] \arrow[dr, "\langle F,\alpha^i\rangle"'] & \mathrm{SEN} \arrow[d, "\langle F,\alpha\rangle"] \\ & \mathrm{SEN}' \end{tikzcd}$$

Clearly, if $\coprod_{i\in I} \mathrm{SEN}^i$ exists, then it is unique up to isomorphism of **Sign**-module systems. The next lemma asserts that the coproduct of a family of **Sign**-module systems always exists.

Given a complete residuated category **Sign** and a family of **Sign**-module systems $\mathrm{SEN}^i : \mathbf{Sign} \to \mathbf{Set}$, $i \in I$, let $\mathrm{SEN} := \prod_{i\in I} \mathrm{SEN}^i$ denote the **Sign**-module system, defined, for all $\Sigma \in |\mathbf{Sign}|$, by $\mathrm{SEN}(\Sigma) = \prod_{i\in I} \mathrm{SEN}^i(\Sigma)$ and, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\mathrm{SEN}(f)(\vec{\phi}) = \langle \mathrm{SEN}^i(f)(\phi_i) : i \in I\rangle$, for all $\vec{\phi} \in \mathrm{SEN}(\Sigma)$.

Lemma 35 *Let* **Sign** *be a complete residuated category and* $\mathrm{SEN}^i : \mathbf{Sign} \to$ **Set**, $i \in I$, *be a family of* **Sign**-*module systems. Then the* **Sign**-*module system* $\coprod_{i\in I} \mathrm{SEN}^i$ *is the direct product* $\prod_{i\in I} \mathrm{SEN}^i$, *with canonical injection* **Sign**-*module system morphisms* $\sigma^i : \mathrm{SEN}^i \to \prod_{j\in I} \mathrm{SEN}^j$ *defined, for all* $i \in I, \Sigma \in |\mathbf{Sign}|$ *and* $p \in \mathrm{SEN}^i(\Sigma)$, *by*

$$\sigma^i_\Sigma(p) = \langle x_j : j \in I\rangle, \text{ where, for all } j \in I,\ x_j = \begin{cases} p, & \text{if } j = i \\ 0^i_\Sigma, & \text{if } j \neq i \end{cases},$$

0^i_Σ *being the least element in the complete lattice* $\mathrm{SEN}^i(\Sigma)$.

Proof:

The maps $\sigma^i : \mathrm{SEN}^i \to \prod_{j\in I}\mathrm{SEN}^j$ are **Sign**-module system morphisms. Suppose that $\langle F,\alpha^i\rangle : \mathrm{SEN}^i \to \mathrm{SEN}'$, $i \in I$, are module system morphisms. Define $\langle F,\alpha\rangle : \prod_{i\in I}\mathrm{SEN}^i \to \mathrm{SEN}'$ by setting, for all $\Sigma \in |\mathbf{Sign}|$ and all $\langle x_i : i \in I\rangle \in \prod_{i\in I}\mathrm{SEN}^i(\Sigma)$, by

$$\alpha_\Sigma(\langle x_i : i \in I\rangle) = \bigvee_{i\in I}^{\Sigma}\alpha^i_\Sigma(x_i).$$

This mapping is residuated with residual $\alpha^* : \mathrm{SEN}' \to \prod_{i\in I}\mathrm{SEN}^i$, given, for all $\Sigma \in |\mathbf{Sign}|$ and all $y \in \mathrm{SEN}'(\Sigma)$, by

$$\alpha^*_\Sigma(y) = \langle(\alpha^i)^*_\Sigma(y) : i \in I\rangle.$$

Since it also preserves the action, it is a module system morphism. □

Our interest in coproducts is the next general result, an adaptation of Lemma 5.12 of [10]. It asserts that the coproduct of projective **Sign**-module systems is also a projective **Sign**-module system.

Lemma 36 *Let* **Sign** *be a complete residuated category. The coproduct of a family of projective* **Sign***-module systems is a projective* **Sign***-module system.*

Proof:

Let $\mathrm{SEN}^i : \mathbf{Sign} \to \mathbf{Set}$, $i \in I$, be a family of projective **Sign**-module systems, $\mathrm{SEN}', \mathrm{SEN}'' : \mathbf{Sign}' \to \mathbf{Set}$ two $\mathbf{Sign}'$-module systems, $g : \mathrm{SEN}' \to \mathrm{SEN}''$ a surjective $\mathbf{Sign}'$-module system morphism and $\langle K,k\rangle : \coprod_{i\in I}\mathrm{SEN}^i \to \mathrm{SEN}''$ a module system morphism. Denote by $\sigma^i : \mathrm{SEN}^i \to \coprod_{i\in I}\mathrm{SEN}^i$ the canonical coproduct **Sign**-module system injections and $\langle K,k_i\rangle := \langle K,k\rangle \circ \sigma^i$.

$$\begin{tikzcd}
& & \mathrm{SEN}' \arrow[dd, "g"] \\
\coprod_{i\in I}\mathrm{SEN}^i \arrow[urr, "\alpha"] \arrow[drr, "k"'] & \mathrm{SEN}^i \arrow[l, "\sigma^i"'] \arrow[ur, "\alpha^i"] \arrow[dr, "k_i"] & \\
& & \mathrm{SEN}''
\end{tikzcd}$$

Since SEN^i is projective, there exists a **Sign**-module system morphism $\langle K,\alpha^i\rangle : \mathrm{SEN}^i \to \mathrm{SEN}'$, such that $\langle K,k_i\rangle = g \circ \langle K,\alpha^i\rangle$. Hence, by the universal property of the coproduct, there exists a **Sign**-module system morphism $\langle K,\alpha\rangle : \coprod_{i\in I}\mathrm{SEN}^i \to \mathrm{SEN}'$, such that $\langle K,\alpha^i\rangle = \langle K,\alpha\rangle \circ \sigma^i$. Thus, $\langle K,k\rangle \circ \sigma^i = \langle K,k_i\rangle = g \circ \langle K,\alpha^i\rangle = g \circ \langle K,\alpha\rangle \circ \sigma^i$, for all $i \in I$. By the uniqueness clause in the universal property, $\langle K,k\rangle = g \circ \langle K,\alpha\rangle$. □

Based on Lemma 36, we will show that multi-term sentence functors give rise to projective module systems. This will be established by showing that these module systems are coproducts of projective cyclic module systems and using the characterization Theorem 32.

Let SEN : $\mathbf{Sign} \to \mathbf{Set}$ be a multi-term sentence functor, with multi-source signature-variable pair $Y : \mathbf{Elt}(\mathrm{SEN}) \to \mathbf{Elt}(\mathrm{SEN})$ and accompanying natural transformation $f : Y \to \mathrm{I}_{\mathbf{Elt}(\mathrm{SEN})}$. Define an equivalence relation $\sim$ on $|\mathbf{Elt}(\mathrm{SEN})|$ by setting, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma), \psi \in \mathrm{SEN}(\Sigma')$,

$$\langle \Sigma, \phi \rangle \sim \langle \Sigma', \psi \rangle \quad \text{iff} \quad Y(\langle \Sigma, \phi \rangle) = Y(\langle \Sigma', \psi \rangle).$$

Let I be an index set for the blocks of the partition corresponding to $\sim$ and denote the partition by $\pi_Y = \{B_i : i \in I\}$. Define, next, a collection of sentence functors $\mathrm{SEN}^i : \mathbf{Sign} \to \mathbf{Set}$ indexed by I, as follows: $\mathrm{SEN}^i(\Sigma) = \{\phi \in \mathrm{SEN}(\Sigma) : \langle \Sigma, \phi \rangle \in B_i\}$ and $\mathrm{SEN}^i(f) = \mathrm{SEN}(f) \restriction_{\mathrm{SEN}^i(\Sigma)}$, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$. The following lemma shows that $\mathcal{P}\mathrm{SEN}^i$ is a projective cyclic $\mathbf{Sign}^{\mathcal{P}}$-module system, for all $i \in I$.

Lemma 37 *Let* $\mathbf{Sign}$ *be a category and* SEN : $\mathbf{Sign} \to \mathbf{Set}$ *a multi-term functor. Then,* $\mathcal{P}\mathrm{SEN}^i : \mathbf{Sign}^{\mathcal{P}} \to \mathbf{Set}$ *is a projective cyclic* $\mathbf{Sign}^{\mathcal{P}}$*-module system, for every* $i \in I$.

Proof:

By construction, there exists $V \in |\mathbf{Sign}|$ and $v \in \mathrm{SEN}(V)$, such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}^i(\Sigma)$, $Y(\langle \Sigma, \phi \rangle) = \langle V, v \rangle$. Moreover, it is clear from the definition of SEN^i that $\mathcal{P}\mathrm{SEN}^i$ is cyclic with generator $\langle V, \{v\} \rangle$. Let $u = f_{\langle V, \{v\} \rangle} \in \mathbf{Sign}^{\mathcal{P}}(V, V)$. It is shown that $V \in |\mathbf{Sign}|$, $\{v\} \in \mathcal{P}\mathrm{SEN}^i(V)$ and the morphism $u \in \mathbf{Sign}^{\mathcal{P}}(V, V)$ satisfy Condition (1) of Theorem 32. Obviously,

$$\begin{array}{rcl} u \star^{V,V} \{v\} & = & \mathcal{P}\mathrm{SEN}^i(u)(\{v\}) \\ & = & \{\mathrm{SEN}^i(f_{\langle V, v \rangle})(v)\} \\ & = & \{v\}. \end{array}$$

On the other hand, if $\Sigma \in |\mathbf{Sign}|$ and $f^{\mathcal{P}} \in \mathbf{Sign}^{\mathcal{P}}(V, \Sigma)$, we get

$$\begin{array}{l} [(f^{\mathcal{P}} \star^{V,\Sigma} \{v\}) /^{V,\Sigma} \{v\}] \circ^{V,V}_{V,\Sigma} f_{\langle V, \{v\} \rangle} \\ \quad = \{g \in \mathbf{Sign}(V, \Sigma) : \mathrm{SEN}(g)(v) \subseteq \{\mathrm{SEN}(f)(v) : f \in f^{\mathcal{P}}\}\} \circ f_{\langle V, \{v\} \rangle} \\ \quad = \{f_{\langle \Sigma, \mathrm{SEN}(g)(v) \rangle} : \mathrm{SEN}(g)(v) \subseteq \{\mathrm{SEN}(f)(v) : f \in f^{\mathcal{P}}\}\} \\ \quad = \{f_{\langle \Sigma, \mathrm{SEN}(f)(v) \rangle} : f \in f^{\mathcal{P}}\} \\ \quad = \{f f_{\langle V, v \rangle} : f \in f^{\mathcal{P}}\} \\ \quad = f^{\mathcal{P}} \circ^{V,V}_{V,\Sigma} u. \end{array}$$

Thus, $\mathcal{P}\mathrm{SEN}^i$ is indeed a projective and cyclic $\mathbf{Sign}^{\mathcal{P}}$-module system. □

The next lemma shows that $\mathcal{P}\mathrm{SEN}$ is the coproduct of the $\mathbf{Sign}^{\mathcal{P}}$-module systems $\mathcal{P}\mathrm{SEN}^i$, $i \in I$.

Lemma 38 *Let* $\mathbf{Sign}$ *be a category and* SEN : $\mathbf{Sign} \to \mathbf{Set}$ *a multi-term functor. Then, the* $\mathbf{Sign}^{\mathcal{P}}$*-module system* $\mathcal{P}\mathrm{SEN}$ *is the coproduct of the projective cyclic* $\mathbf{Sign}^{\mathcal{P}}$*-module systems* $\mathcal{P}\mathrm{SEN}^i$*. Consequently, it is itself projective.*

Proof:
This is clear, since, for every $\Sigma \in |\mathbf{Sign}|$, $\mathrm{SEN}(\Sigma)$ is, by construction, the disjoint union $\mathrm{SEN}(\Sigma) = \bigcup_{i\in I}\{\phi \in \mathrm{SEN}(\Sigma) : \langle \Sigma, \phi\rangle \in B_i\} = \bigcup_{i\in I} \mathrm{SEN}^i(\Sigma)$. This is isomorphic to the product expression, postulated to be the coproduct in Lemma 35. The corresponding injection module system morphisms are the signature-wise injection functions. □

7 Finitary Translations

Let **Sign** be a category and $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ a complete lattice family. A subset X of $\mathrm{SEN}(\Sigma)$ is **(upward) directed** if, for all $x, y \in \mathrm{SEN}(\Sigma)$, there exists $z \in \mathrm{SEN}(\Sigma)$, such that both $x \leq^\Sigma z$ and $y \leq^\Sigma z$. An element $x \in \mathrm{SEN}(\Sigma)$ is **compact** if, for all directed $Y \subseteq \mathrm{SEN}(\Sigma)$, $x \leq^\Sigma \bigvee^\Sigma Y$ implies that $x \leq^\Sigma y$, for some $y \in Y$. A property equivalent to the compactness of $x \in \mathrm{SEN}(\Sigma)$ is that, for all $Z \subseteq \mathrm{SEN}(\Sigma)$, $x \leq^\Sigma \bigvee^\Sigma Z$ implies the existence of a finite $Z_0 \subseteq Z$, such that $x \leq^\Sigma \bigvee^\Sigma Z_0$. Let us use the notation $K_\Sigma(Q)$ to denote the set of compact elements of $\mathrm{SEN}(\Sigma)$ that are contained in $Q \subseteq \mathrm{SEN}(\Sigma)$ and K_Σ to denote the set $K_\Sigma(\mathrm{SEN}(\Sigma))$.

A **finitary lattice family** is a complete lattice family $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $x \in \mathrm{SEN}(\Sigma)$, $x = \bigvee_\Sigma K_\Sigma(\downarrow x)$, i.e., for every $\Sigma \in |\mathbf{Sign}|$, every element of $\mathrm{SEN}(\Sigma)$ is the join of all compact elements below it.

We say that the consequence family $\vdash$ on the finitary complete lattice family $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is **finitary** if, for all $\Sigma \in |\mathbf{Sign}|$ and all $x, y \in \mathrm{SEN}(\Sigma)$, if $x \vdash_\Sigma y$ and y is compact in $\mathrm{SEN}(\Sigma)$, then, there exists a compact element $x_0 \in \mathrm{SEN}(\Sigma)$, such that $x_0 \leq^\Sigma x$ and $x_0 \vdash_\Sigma y$. A closure family γ on a finitary complete lattice family $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is **finitary** iff $\vdash^\gamma$ is finitary, i.e., if, for all $\Sigma \in |\mathbf{Sign}|$, $x, y \in \mathrm{SEN}(\Sigma)$, if $y \leq^\Sigma \gamma_\Sigma(x)$ and y is compact, there exists compact $x_0 \leq^\Sigma x$, such that $y \leq^\Sigma \gamma_\Sigma(x_0)$.

The next lemma forms an analog in the present context of Lemma 6.1 of [10]. It asserts that all fixed points of finitary closure families over finitary complete lattice families are generated by compact elements. Since the proof can be easily obtained by applying signature-wise the same argument as that used to prove Lemma 6.1 of [10], it will be omitted.

Lemma 39 *Let* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *be a finitary complete lattice family and* $\gamma : \mathrm{SEN} \to \mathrm{SEN}$ *a finitary closure family on* SEN. *If* y *is a compact element of* $\mathrm{SEN}^\gamma(\Sigma)$, *then, there exists a compact element* $x \in \mathrm{SEN}(\Sigma)$, *such that* $y = \gamma_\Sigma(x)$. *Thus, for all* $\Sigma \in |\mathbf{Sign}|$, $K^\gamma_\Sigma \subseteq \gamma_\Sigma(K_\Sigma)$.

We also provide an analog of Lemma 6.2 of [10], whose proof we omit, since it can be obtained by applying Lemma 6.2 of [10] signature-wise.

Lemma 40 *Let* **Sign** *be a category,* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *a finitary complete lattice family and* $\gamma : \mathrm{SEN} \to \mathrm{SEN}$ *a closure family on* SEN. *Then, the following are equivalent:*

(1) γ *is finitary;*

(2) γ *preserves directed joins, i.e., for all* $\Sigma \in |\mathbf{Sign}|$ *and all directed* $X \subseteq \mathrm{SEN}(\Sigma)$, $\gamma_\Sigma(\bigvee^\Sigma X) = \bigvee^\Sigma \gamma_\Sigma(X)$*;*

(3) *Arbitrary joins in* SEN^γ *coincide with those in* SEN*, i.e., for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $Y \subseteq \mathrm{SEN}^\gamma(\Sigma)$, $\bigvee_\gamma^\Sigma Y = \bigvee^\Sigma Y$*;*

(4) *For all* $\Sigma \in |\mathbf{Sign}|$ *and all* $x \in \mathrm{SEN}(\Sigma)$, $\gamma_\Sigma(x) = \bigvee^\Sigma \gamma_\Sigma(K_\Sigma(\downarrow x))$*;*

(5) *For all* $\Sigma \in |\mathbf{Sign}|$ *and every compact* $x \in \mathrm{SEN}(\Sigma)$, $\gamma_\Sigma(x)$ *is compact in* $\mathrm{SEN}^\gamma(\Sigma)$*;*

(6) *For all* $\Sigma \in |\mathbf{Sign}|$, $K_\Sigma^\gamma = \gamma_\Sigma(K_\Sigma)$*;*

If (any of) the above statements hold, then SEN^γ *is also finitary.*

A residuated map $\langle F, \alpha\rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2$ between two finitary complete lattice families $\mathrm{SEN}_1 : \mathbf{Sign}_1 \to \mathbf{Set}$, $\mathrm{SEN}_2 : \mathbf{Sign}_2 \to \mathbf{Set}$ is called **finitary** if, for every $\Sigma \in |\mathbf{Sign}|$ and every compact element $x \in \mathrm{SEN}_1(\Sigma)$, $\alpha_\Sigma(x)$ is compact in $\mathrm{SEN}_2(F(\Sigma))$.

The equivalence (1)$\leftrightarrow$(5) of Lemma 40 yields the following

Corollary 41 *Let* $\mathbf{Sign}$ *be a category,* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *a finitary complete lattice family and* $\gamma : \mathrm{SEN} \to \mathrm{SEN}$ *a closure family on* SEN*. Then* γ *is finitary as a closure family iff* $\gamma : \mathrm{SEN} \to \mathrm{SEN}^\gamma$ *is finitary as a residuated map.*

By applying Lemma 6.4 of [10], we obtain the following analog to the effect that the composition of a finitary residuated map between two finitary complete lattice families with its residual generates a finitary closure family.

Lemma 42 *Let* $\mathbf{Sign}_1, \mathbf{Sign}_2$ *be categories,* $\mathrm{SEN}_1 : \mathbf{Sign}_1 \to \mathbf{Set}$, $\mathrm{SEN}_2 : \mathbf{Sign}_2 \to \mathbf{Set}$ *be finitary complete lattice families and* $\langle F, \alpha\rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2$ *a finitary residuated map. Then* $\alpha^*\alpha : \mathrm{SEN}_1 \to \mathrm{SEN}_1$ *is a finitary closure family on* SEN_1*.*

Lemma 43 *Let* $\mathbf{Sign}_1, \mathbf{Sign}_2$ *be categories,* $\mathrm{SEN}_1 : \mathbf{Sign}_1 \to \mathbf{Set}$, $\mathrm{SEN}_2 : \mathbf{Sign}_2 \to \mathbf{Set}$ *be finitary complete lattice families,* $\langle F, \alpha\rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2$ *a finitary residuated map and* $\delta : \mathrm{SEN}_2 \to \mathrm{SEN}_2$ *a finitary closure family on* SEN_2*.*

(1) *The closure family* $\delta^\alpha = \alpha^*\delta\alpha : \mathrm{SEN}_1 \to \mathrm{SEN}_1$ *is finitary.*

(2) *The residuated map* $\langle F, f\rangle : \mathrm{SEN}_1^{\delta^\alpha} \to \mathrm{SEN}_2^\delta$*, with* $f = \delta\alpha \restriction_{\mathrm{SEN}^{\delta^\alpha}}$ *is finitary.*

Proof:

(1) Suppose that $\Sigma \in |\mathbf{Sign}_1|$, $x, y \in \mathrm{SEN}_1(\Sigma)$, with y compact, such that $y \leq^\Sigma \delta^\alpha_\Sigma(x)$. Then we have $y \leq^\Sigma \alpha^*_\Sigma(\delta_{F(\Sigma)}(\alpha_\Sigma(x)))$, which is equivalent to $\alpha_\Sigma(y) \leq^{F(\Sigma)} \delta_{F(\Sigma)}(\alpha_\Sigma(x))$. By the finitarity of $\langle F, \alpha\rangle$, since y is compact, $\alpha_\Sigma(y)$ is compact, whence, by the finitarity of δ, there exists a compact $x' \leq^{F(\Sigma)} \alpha_\Sigma(x)$, such that $\alpha_\Sigma(y) \leq^{F(\Sigma)} \delta_{F(\Sigma)}(x')$. Since SEN_1 is finitary, we have that $x = \bigvee^\Sigma K_\Sigma(\downarrow x)$, whence $\alpha_\Sigma(x) = \bigvee^{F(\Sigma)} \alpha_\Sigma(K_\Sigma(\downarrow x))$. Thus, since $x' \leq^{F(\Sigma)} \alpha_\Sigma(x)$, there exists a compact $x_0 \leq^\Sigma x$, such that $x' \leq^{F(\Sigma)} \alpha_\Sigma(x_0)$. But then $\alpha_\Sigma(y) \leq^{F(\Sigma)} \delta_{F(\Sigma)}(\alpha_\Sigma(x_0))$, i.e., $y \leq \alpha^*_\Sigma(\delta_{F(\Sigma)}(\alpha_\Sigma(x_0))) = \delta^\alpha_\Sigma(x_0)$, for a compact $x_0 \leq^\Sigma x$, which shows that δ^α is indeed finitary.

(2) Suppose that $\Sigma \in |\mathbf{Sign}_1|$ and $x \in \mathrm{SEN}_1^{\delta^\alpha}(\Sigma)$ is compact. By Part (1), δ^α is finitary, whence, by Lemma 39, there exists a compact $y \in \mathrm{SEN}_1(\Sigma)$, such that $x = \delta^\alpha_\Sigma(y)$. By the finitarity of $\langle F, \alpha\rangle$ and δ, $f_\Sigma(x) = f_\Sigma(\delta^\alpha_\Sigma(y)) = \delta_\Sigma(\alpha_\Sigma(y))$ is compact. Hence f is finitary.

□

A **finitary residuated category Sign** is a complete residuated category, such that

- $i_\Sigma : \Sigma \to \Sigma$ is compact, for all $\Sigma \in |\mathbf{Sign}|$;
- if $a \in \mathbf{Sign}(\Sigma, \Sigma')$ and $b \in \mathbf{Sign}(\Sigma', \Sigma'')$ are compact, then $b \circ^{\Sigma,\Sigma'}_{\Sigma',\Sigma''} a \in \mathbf{Sign}(\Sigma, \Sigma'')$ is also compact.

A **finitary module system** is a **Sign**-module system $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, such that

(i) **Sign** is a finitary residuated category;

(ii) SEN is a finitary complete lattice system;

(iii) For every compact $a \in \mathbf{Sign}(\Sigma, \Sigma')$ and every compact $v \in \mathrm{SEN}(\Sigma)$, $a \star^{\Sigma,\Sigma'} v \in \mathrm{SEN}(\Sigma')$ is also compact.

A residuated map $\langle F, \alpha\rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2$ between two finitary module systems $\mathrm{SEN}_1 : \mathbf{Sign}_1 \to \mathbf{Set}$, $\mathrm{SEN}_2 : \mathbf{Sign}_2 \to \mathbf{Set}$ is called **finitary** if it is finitary as a map between finitary complete lattice families and, in addition, for every $\Sigma, \Sigma' \in |\mathbf{Sign}_1|$ and every compact $a \in \mathbf{Sign}_1(\Sigma, \Sigma')$, $F(a)$ is compact in $\mathbf{Sign}_2(F(\Sigma), F(\Sigma'))$.

We denote by $\mathcal{FM}$ the category with objects finitary module systems and morphisms finitary module system morphisms between them.

Next we proceed to formulate an analog of Theorem 6.6 of [10]. This forms an analog of Theorem 23 for morphisms in the category $\mathcal{FM}$ rather than in $\mathcal{M}$. If we consider again the triangle (4) and the square (5) and take into account that, by Corollary 41, finitary closure systems on finitary module systems give rise to morphisms in the category $\mathcal{FM}$, the square (5) may be considered in the category $\mathcal{FM}$.

Theorem 44 *Let* $\mathbf{Sign}_1$ *be a finitary complete residuated category. The objects* $\mathrm{SEN}_1 : \mathbf{Sign}_1 \to \mathbf{Set}$ *of the category* $\mathcal{FM}$ *for which all squares of type (4) can be completed are exactly the projective objects of* $\mathcal{FM}$.

Proof:
We follow the proof of Theorem 23.

$$\begin{tikzcd}
\mathrm{SEN}_1 \arrow[rr, "h=\alpha"] \arrow[dd, "k^*k"'] \arrow[dr, "k"] & & \mathrm{SEN}_2 \arrow[dl, "g"'] \arrow[dd, "g^*g"] \\
& \mathrm{SEN}_2' & \\
\mathrm{SEN}_1^{k^*k} \arrow[ur, "k'"] \arrow[rr, "f"'] & & \mathrm{SEN}_2^{g^*g} \arrow[ul, "g'"']
\end{tikzcd}$$

If all given objects and morphisms are finitary, k^*k is finitary, as a closure operator on SEN_1, by Lemma 42, and as a module morphism $k^*k : \mathrm{SEN}_1 \to \mathrm{SEN}_1^{k^*k}$, by Corollary 41. The module system $\mathrm{SEN}_1^{k^*k}$ is finitary by Lemma 40. The module system morphism $\langle K, k\rangle : \mathrm{SEN}_1^{k^*k} \to \mathrm{SEN}_2'$ is finitary, since, for every $\Sigma \in |\mathbf{Sign}_1|$ and all compact $x \in \mathrm{SEN}_1^{k^*k}(\Sigma)$, $k_\Sigma'(x) = k_\Sigma'(k_\Sigma^* k_\Sigma(x)) = k_\Sigma(k_\Sigma^* k_\Sigma(x)) = k_\Sigma(x)$, which is compact in SEN_2'. Similarly, g^*g, g' and $\mathrm{SEN}_2^{g^*g}$ are finitary. Finally, $\langle F, f\rangle$ is finitary since it is the composition of two finitary maps. □

Corollary 45 *Suppose that* $\mathbf{Sign}$ *is a finitary complete residuated category,* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *an object in* $\mathcal{FM}$ *and* $\gamma : \mathrm{SEN} \to \mathrm{SEN}$ *a finitary closure system on* SEN. *Then* $\mathrm{SEN}^\gamma : \mathbf{Sign} \to \mathbf{Set}$ *is finitary as a* $\mathbf{Sign}$*-module system.*

Proof:
By Corollary 41, SEN^γ is finitary as a complete lattice system. So it suffices to show that the signature action preserves compactness. Let $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $a \in \mathbf{Sign}(\Sigma, \Sigma')$ compact and $\gamma_\Sigma(x) \in \mathrm{SEN}^\gamma(\Sigma)$ compact. By Lemma 39, $x \in \mathrm{SEN}(\Sigma)$ can be taken to be compact. As SEN is finitary $a \star^{\Sigma,\Sigma'} x$ is compact in $\mathrm{SEN}(\Sigma')$. Also, since γ is finitary, $\gamma_{\Sigma'}(a \star^{\Sigma,\Sigma'} x) = a \star_\gamma^{\Sigma,\Sigma'} \gamma_\Sigma(x)$ is compact in $\mathrm{SEN}^\gamma(\Sigma')$. □

By Theorem 32, the projective cyclic module systems in $\mathcal{M}$ are exactly the ones of the form $\mathbf{Sign}^u(V, -)$, with $u \in \mathbf{Sign}(V, V)$ satisfying $u \circ_{V,V}^{V,V} u = u$. Such a module system will be called **regular** if $u \in \mathbf{Sign}(V, V)$ is compact. By Lemma 28 (1), joins in SEN^v coincide with those in SEN, whence such a u is then also compact in $\mathbf{Sign}^u(V-)$.

Lemma 46 *Let* $\mathbf{Sign}$ *be a category and* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *a functor. If* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *is term, then the* $\mathbf{Sign}^{\mathcal{P}}$*-module system* $\mathcal{P}\mathrm{SEN}$ *is regular.*

Proof:
By Corollary 33, we know that $\mathcal{P}\mathrm{SEN}$ is a projective cyclic $\mathbf{Sign}^{\mathcal{P}}$-module system. Note that $u = f_{\langle V,\{v\}\rangle} = \{f_{\langle V,v\rangle}\}$ is finite in $\mathbf{Sign}(V,V)$ and, hence, compact. □

Lemma 47 *Suppose that* **Sign** *is a finitary complete residuated category. If* $V \in |\mathbf{Sign}|$ *and* $u \in \mathbf{Sign}(V,V)$ *is compact, then, for all* $\Sigma \in |\mathbf{Sign}|$*, the compact elements of* $\mathbf{Sign}^u(V,\Sigma)$ *are of the form* $a \circ^{V,V}_{V,\Sigma} u$*, for some compact* $a \in \mathbf{Sign}(V,\Sigma)$*.*

Proof:
If $a \in \mathbf{Sign}(V,\Sigma)$ is compact, then, by definition of a finitary complete residuated category, $a \circ^{V,V}_{V,\Sigma} u \in \mathbf{Sign}(V,\Sigma)$ is compact. Suppose, conversely, that $a \circ^{V,V}_{V,\Sigma} u \in \mathbf{Sign}(V,\Sigma)$ is compact. By the finitarity of **Sign**, $a = \bigvee^{V,\Sigma} C$, where C is the set of compact elements of $\mathbf{Sign}(V,\Sigma)$ lying below a. Thus, we have $a \circ^{V,V}_{V,\Sigma} u = \bigvee^{V,\Sigma}\{c \circ^{V,V}_{V,\Sigma} u : c \in C\} = \bigvee^{V,\Sigma}_u\{c \circ^{V,V}_{V,\Sigma} u : c \in C\}$ and the latter set is a directed set of compact elements of $\mathbf{Sign}^u(V,\Sigma)$, by the compactness of u. Thus, by the compactness of $a \circ^{V,V}_{V,\Sigma} u$, there exists $c \in C$, such that $a \circ^{V,V}_{V,\Sigma} u = c \circ^{V,V}_{V,\Sigma} u$. □

Corollary 48 *Let* **Sign** *be a finitary complete residuated category. Every regular* **Sign***-module system* $\mathbf{Sign}^u(V,-)$ *is finitary.*

Proof:
Let $\Sigma \in |\mathbf{Sign}|$ and $a \in \mathbf{Sign}(V,\Sigma)$. Then $a \circ^{V,V}_{V,\Sigma} u = (\bigvee^{V,\Sigma} K_{V,\Sigma}(\downarrow a)) \circ^{V,V}_{V,\Sigma} u = \bigvee^{V,\Sigma}(K_{V,\Sigma}(\downarrow a) \circ^{V,V}_{V,\Sigma} u)$. By Lemma 47, $K_{V,\Sigma}(\downarrow a) \circ^{V,V}_{V,\Sigma} u$ consists of compact elements in $\mathbf{Sign}^u(V,\Sigma)$, whence, every element of $\mathbf{Sign}^u(V,\Sigma)$ is a join of compact elements. □

Next, an analog of Lemma 6.11 of [10] is presented to the effect that cyclic objects $\mathbf{Sign}^u(V,-)$ with compact u, that are projective in $\mathcal{M}$ are also projective in $\mathcal{FM}$.

Lemma 49 *Let* **Sign** *be a finitary complete residuated category and* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *a regular* **Sign***-module system. Then* SEN *is projective in* $\mathcal{FM}$*.*

Proof:
Let $\mathbf{Sign}'$ be a finitary complete residuated category, $\mathrm{SEN}', \mathrm{SEN}'' : \mathbf{Sign}' \to \mathbf{Set}$ finitary $\mathbf{Sign}'$-module systems, $g : \mathrm{SEN}' \to \mathrm{SEN}''$ a finitary surjective module system morphism and $\langle K,k\rangle : \mathrm{SEN} \to \mathrm{SEN}''$ a finitary module system morphism. We must find a finitary module system morphism $\langle K,h\rangle : \mathrm{SEN} \to \mathrm{SEN}'$, such that $g \circ \langle K,h\rangle = \langle K,k\rangle$.

$$\begin{tikzcd} \mathrm{SEN} \arrow[r, "h"] \arrow[dr, "k"'] & \mathrm{SEN}' \arrow[d, "g"] \\ & \mathrm{SEN}'' \end{tikzcd}$$

Taking into account Theorem 32 and the definition of a regular module system, assume that $\mathrm{SEN} = \mathbf{Sign}^u(V,-)$, with $u \in \mathbf{Sign}(V,V)$ compact, such that $u \circ^{V,V}_{V,V} u = u$. Let $y = k_V(u) \in \mathrm{SEN}''(K(V))$. By the compactness of $\langle K,k\rangle$, y is compact in $\mathrm{SEN}''(K(V))$. By the surjectivity of g, there exists $x \in \mathrm{SEN}'(K(V))$, such that $y = g_{K(V)}(x)$. Since SEN' is finitary, $x = \bigvee^{K(V)} X$, for some set X of compact elements of $\mathrm{SEN}'(K(V))$. This implies that $y = g_{K(V)}(x) = \bigvee^{K(V)} g_{K(V)}(X)$. Since y is compact, there exists finite $Y \subseteq X$, such that $y = g_{K(V)}(x) = \bigvee^{K(V)} g_{K(V)}(Y)$. Setting $w = \bigvee^{K(V)} Y$, which is compact in $\mathrm{SEN}'(K(V))$, we get that $y = g_{K(V)}(w)$. Let $z = K(u) \star^{K(V),K(V)} w$, which is compact in $\mathrm{SEN}'(K(V))$. Define $\langle K,\tau^z\rangle : \mathbf{Sign}^u(V,-) \to \mathrm{SEN}'$, by setting, for all $\Sigma \in |\mathbf{Sign}|$ and all $a \in \mathbf{Sign}(V,\Sigma)$,

$$\tau^z_\Sigma(a \circ^{V,V}_{V,\Sigma} u) = K(a) \star^{K(V),K(\Sigma)} z.$$

We show that $\langle K,\tau^z\rangle$ is a finitary module system morphism, such that $g \circ \langle K,\tau^z\rangle = \langle K,k\rangle$.

- τ^z is well-defined: Assume that, for $\Sigma \in |\mathbf{Sign}|$ and $a,b \in \mathbf{Sign}(V,\Sigma)$, $a \circ^{V,V}_{V,\Sigma} u = b \circ^{V,V}_{V,\Sigma} u$. Then

$$\begin{array}{rcl}
K(a) \star^{K(V),K(\Sigma)} z & = & K(a) \star^{K(V),K(\Sigma)} (K(u) \star^{K(V),K(V)} w)\\
& = & (K(a) \circ^{K(V),K(V)}_{K(V),K(\Sigma)} K(u)) \star^{K(V),K(\Sigma)} w\\
& = & K(a \circ^{V,V}_{V,\Sigma} u) \star^{K(V),K(\Sigma)} w\\
& = & K(b \circ^{V,V}_{V,\Sigma} u) \star^{K(V),K(\Sigma)} w\\
& = & (K(b) \circ^{K(V),K(V)}_{K(V),K(\Sigma)} K(u)) \star^{K(V),K(\Sigma)} w\\
& = & K(b) \star^{K(V),K(\Sigma)} (K(u) \star^{K(V),K(V)} w)\\
& = & K(b) \star^{K(V),K(\Sigma)} z.
\end{array}$$

- τ^z is residuated: Let $\Sigma \in |\mathbf{Sign}|$, $a \in \mathbf{Sign}(V,\Sigma)$ and $x \in \mathrm{SEN}'(K(\Sigma))$. We have

$$\begin{array}{ll}
\multicolumn{2}{l}{\tau^z_\Sigma(a \circ^{V,V}_{V,\Sigma} u) \leq^{K(\Sigma)} x}\\
\text{implies} & K(a) \star^{K(V),K(\Sigma)} z \leq^{K(\Sigma)} x\\
\text{implies} & K(a) \leq^{K(V),K(\Sigma)} x/^{K(V),K(\Sigma)} z\\
\text{implies} & K(a) \star^{K(V),K(\Sigma)} k_V(u) \leq^{K(\Sigma)}\\
& \qquad\qquad (x/^{K(V),K(\Sigma)} z) \star^{K(V),K(\Sigma)} k_V(u)\\
\text{implies} & k_\Sigma(a \circ^{V,V}_{V,\Sigma} u) \leq^{K(\Sigma)} (x/^{K(V),K(\Sigma)} z) \star^{K(V),K(\Sigma)} k_V(u)\\
\text{implies} & a \circ^{V,V}_{V,\Sigma} u \leq^{\Sigma} k^*_\Sigma((x/^{K(V),K(\Sigma)} z) \star^{K(V),K(\Sigma)} k_V(u)).
\end{array}$$

  This proves that τ^z is residuated with τ^{z^*} defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $x \in \mathrm{SEN}'(\Sigma)$, by

$$\tau^{z^*}_\Sigma(x) = k^*_\Sigma((x/^{K(V),K(\Sigma)} z) \star^{K(V),K(\Sigma)} k_V(u)).$$

- τ^z is a **Sign**-module system morphism: Let $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $a \in \mathbf{Sign}(V, \Sigma)$ and $b \in \mathbf{Sign}(\Sigma, \Sigma')$. Then

$$\begin{aligned} K(b) \star^{K(\Sigma),K(\Sigma')} \tau^z_\Sigma(a \circ^{V,V}_{V,\Sigma} u) \\ &= K(b) \star^{K(\Sigma),K(\Sigma')} (K(a) \star^{K(V),K(\Sigma)} z) \\ &= (K(b) \circ^{V,\Sigma}_{\Sigma,\Sigma'} K(a)) \star^{V,\Sigma} z \\ &= K(b \circ^{V,\Sigma}_{\Sigma,\Sigma'} a) \star^{V,\Sigma} z \\ &= \tau^z_{\Sigma'}((b \circ^{V,\Sigma}_{\Sigma,\Sigma'} a) \circ^{V,V}_{V,\Sigma} u) \\ &= \tau^z_{\Sigma'}(b \circ^{V,\Sigma}_{\Sigma,\Sigma'} (a \circ^{V,V}_{V,\Sigma} u)). \end{aligned}$$

- τ^z is finitary: By Lemma 47, given $\Sigma \in |\mathbf{Sign}|$ and $a \in \mathbf{Sign}(V, \Sigma)$, such that $a \circ^{V,V}_{V,\Sigma} u$ is compact in $\mathbf{Sign}^u(V, \Sigma)$, there exists a compact $c \in \mathbf{Sign}(V, \Sigma)$, such that $a \circ^{V,V}_{V,\Sigma} u = c \circ^{V,V}_{V,\Sigma} u$. Thus, if $a \circ^{V,V}_{V,\Sigma} u \in \mathbf{Sign}^u(V, \Sigma)$ is compact, we may assume without loss of generality that $a \in \mathbf{Sign}(V, \Sigma)$ is compact. Then $\tau^z_\Sigma(a \circ^{V,V}_{V,\Sigma} u) = K(a) \star^{K(V),K(\Sigma)} z$ is compact in $\mathrm{SEN}'(\Sigma)$, because z is compact in $\mathrm{SEN}'(V)$, a is compact in $\mathbf{Sign}(V, \Sigma)$, $\langle K, k\rangle$ is a finitary module system morphism and SEN' is a finitary module system. Therefore τ^z is, indeed, finitary.

$\square$

Lemma 49 together with Lemma 46 have the following consequence:

Corollary 50 *Let* $\mathbf{Sign}_1, \mathbf{Sign}_2$ *be categories and* $\mathrm{SEN}_1 : \mathbf{Sign}_1 \to \mathbf{Set}$ *and* $\mathrm{SEN}_2 : \mathbf{Sign}_2 \to \mathbf{Set}$ *two term sentence functors. Then, every finitary structural representation between finitary consequence systems on the* $\mathbf{Sign}_1^{\mathcal{P}}$*-module system* $\mathcal{P}\mathrm{SEN}_1$ *and the* $\mathbf{Sign}_2^{\mathcal{P}}$*-module system* $\mathcal{P}\mathrm{SEN}_2$ *is induced by a finitary module system morphism.*

The next lemma is an analog of Lemma 6.14 of [10] and may be proved by applying the same proof signature-wise. Its proof is therefore omitted.

Lemma 51 *Let* **Sign** *be a category and* $\mathrm{SEN}^i : \mathbf{Sign} \to \mathbf{Set}$ *be finitary lattice families, for all* $i \in I$. *Consider the product* $\prod_{i \in I} \mathrm{SEN}^i$. *For all* $\Sigma \in |\mathbf{Sign}|$, $\langle x_i : i \in I\rangle \in \prod_{i\in I} \mathrm{SEN}^i(\Sigma)$ *is compact iff, there exists finite* $J \subseteq I$, *such that* $x_i = 0^i_\Sigma$, *for all* $i \in I \backslash J$ *and* x_j *is compact in* $\mathrm{SEN}^j(\Sigma)$, *for all* $j \in J$.

Theorem 52 *The coproduct in* $\mathcal{M}$ *of a family of regular module systems is projective in* $\mathcal{F}M$.

Proof:
Let **Sign** be a finitary complete residuated category and $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ the coproduct of a family of regular **Sign**-module systems $\mathrm{SEN}^i : \mathbf{Sign} \to \mathbf{Set}$, $i \in I$. Let, also, $\mathbf{Sign}'$ be a complete residuated category, $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ a module system, $\gamma : \mathrm{SEN} \to \mathrm{SEN}$ a closure system on SEN, $\delta : \mathrm{SEN}' \to$

SEN' a finitary closure system on SEN' and $\langle F, f\rangle : \mathrm{SEN}^\gamma \to \mathrm{SEN}'^\delta$ a finitary representation of γ in δ. It will be shown that $\langle F, f\rangle$ is induced by a finitary module system morphism $\langle F, \alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$

Assume that $\sigma^i : \mathrm{SEN}^i \to \mathrm{SEN}$ are the canonical **Sign**-module system injections associated with the coproduct $\mathrm{SEN} = \coprod_{i\in I} \mathrm{SEN}^i$.

$$\begin{tikzcd}
\mathrm{SEN}^i \arrow[d, "\sigma^i"'] \arrow[dr, "\alpha^i"] & \\
\mathrm{SEN} \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & \mathrm{SEN}' \arrow[d, "\delta"] \\
\mathrm{SEN}^\gamma \arrow[r, "f"'] & \mathrm{SEN}'^\delta
\end{tikzcd}$$

The morphism $\langle F, f\rangle \circ \gamma\sigma^i : \mathrm{SEN}^i \to \mathrm{SEN}'$ is finitary, whence, there exists, by Lemma 49, a finitary $\langle F, \alpha^i\rangle : \mathrm{SEN}^i \to \mathrm{SEN}'$, such that $\langle F, f\rangle \circ \gamma\sigma^i = \delta \circ \langle F, \alpha^i\rangle$. By the universal property of the coproduct, there exists $\langle F, \alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$, such that $\langle F, \alpha\rangle \circ \sigma^i = \langle F, \alpha^i\rangle$.

It suffices to show that $\langle F, \alpha\rangle$ is finitary. To this end, let $\Sigma \in |\mathbf{Sign}|$, $x = \langle x_i : i \in I\rangle \in \mathrm{SEN}(\Sigma)$ compact. By Lemma 51, there exists finite $J \subseteq I$, such that $x_i = 0^i_\Sigma$, for all $i \notin J$ and x_j compact in SEN^j, for all $j \in J$. By the compactness of $\langle F, \alpha^i\rangle$, we get that $\alpha^j_\Sigma(x_j)$ is compact in $\mathrm{SEN}'(\Sigma)$, for all $j \in J$, and, also, $\alpha^i_\Sigma(x_i) = \alpha^i_\Sigma(0^i_\Sigma) = 0'_\Sigma$, for all $i \notin J$. Thus, by Lemma 35, $\alpha_\Sigma(\langle x_i : i \in I\rangle) = \bigvee^\Sigma_{i\in I} \alpha^i_\Sigma(x_i) = \bigvee^\Sigma_{j\in J} \alpha^j_\Sigma(x_j)$, which is compact as a finite join of compact elements. □

Corollary 53 *Let* $\mathbf{Sign}_1$ *and* $\mathbf{Sign}_2$ *be finitary complete residuated categories,* $\mathrm{SEN}_1 : \mathbf{Sign}_1 \to \mathbf{Set}$, $\mathrm{SEN}_2 : \mathbf{Sign}_2 \to \mathbf{Set}$ *coproducts of regular* $\mathbf{Sign}_1$*- and* $\mathbf{Sign}_2$*-module systems, respectively, and* $\gamma : \mathrm{SEN}_1 \to \mathrm{SEN}_1$ *and* $\delta : \mathrm{SEN}_2 \to \mathrm{SEN}_2$ *finitary closure systems on* SEN_1 *and* SEN_2*, respectively. Then, every equivalence between* γ *and* δ *consisting of the module system morphisms* $\langle F, f\rangle : \mathrm{SEN}_1^\gamma \to \mathrm{SEN}_2^\delta$ *and* $\langle G, g\rangle : \mathrm{SEN}_2^\delta \to \mathrm{SEN}_1^\gamma$ *and the adjoint equivalence* $\langle F, G, \eta, \epsilon\rangle : \mathbf{Sign}_1 \to \mathbf{Sign}_2$ *is induced by finitary module system morphisms* $\langle F, \alpha\rangle : \mathrm{SEN}_1 \to \mathrm{SEN}_2$ *and* $\langle G, \beta\rangle : \mathrm{SEN}_2 \to \mathrm{SEN}_1$ *and the adjoint equivalence* $\langle F, G, \eta, \epsilon\rangle$.

Taking into account Theorem 38, Corollary 46 and Theorem 52, we also obtain

Corollary 54 *Let* $\mathbf{Sign}_1, \mathbf{Sign}_2$ *be categories and* $\mathrm{SEN}_1 : \mathbf{Sign}_1 \to \mathbf{Set}$, $\mathrm{SEN}_2 : \mathbf{Sign}_2 \to \mathbf{Set}$ *multi-term sentence functors. Then, every finitary structural representation between consequence systems on the* $\mathbf{Sign}_1^{\mathcal{P}}$*-module system* $\mathcal{P}\mathrm{SEN}_1$ *and the* $\mathbf{Sign}_2^{\mathcal{P}}$*-module system* $\mathcal{P}\mathrm{SEN}_2$ *is induced by a finitary module system morphism.*

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