Categorical Abstract Algebraic Logic: Bloom's Theorem for Rule-Based π-Institutions

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Abstract

A syntactic machinery is developed for π-institutions based on the notion of a category of natural transformations on their sentence functors. Rules of inference, similar to the ones traditionally used in the sentential logic framework to define the best known sentential logics, are, then, introduced for π-institutions. A π-institution is said to be rule-based if its closure system is induced by a collection of rules of inference. A logical matrix-like semantics is introduced for rule-based π-institutions and a version of Bloom's Lemma and Bloom's Theorem are proved for rule-based π-institutions.

Keywords: Closure Operators, Deductive Systems, Logical Matrices, Universal Horn Logic Without Equality, Bloom's Theorem, π-Institutions, Rules of Inference, Filtered Products, Ultraproducts

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1 Introduction

In his work in the 1930's (see, e.g., [27]) Tarski established the notion of a deductive system as a finite and structural closure operator on the collection $\mathrm{Fm}_{\mathcal{L}}(V)$ of $\mathcal{L}$-formulas over a denumerable set of variables V. In a more general form, used extensively in modern studies in abstract algebraic logic, a deductive system is a structural, but not necessarily finitary, consequence operation on $\mathrm{Fm}_{\mathcal{L}}(V)$. In the most commonly used form, this structural consequence operation on the set of formulas of a given sentential language is induced by a collection of axioms and rules of inference that determine the logic under study. In the 1950's, Łoś [23] developed the method of ultraproducts, which has become a fundamental tool in the study of the model theory of first-order logic. The general references [9, 22, 13] in model theory include a detailed treatment and many applications of this method, and the reader is advised to consult them for additional background and bibliographic information concerning the method. In 1975, Bloom [7] pioneered the idea of recasting the axioms and the rules of inference, used to define a sentential logic, in the form of universal strict basic Horn sentences over a specially built first-order language, having as individual variables the variables in V, as function symbols the symbols in the sentential language $\mathcal{L}$ and, finally, one unary relation symbol D, intended to denote truth. In this translation, a given rule of inference

$$\frac{\phi_0, \ldots, \phi_{n-1}}{\phi}$$

with $\phi_0, \ldots, \phi_{n-1}, \phi \in \mathrm{Fm}_{\mathcal{L}}(V)$, is recast as the sentence

$$(\forall \cdots \forall)(D(\phi_0) \wedge \ldots \wedge D(\phi_{n-1}) \Rightarrow D(\phi)),$$

where $(\forall \cdots \forall)$ denotes the universal closure with respect to all individual variables appearing in any of $\phi_0, \ldots, \phi_{n-1}, \phi$. By performing this translation, sentential logics may be studied inside the framework of universal Horn logic without equality. As a consequence, the powerful methods of the model theory of first-order logic, such as the method of ultraproducts, may be used to derive interesting results pertaining to sentential logics.

In his treatise on protoalgebraic logics [11], Czelakowski revisits Bloom's idea and presents some important results obtained by applying this method to deductive systems. Bloom's Lemma (Lemma 0.4.1 of [11]) establishes, roughly speaking, that the class of all matrix models, in the usual matrix semantics sense, of a given sentential logic, defined via standard rules of inference, coincides with the class of all first-order models of the translates of these rules. Bloom's Theorem (Theorem 0.4.2 of [11]) characterizes those structural consequence operations that are induced by a collection of standard rules of inference. They are exactly those structural consequence operations that are finitary, or, equivalently, whose model class is axiomatizable by sentences in the corresponding first-order language, or, what also amounts to the same thing, whose model class is closed under ultraproducts.

The interest of [11] on Bloom's results stems from the fact that sentential logics form the basic structure on which the theory of abstract algebraic logic is developed. The central notions of the theory, that of a protoalgebraic logic [4], of an equivalential logic [25, 10] and of an algebraizable logic [5, 6, 19, 20, 21] all refer to classes of sentential logics.

The author has recently further abstracted the theory of abstract algebraic logic [28]-[37] to cover, apart from sentential logics, also logics that are formalized as π-institutions [14]. In the present work, based on the notion of a category of natural transformations, that has proven key to the development of the theory of categorical abstract algebraic logic, a syntax is developed for π-institutions. Based on this syntax, the concept of a rule of inference, paralleling that for sentential logics, is formulated. If the closure system of a π-institution is induced by a collection of rules of inference, then the π-institution is said to be rule-based. For rule-based π-institutions, a translation of their rules into a first-order language, inspired by Bloom's translation, is performed. A major difference in the treatment here, is that the models of the resulting first-order sentences are not the usual first-order models but, rather, matrix systems, a variant of the usual logical matrices. Using this framework, a result abstracting Bloom's Lemma in the categorical framework is obtained. Further, using a reduced product construction on sentence functors and π-institutions, introduced in [38] and [39, 40], a theorem abstracting Bloom's Theorem for the framework of rule-based π-institutions is also established.

It should be mentioned, in closing, that in the framework of abstract model theory, part of which is based on the concept of an institution [17, 18], there have been several suggestions on possible ways of introducing a syntax that abstracts the syntax of first-order logic, of formulating a notion of ultraproduct and, then, studying the interaction between syntax and semantics in the context of ultraproduct constructions. A few pointers to that literature are the works of Andréka and Németi [1, 2], and the paper by Diaconescu [12]. The paper by Tarlecki [26], although not providing specific information about ultraproducts, is a very useful early reference in the theory of institutions. Diaconescu's paper is the closest in spirit to our foundations, but differs significantly in two aspects: First, models that are available in the institution framework, but not in the π-institution framework, are used to define the syntax and, second, the focus is on "internal" ultraproducts in the given institution, rather than ultraproducts of several related institutions. To the best of the author's knowledge, both consideration of sentences in π-institutions and construction of "external" ultraproducts are two novel ideas in the theory of categorical abstract algebraic logic.

For all unexplained categorical terminology and notation the reader is referred to any of [3, 8, 24]. For the definitions pertaining to institutions see [17, 18], whereas π-institutions were introduced in [14]. For background on the theory of abstract algebraic logic and discussion of the classes of the abstract algebraic hierarchy, some of which were mentioned in this introduction, the reader is referred to the review article [16], the monograph [15] and the comprehensive treatise [11].

2 A Few Preliminaries

Before embarking on the main developments, we review briefly the concept of a π-institution, that of a category of natural transformations on a set-valued functor as well as the notions of an epimorphic translation between two set-valued functors and of a logical morphism between two π-institutions. This will, hopefully, facilitate reading the paper by making it more self-contained. Original references to these notions will be interjected, as needed, so as to enable the reader to find out more details and results pertaining to the underlying framework.

Recall from [14] that a **π-institution** $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|} \rangle$, sometimes abbreviated as $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, is a triple consisting of

(i) a category **Sign**, whose objects are called **signatures** and whose morphisms are called **assignments**,

(ii) a functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ from the category of signatures to the category of small sets, giving, for each $\Sigma \in |\mathbf{Sign}|$, the set of **$\Sigma$-sentences** $\mathrm{SEN}(\Sigma)$ and mapping an assignment $f : \Sigma_1 \to \Sigma_2$ to a **substitution** $\mathrm{SEN}(f) : \mathrm{SEN}(\Sigma_1) \to \mathrm{SEN}(\Sigma_2)$,

(iii) a mapping $C_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \to \mathcal{P}(\mathrm{SEN}(\Sigma))$, for each $\Sigma \in |\mathbf{Sign}|$, called **$\Sigma$-closure,** such that
   (a) $A \subseteq C_\Sigma(A)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq \mathrm{SEN}(\Sigma)$,
   (b) $C_\Sigma(C_\Sigma(A)) = C_\Sigma(A)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq \mathrm{SEN}(\Sigma)$,
   (c) $C_\Sigma(A) \subseteq C_\Sigma(B)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq B \subseteq \mathrm{SEN}(\Sigma)$,
   (d) $\mathrm{SEN}(f)(C_{\Sigma_1}(A)) \subseteq C_{\Sigma_2}(\mathrm{SEN}(f)(A))$, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, $A \subseteq \mathrm{SEN}(\Sigma_1)$.

Given a set-valued functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, as above, recall from, e.g., [34], that the **clone of all natural transformations on** SEN is defined to be the locally small category with collection of objects $\{\mathrm{SEN}^\alpha : \alpha \text{ an ordinal}\}$ and collection of morphisms $\tau : \mathrm{SEN}^\alpha \to \mathrm{SEN}^\beta$ β-sequences of natural transformations $\tau_i : \mathrm{SEN}^\alpha \to \mathrm{SEN}$. Composition

$$\mathrm{SEN}^\alpha \xrightarrow{\langle\tau_i : i<\beta\rangle} \mathrm{SEN}^\beta \xrightarrow{\langle\sigma_j : j<\gamma\rangle} \mathrm{SEN}^\gamma$$

is defined by

$$\langle\sigma_j : j<\gamma\rangle \circ \langle\tau_i : i<\beta\rangle = \langle\sigma_j(\langle\tau_i : i<\beta\rangle) : j<\gamma\rangle.$$

A subcategory N of this category containing *all* objects of the form SEN^k for $k < \omega$, and all projection morphisms $p^{k,i} : \mathrm{SEN}^k \to \mathrm{SEN}, i < k, k < \omega$, with $p^{k,i}_\Sigma : \mathrm{SEN}(\Sigma)^k \to \mathrm{SEN}(\Sigma)$ given by

$$p^{k,i}_\Sigma(\vec{\phi}) = \phi_i, \quad \text{for all} \quad \vec{\phi} \in \mathrm{SEN}(\Sigma)^k,$$

and such that, for every family $\{\tau_i : \mathrm{SEN}^k \to \mathrm{SEN} : i < l\}$ of natural transformations in N, the sequence $\langle\tau_i : i < l\rangle : \mathrm{SEN}^k \to \mathrm{SEN}^l$ is also in N, is referred to as a **category of natural transformations on** SEN.

Given two set-valued functors $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ and $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ a **translation** $\langle F, \alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$ consists of a functor $F : \mathbf{Sign} \to \mathbf{Sign}'$ and a natural transformation $\alpha : \mathrm{SEN} \to \mathcal{P}\mathrm{SEN}' \circ F$, where by $\mathcal{P}$ is denoted the powerset functor. A translation is a **singleton translation**, denoted $\langle F, \alpha\rangle : \mathrm{SEN} \to^s \mathrm{SEN}'$ if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $|\alpha_\Sigma(\phi)| = 1$. In this case $\alpha_\Sigma(\phi)$ will be identified with the single element that it contains. Given two functors $\mathrm{SEN}, \mathrm{SEN}'$, as before, and categories of natural transformations N, N' on $\mathrm{SEN}, \mathrm{SEN}'$, respectively, a singleton translation $\langle F, \alpha\rangle$ from SEN to SEN' is said to be an (N, N')-**epimorphic translation**, denoted $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$ if there exists a correspondence $\sigma \mapsto \sigma'$ between the natural transformations in N and those in N', that preserves projections, such that, for every n-ary natural transformation σ in N, every $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^n$,

$$\begin{tikzcd}
\mathrm{SEN}(\Sigma)^n \arrow[r, "\sigma_\Sigma"] \arrow[d, "\alpha^n_\Sigma"'] & \mathrm{SEN}(\Sigma) \arrow[d, "\alpha_\Sigma"] \\
\mathrm{SEN}'(F(\Sigma))^n \arrow[r, "\sigma'_{F(\Sigma)}"'] & \mathrm{SEN}'(F(\Sigma))
\end{tikzcd}$$

$$\alpha_\Sigma(\sigma_\Sigma(\vec{\phi})) = \sigma'_{F(\Sigma)}(\alpha^n_\Sigma(\vec{\phi})).$$

Finally, recall from [29] that, given two π-institutions $\mathcal{I} = \langle\mathbf{Sign}, \mathrm{SEN}, C\rangle$ and $\mathcal{I}' = \langle\mathbf{Sign}', \mathrm{SEN}', C'\rangle$, with categories of natural transformations N, N' on $\mathrm{SEN}, \mathrm{SEN}'$, respectively, an (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$ is said to be an (N, N')-**logical morphism from** $\mathcal{I}$ **to** $\mathcal{I}'$, denoted $\langle F, \alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$, if, for every $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\phi \in C_\Sigma(\Phi) \quad \text{implies} \quad \alpha_\Sigma(\phi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

This is equivalent to saying that, for every $\Sigma \in |\mathbf{Sign}|$ and every $\Phi \subseteq \mathrm{SEN}(\Sigma)$, $\alpha_\Sigma(C_\Sigma(\Phi)) \subseteq C'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$.

As far as set-theoretic notation goes, by ω will be denoted the first infinite cardinal and the corresponding ordinal. Moreover, given a cardinal κ, by κ^+ will be denoted the least cardinal greater than κ. Therefore, given a set X, by $|X|^+$ will be denoted the least cardinal that exceeds the cardinality of the set X.

3 Syntax, Rules of Inference and Proofs

In this section a fixed but arbitrary functor SEN : **Sign** $\rightarrow$ **Set** is considered and N is a fixed but arbitrary category of natural transformations on SEN. The aim is to develop a syntactical framework in which to be able to study the deductive mechanism of a π-institution based on SEN in a way similar to the one used for deductive systems represented by axioms and rules of inference.

The collection $\mathrm{Te}^N(\mathrm{SEN})$ of N-**formulas** over SEN or, in accordance with the theory of sentential logics, N-**terms** over SEN is defined, in the present context, to be the collection of all natural transformations in N of the form $\sigma : \mathrm{SEN}^k \rightarrow \mathrm{SEN}$, for some $k \in \omega$.

By an N-**rule of inference**, or, simply, an N-**rule**, of SEN it is understood a member r of the cartesian product $\mathcal{P}(\mathrm{Te}^N(\mathrm{SEN})) \times \mathrm{Te}^N(\mathrm{SEN})$. Such a rule is denoted by $r = \langle X, \sigma\rangle$, where $X \subseteq \mathrm{Te}^N(\mathrm{SEN})$ and $\sigma \in \mathrm{Te}^N(\mathrm{SEN})$. The **length** of the N-rule $r = \langle X, \sigma\rangle$ is the cardinal number $|r| = |X|^+$. The N-rule r is **axiomatic** if its length is equal to 1. Otherwise, it is said to be a **proper** N-rule of inference. This means that, if an N-rule $r = \langle X, \sigma\rangle$ is axiomatic, then $X = \emptyset$. Finally, the N-rule $r = \langle X, \sigma\rangle$ is **finitary** if $|r| < \omega$.

An N-rule $r = \langle X, \sigma\rangle$ will be usually given, using a variant of a well-known notation from sentential logics, in the form X/σ or $\frac{X}{\sigma}$. An axiomatic N-rule $r = \langle \emptyset, \sigma\rangle$ has a representation $/\sigma$ or $\frac{}{\sigma}$ or, sometimes, simply σ.

As an illustration, if SEN is the sentence functor of the π-institution representing propositional logic, with N the category of natural transformations representing the whole clone of propositional operations, then the rule of Modus Ponens may be expressed by $\{p^{2,0}, \rightarrow\}/p^{2,1}$, where $p^{2,0}$ and $p^{2,1}$ denote the first and second projections, respectively, in 2 arguments and $\rightarrow$ denotes the implication operation, and if $\mathcal{I}$ is a π-institution representing some modal logic, with N the category of natural transformations representing the whole clone of operations, then the rule of Necessitation may be expressed by $\{\iota\}/\Box$, where, by analogy, $\iota = p^{1,0}$ denotes the identity natural transformation in N.

An axiom family $T = \{T_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ on SEN, i.e., a collection such that $T_\Sigma \subseteq \mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$, is said to be **closed under the** N-**rule** $r = \langle X, \sigma\rangle$ if, for all $\Sigma \in |\mathbf{Sign}|$, and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^\omega$, if $X_\Sigma(\vec{\phi}) \subseteq T_\Sigma$, then $\sigma_\Sigma(\vec{\phi}) \in T_\Sigma$. If T is closed under the rule r, then, we also say that r **preserves** the axiom family T. An N-rule $r = \langle X, \sigma\rangle$ of SEN is an N-**rule of the** π-**institution** $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ or **of the closure system** C on SEN if $\sigma_\Sigma(\vec{\phi}) \subseteq C_\Sigma(X_\Sigma(\vec{\phi}))$, for all $\Sigma \in |\mathbf{Sign}|, \vec{\phi} \in \mathrm{SEN}(\Sigma)^\omega$. If this is the case, then r is said to be **sound for** $\mathcal{I}$ or **for** C.

PROPOSITION 3.1
Let SEN : **Sign** $\rightarrow$ **Set** be a functor, with N a category of natural transformations on SEN. An N-rule $r = \langle X, \sigma\rangle$ of SEN is a rule of a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ iff it preserves all theory families of $\mathcal{I}$.

Proof:
Suppose that $r = \langle X, \sigma\rangle$ is a rule of $\mathcal{I}$, i.e., for all $\Sigma \in |\mathbf{Sign}|, \vec{\phi} \in \mathrm{SEN}(\Sigma)^\omega$,

$$\sigma_\Sigma(\vec{\phi}) \subseteq C_\Sigma(X_\Sigma(\vec{\phi})).$$

Then, if $T \in \mathrm{ThFam}(\mathcal{I})$, we have, for all $\Sigma \in |\mathbf{Sign}|, \vec{\phi} \in \mathrm{SEN}(\Sigma)^\omega$, such that $X_\Sigma(\vec{\phi}) \subseteq$

T_Σ, $\sigma_\Sigma(\vec{\phi}) \in C_\Sigma(X_\Sigma(\vec{\phi})) \subseteq C_\Sigma(T_\Sigma) = T_\Sigma$. Thus r preserves T.

Suppose, conversely, that $r = \langle X, \sigma\rangle$ preserves T, for all $T \in \mathrm{ThFam}(\mathcal{I})$. Then we have, for all $\Sigma \in |\mathbf{Sign}|, \vec{\phi} \in \mathrm{SEN}(\Sigma)^\omega$,

$$\begin{array}{rcl} C_\Sigma(X_\Sigma(\vec{\phi})) & = & \bigcap\{T_\Sigma : T \in \mathrm{ThFam}(\mathcal{I}) \text{ and } X_\Sigma(\vec{\phi}) \subseteq T_\Sigma\} \\ & \supseteq & \bigcap\{T_\Sigma : T \in \mathrm{ThFam}(\mathcal{I}) \text{ and } \sigma_\Sigma(\vec{\phi}) \in T_\Sigma\} \\ & = & C_\Sigma(\sigma_\Sigma(\vec{\phi})) \\ & \ni & \sigma_\Sigma(\vec{\phi}). \end{array}$$

■

For a given set R of N-rules of SEN, there exists a smallest closure system C on SEN, such that every N-rule in R is a rule of C.

Proposition 3.2
Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a functor, N a category of natural transformations on SEN and R a collection of N-rules of SEN. Then there exists a smallest closure system C on SEN, such that, for every $r \in R$, r is an N-rule of C.

Proof:

It is clear that, for every $r \in R$, r is an N-rule of the largest closure system on SEN. So it suffices to show that any given collection $\{C^i : i \in I\}$ of closure systems $C^i, i \in I$, on SEN, such that r is an N-rule of C^i, for all $r \in R$ and all $i \in I$, is closed under intersections. This is, however, easy to see. Since, for all $r = \langle X, \sigma\rangle, \Sigma \in |\mathbf{Sign}|, \vec{\phi} \in \mathrm{SEN}(\Sigma)^\omega$, we have $\sigma_\Sigma(\vec{\phi}) \in C^i_\Sigma(X_\Sigma(\vec{\phi}))$, for all $i \in I$, we obtain $\sigma_\Sigma(\vec{\phi}) \in \bigcap_{i\in I} C^i_\Sigma(X_\Sigma(\vec{\phi})) = (\bigcap_{i\in I} C^i)_\Sigma(X_\Sigma(\vec{\phi}))$, and, therefore r is also an N-rule of $\bigcap_{i\in I} C^i$. ■

The closure system of Proposition 3.2 is denoted by C^R. Based on this notion, the key concept of a rule-based π-institution is now introduced. It abstracts a property possessed by default by all sentential logics.

Given a functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, with N a category of natural transformations on SEN, and an N-rule $r = \langle X, \sigma\rangle$ of SEN, a Σ-sentence ϕ is said to **follow from** a set of Σ-sentences Φ **by an application of** the rule r, if, for some $\vec{\psi} \in \mathrm{SEN}(\Sigma)^\omega$, $X_\Sigma(\vec{\psi}) \subseteq \Phi$ and $\sigma_\Sigma(\vec{\psi}) = \phi$.

Definition 3.3
Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a functor and N a category of natural transformations on SEN. A closure system C on SEN and the corresponding π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ are said to be N-**rule-based** if, for all $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$, there exists an N-rule $\langle X, \sigma\rangle$ of C of length at most $|\Phi|^+$, and $\vec{\psi} \in \mathrm{SEN}(\Sigma)^\omega$, such that $X_\Sigma(\vec{\psi}) \subseteq \Phi$ and $\sigma_\Sigma(\vec{\psi}) = \phi$, i.e., such that ϕ follows from Φ by an application of $\langle X, \sigma\rangle$.

Let R be a set of N-rules, $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$. An R-**proof of** ϕ **from** Φ is a finite sequence $\phi_0, \ldots, \phi_n$ of Σ-sentences, such that

- for all $i = 0, \ldots, n$, $\phi_i \in \Phi$ or ϕ_i follows from previous sentences in the sequence by an application of a rule of R and
- $\phi_n = \phi$.

PROPOSITION 3.4
Let R be a set of finitary N-rules. Then, for all $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\phi \in C^R_\Sigma(\Phi) \quad \text{iff} \quad \text{there exists an } R\text{-proof of } \phi \text{ from } \Phi.$$

Proof:
Suppose, first, that $\phi_0, \phi_1, \ldots, \phi_n = \phi$ is an R-proof of ϕ from Φ. It will be shown by induction on $k = 0, 1, \ldots, n$ that $\phi_i \in C^R_\Sigma(\Phi)$.

For $k = 0$, either $\phi_0 \in \Phi$, whence $\phi_0 \in \Phi \subseteq C^R_\Sigma(\Phi)$, or there exists an axiomatic rule $/\sigma$, and $\vec{\psi} \subseteq \mathrm{SEN}(\Sigma)^\omega$, such that $\sigma_\Sigma(\vec{\psi}) = \phi_0$. But then $\phi_0 = \sigma_\Sigma(\vec{\psi}) \in C^R_\Sigma(\emptyset) \subseteq C^R_\Sigma(\Phi)$.

Suppose, now, as the induction hypothesis, that $\phi_i \in C^R_\Sigma(\Phi)$, for all $i < k$. Then either $\phi_k \in \Phi$, in which case $\phi_k \in \Phi \subseteq C^R_\Sigma(\Phi)$, or ϕ_k follows from $\phi_0, \ldots, \phi_{k-1}$ by an application of an R-rule $r = \langle X, \sigma \rangle$. Thus, there exists $\vec{\psi} \in \mathrm{SEN}(\Sigma)^\omega$, such that $X_\Sigma(\vec{\psi}) \subseteq \{\phi_0, \ldots, \phi_{k-1}\}$ and $\sigma_\Sigma(\vec{\psi}) = \phi_k$. Hence, we obtain

$$\begin{array}{lll}
\phi_k & = & \sigma_\Sigma(\vec{\psi}) \\
& \in & C^R_\Sigma(X_\Sigma(\vec{\psi})) \quad (\text{since } \langle X, \sigma \rangle \text{ is an } R\text{-rule}) \\
& \subseteq & C^R_\Sigma(\{\phi_0, \ldots, \phi_{k-1}\}) \quad (\text{since } X_\Sigma(\vec{\psi}) \subseteq \{\phi_0, \ldots, \phi_{k-1}\}) \\
& \subseteq & C^R_\Sigma(C^R_\Sigma(\Phi)) \quad (\text{by the induction hypothesis}) \\
& = & C^R_\Sigma(\Phi).
\end{array}$$

Therefore $\phi_k \in C^R_\Sigma(\Phi)$, for all $k = 0, \ldots, n$, which shows that $\phi = \phi_n \in C^R_\Sigma(\Phi)$.

Suppose, conversely, that $\phi \in C^R_\Sigma(\Phi)$. To see that there exists an R-proof of ϕ from Φ, it will be shown that the collection $C = \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, of operators $C_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \rightarrow \mathcal{P}(\mathrm{SEN}(\Sigma))$, defined, for all $\Sigma \in |\mathbf{Sign}|$, $\Psi \cup \{\psi\} \subseteq \mathrm{SEN}(\Sigma)$, by

$$\psi \in C_\Sigma(\Psi) \quad \text{iff} \quad \text{there exists an } R\text{-proof of } \psi \text{ from } \Psi$$

forms a closure system on SEN, such that every $r \in R$ is a rule of C. Then, by the minimality of C^R, it will follow that $C^R \leq C$, whence, since $\phi \in C^R_\Sigma(\Phi)$, we will also have $\phi \in C_\Sigma(\Phi)$, i.e., there exists an R-proof of ϕ from Φ.

For reflexivity, if $\Sigma \in |\mathbf{Sign}|, \Phi \subseteq \mathrm{SEN}(\Sigma)$ and $\phi \in \Phi$, then ϕ is an R-proof of ϕ from Φ, whence $\phi \in C_\Sigma(\Phi)$.

For monotonicity, suppose that $\Sigma \in |\mathbf{Sign}|, \Phi \subseteq \Psi \subseteq \mathrm{SEN}(\Sigma)$. If $\phi \in C_\Sigma(\Phi)$, then, there exists an R-proof of ϕ from Φ, which is also an R-proof of ϕ from Ψ, and, therefore, $\phi \in C_\Sigma(\Psi)$. Hence $C_\Sigma(\Phi) \subseteq C_\Sigma(\Psi)$.

For idempotency, suppose that $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\phi \in C_\Sigma(C_\Sigma(\Phi))$. Then, there exists an R-proof $\phi_0, \phi_1, \ldots, \phi_n = \phi$ of ϕ from $C_\Sigma(\Phi)$. Thus, for those ϕ_i's, $i = 0, \ldots, n$, such that $\phi \in C_\Sigma(\Phi)$, there exists an R-proof $\psi^i_0, \ldots, \psi^i_{n_i} = \phi_i$ of ϕ_i from Φ. If one interjects this proof in the main proof $\phi_0, \ldots, \phi_n$, at the place of ϕ_i, for all i, such that $\phi_i \in C_\Sigma(\Phi)$, then an R-proof of $\phi_n = \phi$ is obtained from Φ, whence $\phi \in C_\Sigma(\Phi)$ and C is idempotent.

Finally, for structurality, if $\Sigma, \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma'), \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$, then, there exists an R-proof $\phi_0, \phi_1, \ldots, \phi_n = \phi$ of ϕ from Φ. Now it is not difficult to verify that $\mathrm{SEN}(f)(\phi_0), \mathrm{SEN}(f)(\phi_1), \ldots, \mathrm{SEN}(f)(\phi_n) = \mathrm{SEN}(f)(\phi)$ is an R-proof of $\mathrm{SEN}(f)(\phi)$ from $\mathrm{SEN}(f)(\Phi)$, whence we get that $\mathrm{SEN}(f)(\phi) \in C_{\Sigma'}(\mathrm{SEN}(f)(\Phi))$ and C is indeed structural.

It has now been shown that C is a closure operator on SEN. To complete the proof, it suffices to show that every $r \in R$ is a rule of C. To see this, suppose that $r = \langle X, \sigma \rangle \in R$, with $X = \{\tau^0, \ldots, \tau^{m-1}\}$ and let $\Sigma \in |\mathbf{Sign}|, \vec{\phi} \in \mathrm{SEN}(\Sigma)^\omega$. Then $\tau^0_\Sigma(\vec{\phi}), \ldots, \tau^{m-1}_\Sigma(\vec{\phi}), \sigma_\Sigma(\vec{\phi})$ is an R-proof of $\sigma_\Sigma(\vec{\phi})$ from $X_\Sigma(\vec{\phi})$ and, therefore, we get that $\sigma_\Sigma(\vec{\phi}) \in C_\Sigma(X_\Sigma(\vec{\phi}))$, i.e., r is a rule of C. ■

Recall that, given a π institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, C is *finitary*, written $|C| = \omega$, if, for all $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, $\phi \in C_\Sigma(\Phi)$ implies that there exists $\Psi \subseteq_\omega \Phi$, such that $\phi \in C_\Sigma(\Psi)$, where $\subseteq_\omega$ denotes the finite subset relation.

Theorem 3.5
Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, with N a category of natural transformations on SEN, is an N-rule-based π-institution. Then C is finitary iff $C = C^R$ for some set R of finitary N-rules.

Proof:
Suppose, first, that $\mathcal{I}$ is finitary. Let R be the set of all finitary N-rules $\langle X, \sigma \rangle$ of $\mathcal{I}$, i.e., such that, for all $\Sigma \in |\mathbf{Sign}|, \vec{\phi} \subseteq \mathrm{SEN}(\Sigma)^\omega$, $\sigma_\Sigma(\vec{\phi}) \subseteq C_\Sigma(X_\Sigma(\vec{\phi}))$. Then, obviously, $C^R \leq C$, by the minimality of C^R. If, conversely, $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$, then, by finitarity, there exists $\Psi \subseteq_\omega \Phi$, such that $\phi \in C_\Sigma(\Psi)$. Since C is N-rule-based, there exists a finitary N-rule $\langle X, \sigma \rangle$ of C, and $\vec{\psi} \in \mathrm{SEN}(\Sigma)^\omega$, such that $X_\Sigma(\vec{\psi}) \subseteq \Psi$ and $\sigma_\Sigma(\vec{\psi}) = \phi$. Therefore $\phi \in C^R_\Sigma(\Psi) \subseteq C^R_\Sigma(\Phi)$, which yields that $C \leq C^R$. Thus, we obtain $C = C^R$.
Suppose, conversely, that $C = C^R$, for some set R of finitary N-rules. Then Proposition 3.4 shows that $\mathcal{I}$ is finitary in this case. ■

4 Matrix Systems and Filtered Products

Suppose that $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is a functor, with N a category of natural transformations on SEN. A **matrix system** for SEN consists of a functor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, with N' a category of natural transformations on SEN', an (N, N')-epimorphic translation $\langle F, \alpha \rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$ and an axiom family $T' = \{T'_\Sigma\}_{\Sigma \in |\mathbf{Sign}'|}$ of SEN'. Such a matrix system will be denoted by $\langle \langle \mathrm{SEN}', \langle F, \alpha \rangle \rangle, T' \rangle$.
Given a family $\mathrm{SEN}^i : \mathbf{Sign}^i \to \mathbf{Set}, i \in I$, of sentence functors, with N^i a category of natural transformations on $\mathrm{SEN}^i, i \in I$, the $N^i, i \in I$, will be said to be **compatible** if there exists a functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ and a category of natural transformations N on SEN, such that, for all $i \in I$, N^i is a homomorphic image of N via a surjective functor $F^i : N \to N^i$ that preserves all projections. This implies that F^i also preserves the arities of all natural transformations involved. In this case, we will tacitly assume that, given $\sigma : \mathrm{SEN}^i \to \mathrm{SEN}$ in N, by $\sigma^i : (\mathrm{SEN}^i)^n \to \mathrm{SEN}^i$ in N^i is denoted the image of σ under F^i. It was shown in [38] how, given such a family of functors with compatible categories of natural transformations, the product functor $\prod_{i \in I} \mathrm{SEN}^i$ may be constructed and endowed with a compatible category of natural transformations, denoted by $\prod_{i \in I} N^i$. Moreover, it has been shown that given (N, N^i)-epimorphic translations $\langle F^i, \alpha^i \rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}^i, i \in I$, there exists an $(N, \prod_{i \in I} N^i)$-epimorphic translation $\prod_{i \in I} \langle F^i, \alpha^i \rangle : \mathrm{SEN} \to^{se} \prod_{i \in I} \mathrm{SEN}^i$, such that

the following triangle commutes:

$$\begin{tikzcd} \prod_{i\in I}\mathrm{SEN}^i \arrow[r, "\langle P^i,\pi^i\rangle"] & \mathrm{SEN}^i \\ \mathrm{SEN} \arrow[u, "\prod_{i\in I}\langle F^i,\alpha^i\rangle"] \arrow[ur, "\langle F^i,\alpha^i\rangle"'] & \end{tikzcd}$$

If $\langle\langle \mathrm{SEN}^i, \langle F^i, \alpha^i\rangle\rangle, T^i\rangle, i \in I$, are matrix systems for SEN, with $\langle F^i, \alpha^i\rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}^i, i \in I$, then the **product matrix system** of the $\langle\langle \mathrm{SEN}^i, \langle F^i, \alpha^i\rangle\rangle, T^i\rangle, i \in I$, is the matrix system $\langle\langle \prod_{i\in I} \mathrm{SEN}^i, \prod_{i\in I}\langle F^i, \alpha^i\rangle\rangle, \prod_{i\in I} T^i\rangle$, where, for all $\Sigma_i \in |\mathbf{Sign}^i|$, $i \in I$,

$$\prod_{i\in I} T^i_{\prod_{i\in I}\Sigma_i} = \prod_{i\in I} T^i_{\Sigma_i}.$$

If, in addition to the functors $\mathrm{SEN}^i, i \in I$, with compatible categories of natural transformations $N^i, i \in I$, a proper filter F on I is also given, then one may define the filtered product $\prod^F_{i\in I} \mathrm{SEN}^i$, by considering the equivalence system $\equiv^F$ on $\prod_{i\in I} \mathrm{SEN}^i$, defined, for all $\Sigma_i \in |\mathbf{Sign}^i|, \phi_i, \psi_i \in \mathrm{SEN}^i(\Sigma_i), i \in I$,

$$\vec{\phi} \equiv^F_{\prod_{i\in I}\Sigma_i} \vec{\psi} \quad \text{iff} \quad \{i \in I : \phi_i = \psi_i\} \in F.$$

Then $\prod^F_{i\in I} \mathrm{SEN}^i : \prod_{i\in I} \mathbf{Sign}^i \rightarrow \mathbf{Set}$ is given, for all $\Sigma_i \in |\mathbf{Sign}^i|, i \in I$, by

$$\prod^F_{i\in I} \mathrm{SEN}^i(\prod_{i\in I}\Sigma_i) = \prod_{i\in I} \mathrm{SEN}^i(\Sigma_i)/{\equiv^F_{\prod_{i\in I}\Sigma_i}},$$

and, given $\Sigma_i, \Sigma'_i \in |\mathbf{Sign}^i|, f_i \in \mathbf{Sign}^i(\Sigma_i, \Sigma'_i)$, $\vec{\phi} \in \prod_{i\in I} \mathrm{SEN}^i(\Sigma_i)$,

$$\prod^F_{i\in I} \mathrm{SEN}^i(\prod_{i\in I} f_i)(\vec{\phi}/{\equiv^F_{\prod_{i\in I}\Sigma_i}}) = \prod_{i\in I} \mathrm{SEN}^i(f_i)(\phi_i)/{\equiv^F_{\prod_{i\in I}\Sigma'_i}}.$$

This is a well-defined functor $\prod^F_{i\in I} \mathrm{SEN}^i : \prod_{i\in I} \mathbf{Sign}^i \rightarrow \mathbf{Set}$. Moreover, given (N, N^i)-epimorphic translations $\langle F^i, \alpha^i\rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}^i, i \in I$, there exists an $(N, (\prod_{i\in I} N^i)^{\equiv^F})$-epimorphic translation $\prod^F_{i\in I}\langle F^i, \alpha^i\rangle : \mathrm{SEN} \rightarrow^{se} \prod^F_{i\in I} \mathrm{SEN}^i$.

$$\begin{tikzcd} \mathrm{SEN}^i & \prod_{i\in I}\mathrm{SEN}^i \arrow[l, "\langle P^i,\pi^i\rangle"'] \arrow[r, "\langle \mathrm{I},\pi^F\rangle"] & \prod^F_{i\in I}\mathrm{SEN}^i \\ & \mathrm{SEN} \arrow[ul, "\langle F^i,\alpha^i\rangle"] \arrow[u, "\prod_{i\in I}\langle F^i,\alpha^i\rangle"] \arrow[ur, "\prod^F_{i\in I}\langle F^i,\alpha^i\rangle"'] & \end{tikzcd}$$

Suppose, now, that, $\langle\langle \mathrm{SEN}^i, \langle F^i, \alpha^i\rangle\rangle, T^i\rangle, i \in I$, are matrix systems for SEN, with $\langle F^i, \alpha^i\rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}^i$. Then the **filtered matrix system product**

$\langle\langle\prod_{i\in I}^F \mathrm{SEN}^i, \prod_{i\in I}^F \langle F^i,\alpha^i\rangle\rangle, \prod_{i\in I}^F T^i\rangle$, $i \in I$, is defined by taking $\prod_{i\in I}^F T^i$ to be the axiom family, defined, for all $\Sigma_i \in |\mathbf{Sign}^i|, i \in I$, by

$$\prod_{i\in I}^F T^i_{\prod_{i\in I}\Sigma_i} = \{\vec{\phi}/{\equiv^F_{\prod_{i\in I}\Sigma_i}} : \{i \in I : \phi_i \in T^i_{\Sigma_i}\} \in F\}.$$

It is not difficult to verify that this definition is independent of representatives and, therefore, the notion of filtered matrix system product is well-defined. It is worth mentioning here that, in many works in model theory, in lieu of the term *filtered product* in constructions on models, similar to the one presented here, the term *reduced product* is sometimes applied.

If F happens to be an ultrafilter on I, then the corresponding reduced matrix system product is termed, as is customary, a **matrix system ultraproduct**.

We use matrix systems in the model theory of the language that is developed in the next section and filtered matrix system products in the main theorem of the last section.

5 Model Theory

Consider again a functor SEN : $\mathbf{Sign} \to \mathbf{Set}$, with N a category of natural transformations on SEN. Let $\mathcal{L}^N(\mathrm{SEN})$ be the first-order language *without equality*, whose set of formulas $\mathrm{Fm}^N(\mathrm{SEN})$ is determined as follows:

- The set of $\mathcal{L}^N(\mathrm{SEN})$-terms is exactly the collection $\mathrm{Te}^N(\mathrm{SEN})$ of the N-terms of SEN.
- $\mathcal{L}^N(\mathrm{SEN})$ contains only one unary relation symbol D. Thus, its atomic formulas have the form $D(\tau)$, where $\tau \in \mathrm{Te}^N(\mathrm{SEN})$.
- For all formulas $\theta_1, \theta_2 \in \mathrm{Fm}^N(\mathrm{SEN})$, $(\theta_1 \wedge \theta_2), (\neg\theta_1) \in \mathrm{Fm}^N(\mathrm{SEN})$.
- For all formulas $\theta \in \mathrm{Fm}^N(\mathrm{SEN})$ and all $i \in \omega$, $(\forall i)\theta \in \mathrm{Fm}^N(\mathrm{SEN})$.

Clearly, the remaining connectives $\vee, \to, \leftrightarrow$, etc., may be defined, as usual, based on the connectives described formally above.

What is fundamentally different here from the usual, Bloom style ([7] and [11]), first-order treatment of these languages is the model theory.

A **model for** $\mathcal{L}^N(\mathrm{SEN})$ is a matrix system $\langle\langle \mathrm{SEN}', \langle F,\alpha\rangle\rangle, T'\rangle$ for SEN, where $\langle F,\alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$ is an (N,N')-epimorphic translation and T' an axiom family on SEN'.

Given such a model for $\mathcal{L}^N(\mathrm{SEN})$, $\tau \in \mathrm{Te}^N(\mathrm{SEN})$, $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi} \in \mathrm{SEN}(\Sigma)^\omega$, the **value of** τ **at** $\vec{\phi}$ **in** SEN' **via** $\langle F,\alpha\rangle$, denoted by $\tau_\Sigma^{\langle F,\alpha\rangle}(\vec{\phi})$, is defined by

$$\tau_\Sigma^{\langle F,\alpha\rangle}(\vec{\phi}) = \tau'_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})),$$

where by σ' is denoted the natural transformation in N' corresponding to σ in N via the (N,N')-epimorphic property.

By the definition of the (N,N')-epimorphic property of $\langle F,\alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, it follows that, for every $\tau \in \mathrm{Te}^N(\mathrm{SEN})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^\omega$,

$$\tau_\Sigma^{\langle F,\alpha\rangle}(\vec{\phi}) = \alpha_\Sigma(\tau_\Sigma(\vec{\phi})). \tag{5.1}$$

Next, the task of defining a Tarskian satisfaction relation is undertaken. The model theory of first-order logic is guiding the various steps carried out in the present framework.

Satisfaction of a formula $\theta \in \mathrm{Fm}^N(\mathrm{SEN})$ **at a** Σ-tuple $\vec{\phi} \in \mathrm{SEN}(\Sigma)^\omega$ **in a model** $\langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, T'\rangle$ for $\mathcal{L}^N(\mathrm{SEN})$, denoted $\langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, T'\rangle \models_\Sigma \theta[\vec{\phi}]$, is defined recursively as follows:

- If $\theta = D(\tau)$ is atomic, $\langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, T'\rangle \models_\Sigma D(\tau)[\vec{\phi}]$ if and only if $\tau_\Sigma^{\langle F,\alpha\rangle}(\vec{\phi}) \in T'_{F(\Sigma)}$.
- If $\theta = \theta_1 \wedge \theta_2$, then $\langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, T'\rangle \models_\Sigma \theta[\vec{\phi}]$ iff $\langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, T'\rangle \models_\Sigma \theta_1[\vec{\phi}]$ and $\langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, T'\rangle \models_\Sigma \theta_2[\vec{\phi}]$.
- If $\theta = \neg\theta'$, then $\langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, T'\rangle \models_\Sigma \theta[\vec{\phi}]$ iff $\langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, T'\rangle \not\models_\Sigma \theta'[\vec{\phi}]$.
- If $\theta = (\forall i)\theta'$, then $\langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, T'\rangle \models_\Sigma \theta[\vec{\phi}]$ iff $\langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, T'\rangle \models_\Sigma \theta'[\vec{\psi}]$, for all $\vec{\psi} \in \mathrm{SEN}(\Sigma)^\omega$, such that $\psi_j = \phi_j$, for all $j \neq i$.

The interpretations of the remaining connectives, that are defined in terms of the basic connectives, also follow from the formal interpretations, given above, in the usual way.

Extending further this definition, if SEN' is a functor, with N' a category of natural transformations on SEN', $\langle F, \alpha\rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}'$ is an (N, N')-epimorphic translation, and C' is a closure system on SEN', i.e., $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ is a π-institution, then **satisfaction of a formula** $\theta \in \mathrm{Fm}^N(\mathrm{SEN})$ **at a** Σ-tuple $\vec{\phi} \in \mathrm{SEN}(\Sigma)^\omega$ **in** $\langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, C'\rangle$, denoted $\langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, C'\rangle \models_\Sigma \theta[\vec{\phi}]$, is understood to mean satisfaction of θ at $\vec{\phi}$ in $\langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, T'\rangle$, for all $T' \in \mathrm{ThFam}(\mathcal{I}')$. Finally, $\langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, T'\rangle \models \theta$ means $\langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, T'\rangle \models_\Sigma \theta[\vec{\phi}]$, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^\omega$, and $\langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, C'\rangle \models \theta$ means $\langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, C'\rangle \models_\Sigma \theta[\vec{\phi}]$, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^\omega$.

Based on this definition of satisfaction, we may define, given a collection Γ of $\mathcal{L}^N(\mathrm{SEN})$-sentences, the collection of all models of Γ, denoted by $\mathbf{M}(\Gamma)$, and given a collection M of models for $\mathcal{L}^N(\mathrm{SEN})$, the collection $\mathbf{\Gamma}(M)$ of all sentences true in every model $\langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, T'\rangle \in M$. The definition extends in a natural way when models with closure systems instead of axiom systems are considered.

Let $r = \langle X, \sigma\rangle$, with $X = \{\tau^0, \ldots, \tau^{m-1}\}$, be a finitary N-rule of SEN. Then, denote by (r) the following formula of $\mathcal{L}^N(\mathrm{SEN})$:

$$(\forall \ldots \forall)(D(\tau^0) \wedge \ldots \wedge D(\tau^{m-1}) \rightarrow D(\sigma)), \tag{5.2}$$

where $(\forall \ldots \forall)$ is the universal closure of the formula with respect to all natural numbers that are less that the maximum arity of the natural transformations in $X \cup \{\sigma\}$. Then we have the following analog of Bloom's Lemma 0.4.1 of [11] for π-institutions:

Lemma 5.1 (Bloom's Lemma for π-Institutions)
Let $\mathrm{SEN} : \mathbf{Sign} \rightarrow \mathbf{Set}$ be a functor and N a category of natural transformations on SEN. If R is a set of finitary N-rules of SEN, then a π-institution $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ is a model of the π-institution $\mathcal{I}^R = \langle \mathbf{Sign}, \mathrm{SEN}, C^R\rangle$ via an (N, N')-logical morphism $\langle F, \alpha\rangle : \mathcal{I}^R \vdash^{se} \mathcal{I}'$ if and only if $\langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, C'\rangle$ is a model of the collection of rules $\{(r) : r \in R\}$ (in the previously defined model-theoretic sense).

Proof:

Let $\langle F,\alpha\rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}'$ be an (N,N')-epimorphic translation.

Suppose, first, that $\langle F,\alpha\rangle : \mathcal{I}^R \vdash^{se} \mathcal{I}'$ is an (N,N')-logical morphism. We need to show that for all $r = \langle X,\sigma\rangle \in R$, with $X = \{\tau^0,\ldots,\tau^{m-1}\}$, we have that

$$\langle\langle \mathrm{SEN}', \langle F,\alpha\rangle\rangle, C'\rangle \models (r).$$

To this end, suppose that $\Sigma \in |\mathbf{Sign}|, \vec{\phi} \in \mathrm{SEN}(\Sigma)^\omega$ and $T' \in \mathrm{ThFam}(\mathcal{I}')$ are such that $\langle\langle \mathrm{SEN}', \langle F,\alpha\rangle\rangle, T'\rangle \models_\Sigma D(\tau^i)[\vec{\phi}]$, for all $i < m$. This yields that

$$\tau_\Sigma^{i^{\langle F,\alpha\rangle}}(\vec{\phi}) \in T'_{F(\Sigma)}, \quad \text{for all } i = 0,\ldots,m-1. \tag{5.3}$$

Since $\langle X,\sigma\rangle$ is a rule of C^R, by definition, we have that $\sigma_\Sigma(\vec{\phi}) \in C_\Sigma^R(X_\Sigma(\vec{\phi}))$, whence, since $\langle F,\alpha\rangle$ is an (N,N')-logical morphism by the hypothesis, we get that $\alpha_\Sigma(\sigma_\Sigma(\vec{\phi})) \in C'_{F(\Sigma)}(\alpha_\Sigma(X_\Sigma(\vec{\phi})))$, i.e., by (5.1), that $\sigma_\Sigma^{\langle F,\alpha\rangle}(\vec{\phi}) \in C'_{F(\Sigma)}(X_\Sigma^{\langle F,\alpha\rangle}(\vec{\phi}))$. Thus, by Condition (5.3), we obtain $\sigma_\Sigma^{\langle F,\alpha\rangle}(\vec{\phi}) \in T'_{F(\Sigma)}$ and, hence,

$$\langle\langle \mathrm{SEN}', \langle F,\alpha\rangle\rangle, T'\rangle \models_\Sigma D(\sigma)[\vec{\phi}].$$

Therefore $\langle\langle \mathrm{SEN}', \langle F,\alpha\rangle\rangle, T'\rangle \models (r)$, for all $T' \in \mathrm{ThFam}(\mathcal{I}')$, which implies that $\langle\langle \mathrm{SEN}', \langle F,\alpha\rangle\rangle, C'\rangle \models (r)$, for all $r \in R$.

Suppose, conversely, that $\langle\langle \mathrm{SEN}', \langle F,\alpha\rangle\rangle, C'\rangle \models (r)$, for all $r = \langle X,\sigma\rangle \in R$, with $X = \{\tau^0,\ldots,\tau^{m-1}\}$. That is, for all $T' \in \mathrm{ThFam}(\mathcal{I}')$, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^\omega$,

$$\langle\langle \mathrm{SEN}', \langle F,\alpha\rangle\rangle, T'\rangle \models_\Sigma D(\tau^0) \wedge \ldots \wedge D(\tau^{m-1}) \rightarrow D(\sigma)[\vec{\phi}].$$

Suppose that $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\phi \in C_\Sigma^R(\Phi)$. We need to show that $\alpha_\Sigma(\phi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$. Since $\phi \in C_\Sigma^R(\Phi)$, there exists, by Proposition 3.4, an R-proof $\phi_0,\phi_1,\ldots,\phi_n = \phi$ of ϕ from Φ. It will be shown by induction on $k = 0,\ldots,n$ that $\alpha_\Sigma(\phi_k) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$.

For $k = 0$, either $\phi_0 \in \Phi$ or there exists an axiomatic rule $/\sigma \in R$ and $\vec{\psi} \in \mathrm{SEN}(\Sigma)^\omega$, such that $\sigma_\Sigma(\vec{\psi}) = \phi_0$. In the first case, $\alpha_\Sigma(\phi_0) \in \alpha_\Sigma(\Phi) \subseteq C'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$. In the second case, we have that

$$\begin{array}{rcl}
\alpha_\Sigma(\phi_0) & = & \alpha_\Sigma(\sigma_\Sigma(\vec{\psi}))\\
& = & \sigma_\Sigma^{\langle F,\alpha\rangle}(\vec{\psi}) \quad \text{(by (5.1))}\\
& \subseteq & C'_{F(\Sigma)}(\emptyset) \quad \text{(since } \langle\langle \mathrm{SEN}', \langle F,\alpha\rangle\rangle, C'\rangle \text{ is a model of } (R))\\
& \subseteq & C'_{F(\Sigma)}(\alpha_\Sigma(\Phi)). \quad \text{(by monotonicity)}
\end{array}$$

Suppose, as the inductive hypothesis, that $\alpha_\Sigma(\phi_i) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$, for all $i < k$. Then either $\phi_k \in \Phi$ or ϕ_k follows from $\{\phi_0,\ldots,\phi_{k-1}\}$ by an application of an R-rule $\langle X,\sigma\rangle$, with $X = \{\tau^0,\ldots,\tau^{m-1}\}$. In the first case $\alpha_\Sigma(\phi_k) \in \alpha_\Sigma(\Phi) \subseteq C'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$. In the second case, there exists $\vec{\psi} \in \mathrm{SEN}(\Sigma)^\omega$, such that $X_\Sigma(\vec{\psi}) \subseteq \{\phi_0,\ldots,\phi_{k-1}\}$ and

$\sigma_\Sigma(\vec{\psi}) = \phi_k$. Therefore, we obtain

$$\begin{array}{rcll}
\alpha_\Sigma(\phi_k) & = & \alpha_\Sigma(\sigma_\Sigma(\vec{\psi})) & \\
& = & \sigma_\Sigma^{\langle F,\alpha\rangle}(\vec{\psi}) \quad \text{(by (5.1))} & \\
& \subseteq & C'_{F(\Sigma)}(\tau_\Sigma^{0\langle F,\alpha\rangle}(\vec{\psi}), \ldots, \tau_\Sigma^{m-1\langle F,\alpha\rangle}(\vec{\psi})) & \text{(since } \langle\langle \mathrm{SEN}', \langle F,\alpha\rangle\rangle, C'\rangle \\
& & & \text{is a model of } (R)) \\
& = & C'_{F(\Sigma)}(\alpha_\Sigma(\tau_\Sigma^0(\vec{\psi})), \ldots, \alpha_\Sigma(\tau_\Sigma^{m-1}(\vec{\psi}))) \quad \text{(by (5.1))} & \\
& \subseteq & C'_{F(\Sigma)}(\alpha_\Sigma(\phi_0), \ldots, \alpha_\Sigma(\phi_{k-1})) \quad \text{(by hypothesis)} & \\
& \subseteq & C'_{F(\Sigma)}(C'_{F(\Sigma)}(\alpha_\Sigma(\Phi))) \quad \text{(by the induction hypothesis)} & \\
& = & C'_{F(\Sigma)}(\alpha_\Sigma(\Phi)). \quad \text{(by idempotency)} &
\end{array}$$

■

6 Rule-Based π-Institutions and Bloom's Theorem

In this last section of the paper, an analog of Bloom's Theorem, characterizing those consequence operations that are finitary, is established in the context of rule-based π-institutions. Again for the original Bloom's Theorem [7] is the original source and [11], Section 0.4, contains an exposition in the context of abstract algebraic logic.

Given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, by $\mathbf{Mod}(\mathcal{I})$ or $\mathbf{Mod}(C)$ is denoted the collection

$$\mathbf{Mod}(C) = \{\langle\langle \mathrm{SEN}', \langle F,\alpha\rangle\rangle, T'\rangle : T' \in \mathrm{ThFam}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')\},$$

where, as is customary, by $\mathrm{ThFam}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$ is denoted the collection of all theory families of the $\langle F,\alpha\rangle$-min (N,N')-model of $\mathcal{I}$ on SEN'. The reader should consult [29] for the definition of min models, their origin in the concept of a basic full model [15] in the theory of abstract algebraic logic and further information on the role they play in categorical abstract algebraic logic.

THEOREM 6.1 (Bloom's Theorem for Rule-Based π-Institutions)
Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformation on SEN, be an N-rule-based π-institution. Then the following statements are equivalent:

1. $\mathbf{Mod}(C)$ is axiomatizable by universal sentences of the form (r).
2. $\mathbf{Mod}(C)$ is closed under filtered matrix system products.
3. $\mathbf{Mod}(C)$ is closed under matrix system ultraproducts.
4. C is finitary.
5. $C = C^R$, for some set R of finitary rules.

Proof:

$1 \rightarrow 2$ Suppose that $\langle F^i, \alpha^i\rangle : \mathrm{SEN} \rightarrow \mathrm{SEN}^i$ is an (N,N^i)-epimorphic translation, that $T^i \in \mathrm{ThFam}_{\mathcal{I}}^{\langle F^i,\alpha^i\rangle}(\mathrm{SEN}^i)$, for all $i \in I$, and F a proper filter on I. Consider the matrix system $\langle\langle \prod_{i\in I}^F \mathrm{SEN}^i, \prod_{i\in I}^F \langle F^i, \alpha^i\rangle\rangle, \prod_{i\in I}^F T^i\rangle$. Consider, also, a finitary N-rule $r = \langle X, \sigma\rangle$, with $X = \{\tau^0, \ldots, \tau^{m-1}\}$, such that $\langle\langle \mathrm{SEN}^i, \langle F^i, \alpha^i\rangle\rangle, T^i\rangle \models (r)$, for all $i \in I$. It will be shown that

$$\langle\langle\prod_{i\in I}^F \mathrm{SEN}^i, \prod_{i\in I}^F \langle F^i,\alpha^i\rangle\rangle, \prod_{i\in I}^F T^i\rangle \models (r).$$

$$\begin{tikzcd}
\mathrm{SEN}^i & \prod_{i\in I}\mathrm{SEN}^i \arrow[l, "\langle P^i,\pi^i\rangle"'] \arrow[r, "\langle \mathrm{I},\pi^F\rangle"] & \prod^F_{i\in I}\mathrm{SEN}^i \\
\mathrm{SEN} \arrow[u, "\langle F^i,\alpha^i\rangle"] \arrow[ur, "\prod_{i\in I}\langle F^i,\alpha^i\rangle"'] & &
\end{tikzcd}$$

To this end, suppose that $\Sigma \in |\mathbf{Sign}|, \vec{\phi} \in \mathrm{SEN}(\Sigma)^\omega$, such that

$$\langle\langle\prod_{i\in I}^F \mathrm{SEN}^i, \prod_{i\in I}^F \langle F^i,\alpha^i\rangle\rangle, \prod_{i\in I}^F T^i\rangle \models_\Sigma D(\tau^1) \wedge \ldots \wedge D(\tau^{m-1})[\vec{\phi}].$$

Thus, $\tau_\Sigma^{j\prod_{i\in I}^F\langle F^i,\alpha^i\rangle}(\vec{\phi}) \subseteq \prod_{i\in I}^F T^i_{\prod_{i\in I}F^i(\Sigma)}$, for all $j < m$. This yields that $\{i \in I : \tau_\Sigma^{j\langle F^i,\alpha^i\rangle}(\phi_i) \subseteq T^i_{F^i(\Sigma_i)}\} \subseteq F$, for all $j < m$. Therefore, since $\langle\langle\mathrm{SEN}^i, \langle F^i,\alpha^i\rangle\rangle, T^i\rangle \models (r)$, for all $i \in I$, we obtain that

$$\{i \in I : \sigma_\Sigma^{\langle F^i,\alpha^i\rangle}(\phi_i) \subseteq T^i_{F^i(\Sigma_i)}\} \supseteq \bigcap_{j=0}^{m-1}\{i \in I : \tau_\Sigma^{j\langle F^i,\alpha^i\rangle}(\phi_i) \subseteq T^i_{F^i(\Sigma_i)}\} \in F,$$

which yields that $\{i \in I : \sigma_\Sigma^{\langle F^i,\alpha^i\rangle}(\phi_i) \subseteq T^i_{F^i(\Sigma_i)}\} \in F$, and, hence, $\sigma_\Sigma^{\prod_{i\in I}^F\langle F^i,\alpha^i\rangle}(\vec{\phi}) \subseteq \prod_{i\in I}^F T^i_{\prod_{i\in I}F^i(\Sigma)}$, i.e.,

$$\langle\langle\prod_{i\in I}^F \mathrm{SEN}^i, \prod_{i\in I}^F \langle F^i,\alpha^i\rangle\rangle, \prod_{i\in I}^F T^i\rangle \models_\Sigma D(\sigma)[\vec{\phi}],$$

which shows that $\langle\langle\prod_{i\in I}^F \mathrm{SEN}^i, \prod_{i\in I}^F \langle F^i,\alpha^i\rangle\rangle, \prod_{i\in I}^F T^i\rangle$ satisfies (r).

$2 \to 3$ Since ultraproducts are special cases of filtered products, this implication is trivial.

$3 \to 4$ Suppose that $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\phi \notin C_\Sigma(\Psi)$, for all $\Psi \subseteq_\omega \Phi$. Let I be the family of all finite subsets of Φ and, for all $\Psi \in I$, let $\Psi^* = \{X \in I : \Psi \subseteq X\}$. $\{\Psi^* : \Psi \in I\}$ has the finite intersection property, whence, there exists an ultrafilter F that contains $\{\Psi^* : \Psi \in I\}$. Now, recalling that Thm denotes the theorem system of $\mathcal{I}$ and following notation introduced in [34], let, for all $\Psi \in I$, $\mathrm{Thm}^{[\Psi]} = \{\mathrm{Thm}^{[\Psi]}_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}|}$ be the theory family of $\mathcal{I}$, defined, for all $\Sigma' \in |\mathbf{Sign}|$, by

$$\mathrm{Thm}^{[\Psi]}_{\Sigma'} = \begin{cases} C_\Sigma(\Psi), & \text{if } \Sigma' = \Sigma \\ C_{\Sigma'}(\emptyset), & \text{otherwise.} \end{cases}$$

Consider the ultraproduct $\langle\langle\prod_{\Psi\in I}^F \mathrm{SEN}, \prod_{\Psi\in I}^F \langle \mathrm{I}_{\mathbf{Sign}},\iota\rangle\rangle, \prod_{\Psi\in I}^F \mathrm{Thm}^{[\Psi]}\rangle$ of the matrix systems $\langle\langle\mathrm{SEN}, \langle \mathrm{I}_{\mathbf{Sign}},\iota\rangle, \mathrm{Thm}^{[\Psi]}\rangle, \Psi \in I$. It is clear that $\langle\langle\mathrm{SEN}, \langle\mathrm{I}_{\mathbf{Sign}},$

$\iota\rangle\rangle, \mathrm{Thm}^{[\Psi]}\rangle$ is a model of C, for all $\Psi \in I$, whence, by the hypothesis, $\langle\langle\prod^F_{\Psi\in I} \mathrm{SEN}, \prod^F_{\Psi\in I}\langle \mathrm{I}_{\mathbf{Sign}},\ \iota\rangle\rangle, \prod^F_{\Psi\in I} \mathrm{Thm}^{[\Psi]}\rangle$ is also a model of C. Therefore, to show that $\phi \notin C_\Sigma(\Phi)$, it suffices now to show that, for all $\psi \in \Phi$, $\{\Psi \in I : \psi \in \mathrm{Thm}^{[\Psi]}_\Sigma\} \in F$, whereas $\{\Psi \in I : \phi \in \mathrm{Thm}^{[\Psi]}_\Sigma\} \notin F$. The second condition is obvious since $\{\Psi \in I : \phi \in \mathrm{Thm}^{[\Psi]}_\Sigma\} = \emptyset$. For the first condition, note that, for all $\psi \in \Phi$, we have

$$\{\psi\}^* \subseteq \{\Psi \in I : \psi \in C_\Sigma(\Psi)\} = \{\Psi \in I : \psi \in \mathrm{Thm}^{[\Psi]}_\Sigma\}.$$

From this and the fact that, by the definition of F, $\{\psi\}^* \in F$, it follows that $\{\Psi \in I : \psi \in \mathrm{Thm}^{[\Psi]}_\Sigma\} \in F$.

$4 \to 5$ This is the content of Theorem 3.5.

$5 \to 1$ This is the content of Lemma 5.1. ∎

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