Categorical Abstract Algebraic Logic:
Selfextensional π-Institutions with Implication

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Abstract

The work of Jansana on selfextensional deductive systems with an implication satisfying the deduction-detachment property, that was partially based on the well-known work of Font and Jansana on providing a general algebraic semantics for sentential logics, is abstracted to cover selfextensional logics with implication that are formalized as π-institutions. Analogs are provided in this more general context of the main results of Jansana. In the first, it was shown that the class of algebras canonically associated with a deductive system with an implication having the deduction-detachment property is a variety. In the second, selfextensionality of a deductive system possessing an implication with the deduction-detachment property was seen to imply full selfextensionality. Finally, the existence of a dual isomorphism between selfextensional deductive systems having an implication with the deduction-detachment property, ordered by extension, and subvarieties of the variety, over the same similarity type, axiomatized by the Hilbert equations is demonstrated. In order to prove analogs of these results at the categorical level, the powerful machinery developed in the last few years in this area is brought to bear. In particular, specific use is made for the first time, of the theory of varieties and quasi-varieties of algebraic systems, as previously developed by the author.

1 Introduction

Two classes of deductive systems that have played an important role in the modern theory of abstract algebraic logic are the classes of selfextensional and of Fregean logics. A deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ is said to be *selfextensional* if the interderivability relation is a congruence on the formula algebra, i.e., if, for all $\phi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$ and all $\delta(p) \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\phi \dashv\vdash_{\mathcal{S}} \psi \quad \text{imply} \quad \delta(\phi) \dashv\vdash_{\mathcal{S}} \delta(\psi).$$

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These logics were introduced by Wójcicki in [28]. On the other hand, a deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ is said to be *Fregean* if interderivability modulo any set of formulas $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$ is a congruence relation on the formula algebra, i.e., if, for all $\Gamma \cup \{\phi, \psi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$ and all $\delta(p) \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Gamma, \phi \dashv\vdash_{\mathcal{S}} \Gamma, \psi \quad \text{imply} \quad \Gamma, \delta(\phi) \dashv\vdash_{\mathcal{S}} \Gamma, \delta(\psi).$$

The name Fregean is due to the study by R. Suszko of non-Fregean logic and stems from the fact that this property formalizes, in a certain sense, Frege's compositionality principle for truth. By taking $\Gamma = \emptyset$ in the definition of Fregean deductive systems, it is easily seen that every Fregean deductive system is selfextensional.

One interesting feature of selfextensional and Fregean deductive systems from the point of view of abstract algebraic logic, as exemplified in particular in the overview [], is the fact that they span different classes of the Leibniz or abstract algebraic hierarchy of logics, as established over the last few decades. The review article [10] and, in more detail, the book [5] contain comprehensive treatments of the different classes of logics forming the various steps in this hierarchy. As a consequence, results and techniques that are developed within the classes of selfextensional or Fregean logics are applicable to a variety of logics in different levels of the hierarchy providing intuitions differing in nature from those provided by working inside a single specific class of the hierarchy.

The widest class of deductive systems in the Leibniz hierarchy, whose members form the main objects of study in abstract algebraic logic, is the class of protoalgebraic deductive systems. Given a deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ and an $\mathcal{S}$-theory T, recall that the *Leibniz congruence* $\Omega_{\mathcal{S}}(T)$ *associated with* T is the largest congruence θ on the formula algebra $\mathbf{Fm}_{\mathcal{L}}(V)$ that is compatible with the theory T in the sense that, if $\phi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$, such that $\langle \phi, \psi \rangle \in \theta$ and $\phi \in T$, then $\psi \in T$. According to the original definition of Blok and Pigozzi [3], a deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ is *protoalgebraic* if, for all theories T of $\mathcal{S}$ and all $\phi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$, $\langle \phi, \psi \rangle \in \Omega_{\mathcal{S}}(T)$ implies that $T, \phi \vdash_{\mathcal{S}} \psi$. A better known characterization of protoalgebraic deductive systems asserts that a deductive system is protoalgebraic if and only if the Leibniz operator is monotone on the theories of the deductive system, i.e., if for all theories T_1 and T_2 of $\mathcal{S}$, $T_1 \subseteq T_2$ implies that $\Omega_{\mathcal{S}}(T_1) \subseteq \Omega_{\mathcal{S}}(T_2)$. A stronger property actually holds and also characterizes protoalgebraic deductive systems. But to state it properly, the notion of a matrix model of a deductive system must be introduced first.

An $\mathcal{L}$-*matrix* $\mathfrak{A} = \langle \mathbf{A}, F \rangle$ is a pair consisting of an $\mathcal{L}$-algebra $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$ and a subset $F \subseteq A$ of the carrier A of $\mathbf{A}$. Given a deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$, an $\mathcal{L}$-matrix $\mathfrak{A}$ is said to be an $\mathcal{S}$-*matrix* if F is an $\mathcal{S}$-*filter* on $\mathbf{A}$, i.e., if, for all $\Gamma \cup \{\phi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$, and every $\mathcal{L}$-homomorphism $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$,

$$\Gamma \vdash_{\mathcal{S}} \phi \quad \text{and} \quad h(\Gamma) \subseteq F \quad \text{imply} \quad h(\phi) \in F.$$

The collection of all $\mathcal{S}$-filters on $\mathbf{A}$ is denoted by $\mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$. The *Leibniz congruence* $\Omega_{\mathbf{A}}(F)$, also denoted by $\Omega(\mathfrak{A})$, of an $\mathcal{S}$-matrix $\mathfrak{A} = \langle \mathbf{A}, F \rangle$ is the largest congruence on $\mathbf{A}$ that is *compatible with* the $\mathcal{S}$-filter F, i.e., in a similar way as before, such that, if $\langle a, b \rangle \in \Omega_{\mathbf{A}}(F)$ and $a \in F$, then $b \in F$. Protoalgebraic deductive systems are also characterized as those deductive systems $\mathcal{S}$ for which the Leibniz operator is monotone on the $\mathcal{S}$-filters of any

$\mathcal{L}$-algebra, i.e., for every $\mathcal{L}$-algebra $\mathbf{A}$ and all $\mathcal{S}$-filters F, G of $\mathbf{A}$, $F \subseteq G$ implies that $\Omega_{\mathbf{A}}(F) \subseteq \Omega_{\mathbf{A}}(G)$.

Yet one more alternative characterization of protoalgebraicity comes from the generalized matrices of a deductive system $\mathcal{S}$. A *generalized $\mathcal{L}$-matrix* or *$\mathcal{L}$-g-matrix* $\mathcal{A} = \langle \mathbf{A}, \mathcal{C} \rangle$ is a pair consisting of an $\mathcal{L}$-algebra $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$ together with a closed set system $\mathcal{C} \subseteq \mathcal{P}(A)$ on the universe A of $\mathbf{A}$. Sometimes, the closure operator $C : \mathcal{P}(A) \to \mathcal{P}(A)$ corresponding to the closed set system $\mathcal{C}$ is employed, in which case the g-matrix is denoted by $\mathcal{A} = \langle \mathbf{A}, C \rangle$. Generalized matrices have a long history but were recently brought to the forefront of the work in abstract algebraic logic via the results of Font and Jansana [9], where they were referred to as *abstract logics*. A g-matrix $\mathcal{A}$ is said to be a *g-matrix model of* $\mathcal{S}$, or an *$\mathcal{S}$-g-matrix*, if every closed set in $\mathcal{C}$ is an $\mathcal{S}$-filter, i.e., if $\mathcal{C} \subseteq \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$. When one considers g-matrices instead of matrices as the models of deductive systems, the Leibniz operator is replaced by the Tarski operator. The *Tarski congruence* $\widetilde{\Omega}_{\mathbf{A}}(\mathcal{C})$, or $\widetilde{\Omega}(\mathcal{A})$, of an $\mathcal{L}$-g-matrix $\mathcal{A} = \langle \mathbf{A}, \mathcal{C} \rangle$ is the largest congruence on the algebra $\mathbf{A}$ that is compatible with every closed set $F \in \mathcal{C}$. It is easily seen from the definition that $\widetilde{\Omega}_{\mathbf{A}}(\mathcal{C}) = \bigcap_{F \in \mathcal{C}} \Omega_{\mathbf{A}}(F)$. Moreover, a deductive system is protoalgebraic if and only if, for every $\mathcal{S}$-g-matrix $\mathcal{A} = \langle \mathbf{A}, \mathcal{C} \rangle$, $\widetilde{\Omega}_{\mathbf{A}}(\mathcal{C}) = \Omega_{\mathbf{A}}(\bigcap \mathcal{C})$.

Based on the notion of a g-matrix model of a deductive system $\mathcal{S}$, one may define strengthnenings of the concepts of a selfextensional and of a Fregean deductive system. According to Font and Jansana [9], a *basic g-matrix model of* $\mathcal{S}$, or a *basic $\mathcal{S}$-g-matrix*, is an $\mathcal{S}$-g-matrix $\mathcal{A} = \langle \mathbf{A}, \mathcal{C} \rangle$, such that $\mathcal{C} = \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$, the entire collection of $\mathcal{S}$-filters on $\mathbf{A}$. Moreover, an $\mathcal{S}$-g-matrix is said to be a *full $\mathcal{S}$-g-matrix* if its reduction $\mathcal{A}/\widetilde{\Omega}(\mathcal{A}) = \langle \mathbf{A}/\widetilde{\Omega}(\mathcal{A}), \mathcal{C}/\widetilde{\Omega}(\mathcal{A}) \rangle$ is a basic $\mathcal{S}$-g-matrix. A deductive system $\mathcal{S}$ is then said to be *fully selfextensional* if every full model $\mathcal{A}$ of $\mathcal{S}$ is selfextensional in the sense that the interderivability relation, or Frege relation, $\Lambda(\mathcal{A}) = \{\langle a, b \rangle \in A^2 : C(a) = C(b)\}$ of $\mathcal{A}$ is a congruence relation on $\mathbf{A}$. Since the $\mathcal{S}$-g-matrix $\mathcal{S} = \langle \mathbf{Fm}_{\mathcal{L}}(V), \mathrm{Th}_{\mathcal{S}} \rangle$ is a full model of $\mathcal{S}$, it is obvious that every fully selfextensional deductive system is selfextensional. On the other hand, a deductive system $\mathcal{S}$ is said to be *fully Fregean* if, for every full model $\mathcal{A}$ of $\mathcal{S}$, $\mathcal{A}$ is Fregean in the sense that, for every $F \in \mathcal{C}$, the relation $\Lambda_{\mathcal{A}}(F) = \{\langle a, b \rangle \in A^2 : C(F, a) = C(F, b)\}$ is a congruence relation on $\mathbf{A}$. For a reason analogous to that with fully selfextensional and selfextensional logics, every fully Fregean deductive system is Fregean. Moreover, Babyonyshev [1] has shown that both the inclusion of fully selfextensional logics in the class of all selfextensional logics as well as that of fully Fregean logics in the class of all Fregean logics are proper.

Besides selfextensionality and Fregeanity, another property that is critical in the development of the theory of both [9] and [13] is that of possessing an implication system with the deduction-detachment property. A deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ is said to *have an implication $\Rightarrow$ satisfying the deduction-detachment property*, if there exists a set $\Rightarrow$ of binary terms (either primitive or derived) such that, for every $\phi \in \mathrm{Fm}_{\mathcal{L}}(V)$, $\vdash_{\mathcal{S}} \phi \Rightarrow \phi$ and, for all $\Gamma \cup \{\phi, \psi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Gamma, \phi \vdash_{\mathcal{S}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathcal{S}} \phi \Rightarrow \psi.$$

The implication from left to right is referred to as the *deduction property of* $\Rightarrow$ whereas the implication from right to left as the *detachment property of* $\Rightarrow$.

Having all these notions at hand, it is now possible to give the reader an overview of the main results of [13], that will be at the focus of the investigations in the remainder of the present paper. First, in Theorem 4.27 of [13] it is shown that, if $\mathcal{S}$ is a deductive system with an implication that satisfies the deduction-detachment property, then the class of algebras Alg$\mathcal{S}$, canonically associated with $\mathcal{S}$ via an abstraction of the Lindenbaum-Tarski process, is a variety. Second, in Theorem 4.31 of [13], every selfextensional deductive system having an implication satisfying the deduction-detachment property is shown to be fully selfextensional. Finally, given a similarity type $\mathcal{L}$ and a fixed binary term $\Rightarrow$ of $\mathcal{L}$, Theorem 13 of [13] asserts that there exists a dual isomorphism between the collection of selfextensional deductive systems in which $\Rightarrow$ has the deduction-detachment property, ordered by extension, and the collection of all subvarieties of the variety of all $\mathcal{L}$-algebras that is axiomatized by the Hilbert-algebra equations with respect to $\Rightarrow$.

The goal of the present paper is to prove analogs of these results of Jansana so as to cover selfextensional logics, with an implication satisfying a categorical analog of the deduction-detachment property, that are formalized as π-institutions. In order to prove those analogs at the categorical level, the powerful machinery developed in the last few years in the area of Categorical Abstract Algebraic Logic is brought to bear. In particular, specific use is made for the first time, of the theory of varieties and quasi-varieties of algebraic systems, as previously developed by the author.

A brief overview of the contents and the main results of the paper is provided now. The next section revisits some of the preliminaries that are required in order to state and prove the main results of the paper. In particular, the definition of a π-institution, of a category of natural transformations on a sentence functor, of theory systems, of N-congruence systems and those of N-selfextensionality and of N-Fregeanity are recalled. Section 3 is entirely new and key to the developments not only of the present paper but also of some other forthcoming work by the author. It introduces two classes of N-algebraic systems related to the sentence functor SEN of a given π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, where N is a category of natural transformations on SEN. The first is the variety $\mathtt{K}_{\mathcal{I}}^N$ of N-algebraic systems generated by the N-Tarski reduction SEN^N of SEN. This class corresponds in this framework to the intrinsic variety $\mathtt{K}_{\mathcal{S}}$ associated to a deductive system $\mathcal{S}$ in the universal algebraic framework. The second class is the class $\mathrm{Alg}^N(\mathcal{I})$ of all N-algebraic systems that are the algebraic system reducts of all Tarski reduced models of $\mathcal{I}$ via surjective logical morphisms. This class is the one corresponding to the class Alg$\mathcal{S}$ of all algebraic reducts of the reduced g-matrix models of $\mathcal{S}$ in the deductive system framework. In a very interesting result paralleling one holding for deductive systems, it is shown that $\mathrm{Alg}^N(\mathcal{I})$ is always a subclass of $\mathtt{K}_{\mathcal{I}}^N$ and that, moreover, $\mathtt{K}_{\mathcal{I}}^N$ is the variety of N-algebraic systems that is generated by $\mathrm{Alg}^N(\mathcal{I})$. These results rely on the theory of varieties and quasivarieties of N-algebraic systems studied in detail in [20, 21, 24]. Hilbert-based π-institutions are introduced and studied in Section 4, where it is shown that a finitary π-institution $\mathcal{I}$ is N-Hilbert based if and only if it is N-selfextensional and has the N-uniterm deduction-detachment property. This section also contains analogs of the three main results of [13] that were summarized above. These analogs are formulated in the framework of finitary symmetrically N-rule based π-institutions. Leaving aside for the moment the precise meaning of the terminology, which will be introduced in detail in the following sections, the following results will be obtained:

1. Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, be a finitary π-institution. If $\mathcal{I}$ is N-selfextensional and has an N-implication $\Rightarrow$: $\mathrm{SEN}^2 \to \mathrm{SEN}$ with the deduction-detachment property, then $\mathrm{Alg}^N(\mathcal{I})$ is a variety of N-algebraic systems.

2. Every π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, that is N-selfextensional and has an N-implication $\Rightarrow$: $\mathrm{SEN}^2 \to \mathrm{SEN}$ with the deduction-detachment property is also fully N-selfextensional.

3. Let SEN : $\mathbf{Sign} \to \mathbf{Set}$ be a set-valued functor, and N a category of natural transformations on SEN. Suppose that SEN is symmetrically N-rule based and that $\Rightarrow$: $\mathrm{SEN}^2 \to \mathrm{SEN}$ is a binary natural transformation in N. Then, there exists a dual isomorphism between the collection of all N-selfextensional π-institutions with the deduction-detachment property relative to $\Rightarrow$, ordered by extension, and the collection of all subvarieties $\mathtt{K}$ of the variety of N-algebraic systems axiomatized by the following Hilbert algebra equations (H1)-(H4) and satisfying the technical condition $\mathtt{K} = \mathbf{V}^N(\mathrm{cor}^N(\mathtt{K}))$, ordered by inclusion.

$$\begin{array}{ll}
\text{(H1)} & x \Rightarrow x \approx y \Rightarrow y \\
\text{(H2)} & (x \Rightarrow x) \Rightarrow x \approx x \\
\text{(H3)} & x \Rightarrow (y \Rightarrow z) \approx (x \Rightarrow y) \Rightarrow (x \Rightarrow z) \\
\text{(H4)} & (x \Rightarrow y) \Rightarrow ((y \Rightarrow x) \Rightarrow y) \approx (y \Rightarrow x) \Rightarrow ((x \Rightarrow y) \Rightarrow x)
\end{array}$$

4. Finitary N-selfextensional π-institutions with an implication $\Rightarrow$ in N, that has the deduction-detachment property, are characterized as those π-institutions $\mathcal{I}$ for which, there exists a class $\mathtt{K}$ of N-algebraic systems satisfying (H1)-(H4) and is connected to $\mathcal{I}$ by the following two conditions:

   (a) For all $\Sigma \in |\mathbf{Sign}|$ and all $\phi_0, \ldots, \phi_{n-1}, \phi \in \mathrm{SEN}(\Sigma)$, if $\phi \in C_\Sigma(\phi_0, \ldots, \phi_{n-1})$, then, for all $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}$ and all surjective $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$,

   $$\alpha_\Sigma(\phi_0 \Rightarrow_\Sigma (\cdots \Rightarrow_\Sigma (\phi_{n-1} \Rightarrow \phi)\cdots)) = 1'_{F(\Sigma)};$$

   (b) For all $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$, $\phi \in C_\Sigma(\emptyset)$ if and only if, for all $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}$ and all surjective $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, $\alpha_\Sigma(\phi) = 1'_{F(\Sigma)}$,

   where, by $1'_{F(\Sigma)}$ is denoted the $F(\Sigma)$-sentence $\phi' \Rightarrow'_{F(\Sigma)} \phi'$, for any $\phi' \in \mathrm{SEN}'(F(\Sigma))$. Since $\mathtt{K}$ satisfies (H1), we have, for all $\phi', \psi' \in \mathrm{SEN}'(F(\Sigma))$, $\phi' \Rightarrow'_{F(\Sigma)} \phi' = \psi' \Rightarrow'_{F(\Sigma)} \psi'$, whence this definition of $1'_{F(\Sigma)}$ is unambiguous.

   The π-institutions that are characterized in this last part are referred to as N-Hilbert-based in the paper.

In Section 5 Fregean and fully Fregean π-institutions are studied. The main result obtained in this section is that every finitary N-rule based and N-protoalgebraic π-institution that is N-Fregean is also fully N-Fregean. In the final section, Section 6, of the paper, the special case of N-Fregean π-institutions with the deduction-detachment theorem is considered and some analogs of results holding for Fregean deductive systems with the deduction-detachment theorem are shown to be valid in this more general context.

For all unexplained categorical terminology and notation the reader is referred to any of [2, 4, 14]. For background on the theory of abstract algebraic logic and discussion of the various classes of the abstract algebraic hierarchy, one of which is the class of protoalgebraic deductive systems, the reader is referred to the review article [10], the monograph [9] and the comprehensive treatise [5].

2 Preliminaries

Let $\mathrm{SEN} : \mathbf{Sign} \rightarrow \mathbf{Set}$ be a set-valued functor and N a category of natural transformations on SEN. When such a distinguished functor is under consideration, all varieties or quasi-varieties that will be discussed will be varieties or quasi-varieties of N-algebraic systems defined by collections of N-equations or N-quasi-equations in the sense of either [24] or [25]. As a consequence, if $\mathtt{K}$ is a class of N-algebraic systems and $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F' \rangle\rangle, \mathbf{B} = \langle \mathrm{SEN}'', \langle N'', F'' \rangle\rangle$ are two N-algebraic systems in $\mathtt{K}$, by an N-**morphism** $\langle F, \alpha \rangle : \mathbf{A} \rightarrow \mathbf{B}$ we will always mean an (N', N'')-epimorphic translation $\langle F, \alpha \rangle : \mathrm{SEN}' \rightarrow \mathrm{SEN}''$, such that the following triangle commutes

$$\begin{tikzcd} & N \arrow[dl, "F'"'] \arrow[dr, "F''"] & \\ N' \arrow[rr, dash, dashed] & & N'' \end{tikzcd}$$

where the dotted line denotes the correspondence established between N' and N'' by the (N', N'')-epimorphic property of $\langle F, \alpha \rangle$, i.e., given any $\sigma : \mathrm{SEN}^n \rightarrow \mathrm{SEN}$ in N, it will always be assumed that $\sigma' := F'(\sigma)$ and $\sigma'' := F''(\sigma)$ correspond under the (N', N'')-epimorphic property of $\langle F, \alpha \rangle$.

The basic logical structures that will serve as the underlying structures of our investigations are π-institutions. They were introduced in [8] as a modification of institutions [11, 12]. Recall from [8] that a π-**institution** $\mathcal{I}$ is a triple $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, such that

(i) **Sign** is a category, whose objects are called **signatures**;

(ii) $\mathrm{SEN} : \mathbf{Sign} \rightarrow \mathbf{Set}$, is a set-valued functor from the category **Sign** of signatures, called the **sentence functor** and giving, for each signature Σ, a set whose elements are called **sentences** over that signature Σ or Σ-**sentences**;

(iii) $C_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \rightarrow \mathcal{P}(\mathrm{SEN}(\Sigma))$, for each $\Sigma \in |\mathbf{Sign}|$, is a mapping, called Σ-**closure**, such that

(a) $A \subseteq C_\Sigma(A)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq \mathrm{SEN}(\Sigma)$,
(b) $C_\Sigma(C_\Sigma(A)) = C_\Sigma(A)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq \mathrm{SEN}(\Sigma)$,
(c) $C_\Sigma(A) \subseteq C_\Sigma(B)$, for all $\Sigma \in |\mathbf{Sign}|, A \subseteq B \subseteq \mathrm{SEN}(\Sigma)$,
(d) $\mathrm{SEN}(f)(C_{\Sigma_1}(A)) \subseteq C_{\Sigma_2}(\mathrm{SEN}(f)(A))$, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, $A \subseteq \mathrm{SEN}(\Sigma_1)$.

A π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is said to be **finitary**, if, for all $\Sigma \in |\mathbf{Sign}|$, $C_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)) \to \mathcal{P}(\mathrm{SEN}(\Sigma))$ is a finitary closure operator in the usual sense, i.e., if, for every $\Sigma \in |\mathbf{Sign}|$ and every $\Phi \subseteq \mathrm{SEN}(\Sigma)$,

$$C_\Sigma(\Phi) = \bigcup \{C_\Sigma(\Phi') : \Phi' \subseteq_\omega \Phi\},$$

where by $\subseteq_\omega$ is denoted the finite subset relation.

Given a functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, a collection $\theta = \{\theta_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, such that θ_Σ is an equivalence relation on $\mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$, is called an **equivalence family on** SEN. If, in addition, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, θ satisfies

$$\mathrm{SEN}(f)^2(\theta_{\Sigma_1}) \subseteq \theta_{\Sigma_2},$$

then θ is said to be an **equivalence system on** SEN. If N is a category of natural transformations on SEN and an equivalence system θ on SEN satisfies, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi_0, \psi_0, \ldots, \phi_{n-1}, \psi_{n-1} \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi_i, \psi_i\rangle \in \theta_\Sigma,\ i < n, \quad \text{imply} \quad \langle \sigma_\Sigma(\phi_0, \ldots, \phi_{n-1}), \sigma_\Sigma(\psi_0, \ldots, \psi_{n-1}\rangle \in \theta_\Sigma,$$

then θ is a said to be an N**-congruence system on** SEN.

Given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ and a category N of natural transformations on SEN, one may associate with $\mathcal{I}$ an N-congruence system and an equivalence system that have played very significant roles in the Abstract Algebraic Logic literature in classifying sentential logics and π-institutions. Recall that an **axiom family** $A = \{A_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ on a sentence functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is simply a collection of subsets $A_\Sigma \subseteq \mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$. It is said to be an **axiom system** on SEN if, in addition, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, $\mathrm{SEN}(f)(A_{\Sigma_1}) \subseteq A_{\Sigma_2}$. Recall, also, that a **theory family** $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ of $\mathcal{I}$ is a collection of Σ-theories of $\mathcal{I}$, $\Sigma \in |\mathbf{Sign}|$. Similarly, a theory family T is called a **theory system** if, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, $\mathrm{SEN}(f)(T_{\Sigma_1}) \subseteq T_{\Sigma_2}$. The **Tarski** N**-congruence system** $\widetilde{\Omega}^N(\mathcal{I})$ of $\mathcal{I}$ is the largest N-congruence system on SEN that is compatible with every theory family of $\mathcal{I}$ in the sense that, for every theory family $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ of $\mathcal{I}$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\langle \phi, \psi\rangle \in \widetilde{\Omega}^N_\Sigma(\mathcal{I}) \quad \text{and} \quad \phi \in T_\Sigma \quad \text{imply} \quad \psi \in T_\Sigma.$$

Such an N-congruence system is called a **logical** N**-congruence system** of $\mathcal{I}$. The **Frege equivalence system** on SEN is the equivalence system $\Lambda(\mathcal{I}) = \{\Lambda_\Sigma(\mathcal{I})\}_{\Sigma \in |\mathbf{Sign}|}$ on SEN, defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, by

$$\langle \phi, \psi\rangle \in \Lambda_\Sigma(\mathcal{I}) \quad \text{iff} \quad C_\Sigma(\phi) = C_\Sigma(\psi).$$

In [17] a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, was called N-**selfextensional** if its Frege equivalence system is an N-congruence system. Since it is always the case that $\widetilde{\Omega}(\mathcal{I}) \leq \Lambda(\mathcal{I})$ and $\widetilde{\Omega}(\mathcal{I})$ is the largest logical N-congruence system of $\mathcal{I}$, $\mathcal{I}$ being N-selfextensional is equivalent to the condition that $\widetilde{\Omega}^N(\mathcal{I}) = \Lambda(\mathcal{I})$.

A π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is called **fully** N-**selfextensional** if, for every full (N, N')-model of $\mathcal{I}$ via a surjective (N, N')-logical morphism $\langle F, \alpha\rangle : \mathcal{I} \rangle\!-^{se} \mathcal{I}'$, $\mathcal{I}'$ is N'-selfextensional.

Given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, the **Frege operator** $\Lambda^{\mathcal{I}}$ maps an axiom family $F = \{F_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ of $\mathcal{I}$ to the equivalence system $\Lambda^{\mathcal{I}}(F) = \{\Lambda^{\mathcal{I}}_\Sigma(F)\}_{\Sigma \in |\mathbf{Sign}|}$ of $\mathcal{I}$ that is defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$ by

$$\begin{array}{lll}\langle \phi, \psi\rangle \in \Lambda^{\mathcal{I}}_\Sigma(F) & \text{iff} & C_{\Sigma'}(F_{\Sigma'} \cup \{\mathrm{SEN}(f)(\phi)\}) = C_{\Sigma'}(F_{\Sigma'} \cup \{\mathrm{SEN}(f)(\psi)\}),\\ & & \text{for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma').\end{array}$$

If F happens to be an axiom system (rather than simply an axiom family), we have $\Lambda^{\mathcal{I}}_\Sigma(F) = \Lambda_\Sigma(\mathcal{I}^F)$, where $\mathcal{I}^F = \langle \mathbf{Sign}, \mathrm{SEN}, C^F\rangle$ is given, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, by

$$\phi \in C^F_\Sigma(\Phi) \quad \text{iff} \quad \phi \in C_\Sigma(F_\Sigma \cup \Phi).$$

A π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is called N-**Fregean** if, for every theory family T of $\mathcal{I}$, $\Lambda_{\mathcal{I}}(T)$ is an N-congruence system on SEN. Of course, by considering the theorem system $\mathrm{Thm} = \{\mathrm{Thm}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|} := \{C_\Sigma(\emptyset)\}_{\Sigma \in |\mathbf{Sign}|}$ of $\mathcal{I}$ it is easy to see that, if $\mathcal{I}$ is N-Fregean, then it is also N-selfextensional. Furthermore, $\mathcal{I}$ is called **fully** N-**Fregean** if, for every full (N, N')-model of $\mathcal{I}$ via a surjective (N, N')-logical morphism $\langle F, \alpha\rangle : \mathcal{I} \rangle\!-^{se} \mathcal{I}'$, the π-institution $\mathcal{I}'$ is N'-Fregean.

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a functor and N a category of natural transformations on SEN. A closure system C on SEN and the corresponding π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ are said to be N-**rule based** if, for all $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$, there exists an N-rule $\langle X, \sigma\rangle$ of C of length at most $|\Phi|^+$, and $\vec{\psi} \in \mathrm{SEN}(\Sigma)^\omega$, such that $X_\Sigma(\vec{\psi}) \subseteq \Phi$ and $\sigma_\Sigma(\vec{\psi}) = \phi$, i.e., such that ϕ follows from Φ by an application of $\langle X, \sigma\rangle$. This definition of an N-rule based π-institution is borrowed from [22], where it was used as a platform to discuss a generalized version of Bloom's Theorem for π-institutions. The reader may consult that paper for the definition of an N-rule and for many more details on these two concepts.

A finitary π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ will be said to be **symmetrically** N-**rule based** if it is N-rule based and, in addition, if, for some $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$, $C_\Sigma(\phi) = C_\Sigma(\psi)$, then, there exist natural transformations $\sigma^{\langle \Sigma, \phi\rangle}, \sigma^{\langle \Sigma, \psi\rangle} : \mathrm{SEN}^k \to \mathrm{SEN}$ in N and $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, such that $\sigma^{\langle \Sigma, \phi\rangle}_\Sigma(\vec{\chi}) = \phi, \sigma^{\langle \Sigma, \psi\rangle}_\Sigma(\vec{\chi}) = \psi$ and $\langle \{\sigma^{\langle \Sigma, \phi\rangle}\}, \sigma^{\langle \Sigma, \psi\rangle}\rangle$, $\langle \{\sigma^{\langle \Sigma, \psi\rangle}\}, \sigma^{\langle \Sigma, \phi\rangle}\rangle$ are both N-rules of $\mathcal{I}$.

A set-valued functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, with N a category of natural transformations on SEN, is said to be **symmetrically** N-**rule based** if, for every finitary closure system C on SEN, the π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is symmetrically N-rule based.

3 The Classes $\mathtt{K}_{\mathcal{I}}^N$ and $\mathrm{Alg}^N(\mathcal{I})$

Recall that, given a functor $\mathrm{SEN}:\mathbf{Sign}\to\mathbf{Set}$, with N a category of natural transformations on SEN, an N-**algebraic system** is a triple $\langle \mathrm{SEN}',\langle N,F\rangle\rangle$, such that $\mathrm{SEN}':\mathbf{Sign}'\to\mathbf{Set}$ is a functor, N' is a category of natural transformations on SEN' and $F:N\to N'$ is a surjective functor that preserves objects, i.e., preserves the arities of the operations in the clones N and N'. An N-**algebraic morphism** $\langle F,\alpha\rangle:\langle \mathrm{SEN}',\langle N',F'\rangle\rangle\to\langle \mathrm{SEN}'',\langle N'',F''\rangle\rangle$ is an (N',N'')-epimorphic translation $\langle F,\alpha\rangle:\mathrm{SEN}'\to^{se}\mathrm{SEN}''$, such that, for all $\sigma:\mathrm{SEN}^n\to\mathrm{SEN}$ in N, $\sigma':=F'(\sigma)$ and $\sigma'':=F''(\sigma)$ correspond under the (N',N'')-epimorphic property of $\langle F,\alpha\rangle$.

Let $\mathcal{I}=\langle\mathbf{Sign},\mathrm{SEN},C\rangle$ be a π-institution, with N a category of natural transformations on SEN. Consider the triple $\langle \mathrm{SEN}^N,\langle\overline{N},\overline{F}\rangle\rangle$, where $\mathrm{SEN}^N:\mathbf{Sign}\to\mathbf{Set}$ is the quotient functor $\mathrm{SEN}/\widetilde{\Omega}^N(\mathcal{I})$, $\overline{N}$ is the quotient category of N by $\widetilde{\Omega}^N(\mathcal{I})$ and $\overline{F}:N\to\overline{N}$ maps a natural transformation $\sigma:\mathrm{SEN}^n\to\mathrm{SEN}$ in N to its quotient $\overline{\sigma}:(\mathrm{SEN}^N)^n\to\mathrm{SEN}^N$. All these concepts were defined in [15], where they were shown to be well-defined. The triple $\langle \mathrm{SEN}^N,\langle\overline{N},\overline{F}\rangle\rangle$ is an N-algebraic system. The variety that it generates in the sense of [24] will be denoted by $\mathtt{K}_{\mathcal{I}}^N$ and will be called, by analogy with the intrinsic variety $\mathtt{K}_{\mathcal{S}}$ associated with a deductive system $\mathcal{S}$, the **intrinsic** N-**variety** of the π-institution $\mathcal{I}$. It is shown below that an N-equation $\sigma\approx\tau$, with $\sigma,\tau:\mathrm{SEN}^n\to\mathrm{SEN}$ in N, is an N-identity of the intrinsic variety $\mathtt{K}_{\mathcal{I}}^N$ of a π-institution $\mathcal{I}$ if and only if, for every $\lambda:\mathrm{SEN}^k\to\mathrm{SEN}$ in N, all $\Sigma,\Sigma'\in|\mathbf{Sign}|$, $f\in\mathbf{Sign}(\Sigma,\Sigma')$ and all $\vec{\phi}\in\mathrm{SEN}(\Sigma)^n$, $\vec{\chi}\in\mathrm{SEN}(\Sigma')^{k-1}$,

$$C_{\Sigma'}(\lambda_{\Sigma'}(\mathrm{SEN}(f)(\sigma_\Sigma(\vec{\phi})),\vec{\chi}))=C_{\Sigma'}(\lambda_{\Sigma'}(\mathrm{SEN}(f)(\tau_\Sigma(\vec{\phi})),\vec{\chi})). \tag{1}$$

Note that Equation (1) abbreviates the following sets of equations, for all $i<k$:

$$\begin{aligned}&C_{\Sigma'}(\lambda_{\Sigma'}(\chi_0,\ldots,\chi_{i-1},\mathrm{SEN}(f)(\sigma_\Sigma(\vec{\phi})),\chi_{i+1},\ldots,\chi_{k-1})=\\&\qquad C_{\Sigma'}(\lambda_{\Sigma'}(\chi_0,\ldots,\chi_{i-1},\mathrm{SEN}(f)(\tau_\Sigma(\vec{\phi})),\chi_{i+1},\ldots,\chi_{k-1})).\end{aligned}$$

This abbreviating convention will be followed throughout the paper when it is convenient to shorten these longer expressions.

Proposition 1 *Let $\mathcal{I}=\langle\mathbf{Sign},\mathrm{SEN},C\rangle$ be a π-institution, with N a category of natural transformations on* SEN. *Then, for every $\sigma,\tau:\mathrm{SEN}^n\to\mathrm{SEN}$ in N, $\mathtt{K}_{\mathcal{I}}^N\models\sigma\approx\tau$ if and only if, for every $\lambda:\mathrm{SEN}^k\to\mathrm{SEN}$ in N, all $\Sigma,\Sigma'\in|\mathbf{Sign}|$, $f\in\mathbf{Sign}(\Sigma,\Sigma')$ and all $\vec{\phi}\in\mathrm{SEN}(\Sigma)^n$, $\vec{\chi}\in\mathrm{SEN}(\Sigma')^{k-1}$,*

$$C_{\Sigma'}(\lambda_{\Sigma'}(\mathrm{SEN}(f)(\sigma_\Sigma(\vec{\phi})),\vec{\chi}))=C_{\Sigma'}(\lambda_{\Sigma'}(\mathrm{SEN}(f)(\tau_\Sigma(\vec{\phi})),\vec{\chi})).$$

Proof:

Suppose, first, that $\mathtt{K}_{\mathcal{I}}^N\models\sigma\approx\tau$. Thus, for every N-algebraic system $\mathbf{A}\in\mathtt{K}_{\mathcal{I}}^N$, we must have $\mathbf{A}\models\sigma\approx\tau$. In particular, we obtain that $\langle \mathrm{SEN}^N,\langle\overline{N},\overline{F}\rangle\rangle\models\sigma\approx\tau$. Therefore, for all $\Sigma\in|\mathbf{Sign}|$ and all $\vec{\phi}\in\mathrm{SEN}(\Sigma)^n$, we get that $\overline{\sigma}_\Sigma(\vec{\phi}^N)=\overline{\tau}_\Sigma(\vec{\phi}^N)$. Therefore, we get that $\sigma_\Sigma(\vec{\phi})^N = \tau_\Sigma(\vec{\phi})^N$, i.e., $\langle\sigma_\Sigma(\vec{\phi}), \tau_\Sigma(\vec{\phi})\rangle \in \widetilde{\Omega}^N_\Sigma(\mathcal{I})$. Now the conclusion follows by the characterization of $\widetilde{\Omega}^N(\mathcal{I})$ contained in Theorem 4 of [15].

Suppose, conversely, that, for every $\lambda : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, all $\Sigma, \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^n, \vec{\chi} \in \mathrm{SEN}(\Sigma')^{k-1}$, the displayed equation in the statement of the proposition holds. Then, by Theorem 4 of [15], we get that $\langle\sigma_\Sigma(\vec{\phi}), \tau_\Sigma(\vec{\phi})\rangle \in \widetilde{\Omega}^N_\Sigma(\mathcal{I})$. This means that $\overline{\sigma}_\Sigma(\vec{\phi}^N) = \overline{\tau}_\Sigma(\vec{\phi}^N)$, for every $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^n$. Thus, $\sigma \approx \tau$ holds in $\langle\mathrm{SEN}^N, \langle\overline{N}, \overline{F}\rangle\rangle$. Hence, since $\mathtt{K}^N_\mathcal{I}$ is the variety generated by $\langle\mathrm{SEN}^N, \langle\overline{N}, \overline{F}\rangle\rangle$, we must also have that $\mathtt{K}^N_\mathcal{I} \models \sigma \approx \tau$. ■

From the proof of Proposition 1, we also infer that, if $\mathcal{I}$ happens to be N-selfextensional, then

$$\begin{array}{rll}\mathtt{K}^N_\mathcal{I} \models \sigma \approx \tau & \text{iff} & \langle\sigma_\Sigma(\vec{\phi}), \tau_\Sigma(\vec{\phi})\rangle \in \widetilde{\Omega}^N_\Sigma(\mathcal{I}), \text{ for all } \Sigma \in |\mathbf{Sign}|, \vec{\phi} \in \mathrm{SEN}(\Sigma)^n,\\ & \text{iff} & \langle\sigma_\Sigma(\vec{\phi}), \tau_\Sigma(\vec{\phi})\rangle \in \Lambda_\Sigma(\mathcal{I}), \text{ for all } \Sigma \in |\mathbf{Sign}|, \vec{\phi} \in \mathrm{SEN}(\Sigma)^n,\\ & \text{iff} & C_\Sigma(\sigma_\Sigma(\vec{\phi})) = C_\Sigma(\tau_\Sigma(\vec{\phi})) \text{ for all } \Sigma \in |\mathbf{Sign}|, \vec{\phi} \in \mathrm{SEN}(\Sigma)^n.\end{array} \tag{2}$$

The class $\mathrm{Alg}^N(\mathcal{I})$ is defined next. Note that a class with the same name was defined first in [16], but the definition presented here is slightly different because, here, the model logical morphisms will be required to be surjective, which was not a requirement imposed in [16].

Let $\mathcal{I} = \langle\mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on SEN. The N-algebraic system $\langle\mathrm{SEN}', \langle N', F'\rangle\rangle$ is said to be an $(\mathcal{I}, N)$-**algebraic system** if and only if there exists a surjective (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathcal{I} \to^{se} \mathrm{SEN}'$, such that the $\langle F, \alpha\rangle$-min (N, N')-model $\mathcal{I}' = \langle\mathbf{Sign}', \mathrm{SEN}', C'\rangle$ of $\mathcal{I}$ on SEN' is N'-reduced, i.e., iff $\mathcal{I}'$ is a reduced (N, N')-full model of $\mathcal{I}$ via $\langle F, \alpha\rangle$. Let $\mathrm{Alg}^N(\mathcal{I})$ denote the class of all $(\mathcal{I}, N)$-algebraic systems.

The next proposition relates the two classes $\mathtt{K}^N_\mathcal{I}$ and $\mathrm{Alg}^N(\mathcal{I})$. More specifically, it states that $\mathrm{Alg}^N(\mathcal{I})$ is a subclass of $\mathtt{K}^N_\mathcal{I}$ and that, moreover, the class $\mathtt{K}^N_\mathcal{I}$ is the variety of N-algebraic systems that is generated by the class $\mathrm{Alg}^N(\mathcal{I})$.

Proposition 2 *Let $\mathcal{I} = \langle\mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN. *Then $\mathrm{Alg}^N(\mathcal{I}) \subseteq \mathtt{K}^N_\mathcal{I}$ and, moreover, $\mathtt{K}^N_\mathcal{I} = \mathbf{V}(\mathrm{Alg}^N(\mathcal{I}))$, where* $\mathbf{V}$ *denotes the variety operator (which was shown in Theorem 4 of [24], an analog of Birkhoff's Theorem, to be equal to the operator* **HSP**).

Proof:

Suppose that $\langle\mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathrm{Alg}^N(\mathcal{I})$. Then, there exists, by definition, a surjective (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathcal{I} \to^{se} \mathrm{SEN}'$, such that the $\langle F, \alpha\rangle$-min (N, N')-model $\mathcal{I}' = \langle\mathbf{Sign}', \mathrm{SEN}', C'\rangle$ of $\mathcal{I}$ on SEN' is N'-reduced, i.e., such that $\widetilde{\Omega}^{N'}(C') = \Delta^{\mathrm{SEN}'}$. Now suppose that $\sigma, \tau : \mathrm{SEN}^n \to \mathrm{SEN}$ in N are such that $\mathtt{K}^N_\mathcal{I} \models \sigma \approx \tau$. This means that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^n$, we have that $\langle\sigma_\Sigma(\vec{\phi}), \tau_\Sigma(\vec{\phi})\rangle \in \widetilde{\Omega}^N_\Sigma(\mathcal{I})$. Hence, since $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$ is surjective, by Proposition 8 of [15], we obtain that $\langle\alpha_\Sigma(\sigma_\Sigma(\vec{\phi})), \alpha_\Sigma(\tau_\Sigma(\vec{\phi}))\rangle \in \widetilde{\Omega}^{N'}_{F(\Sigma)}(\mathcal{I}')$. This is equivalent, by the (N, N')-epimorphic property of $\langle F, \alpha\rangle$, to $\langle \sigma_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})), \tau_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi}))\rangle \in \widetilde{\Omega}^{N'}_{F(\Sigma)}(\mathcal{I}')$. This, in turn, once more using the surjectivity of $\langle F, \alpha\rangle$ together with the fact that $\widetilde{\Omega}^{N'}(\mathcal{I}') = \Delta^{\mathrm{SEN}'}$, is equivalent to $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \models \sigma \approx \tau$. Therefore, we obtain that $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}^N_{\mathcal{I}}$, showing that $\mathrm{Alg}^N(\mathcal{I}) \subseteq \mathtt{K}^N_{\mathcal{I}}$.

For the second part, it suffices to observe that $\langle \mathrm{SEN}^N, \langle \overline{N}, \overline{F}\rangle\rangle \in \mathrm{Alg}^N(\mathcal{I})$, which yields that $\mathrm{K}^N_{\mathcal{I}} = \mathbf{V}(\langle \mathrm{SEN}^N, \langle \overline{N}, \overline{F}\rangle\rangle) \subseteq \mathbf{V}(\mathrm{Alg}^N(\mathcal{I}))$. The opposite inclusion follows, of course, from the first part and the fact that $\mathrm{K}^N_{\mathcal{I}}$ is, by definition, a variety. ■

Observe, now, that Proposition 2 yields the following interesting corollary:

Corollary 3 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN. *Then, if the class $\mathrm{Alg}^N(\mathcal{I})$ of all $(\mathcal{I}, N)$-algebraic systems is a variety, it is necessarily equal to the intrinsic N-variety $\mathtt{K}^N_{\mathcal{I}}$ of $\mathcal{I}$.*

4 Hilbert-Based π-Institutions

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on SEN. A binary natural transformation $\Rightarrow: \mathrm{SEN}^2 \to \mathrm{SEN}$ in N will be said to have the **deduction-detachment property** or to be an N-**deduction-detachment term** for $\mathcal{I}$, if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Gamma \cup \{\phi, \psi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\psi \in C_\Sigma(\Gamma, \phi) \quad \text{iff} \quad \phi \Rightarrow_\Sigma \psi \in C_\Sigma(\Gamma),$$

where, of course $\phi \Rightarrow_\Sigma \psi := \Rightarrow_\Sigma (\phi, \psi)$.

A π-institution $\mathcal{I}$, as above, with N a category of natural transformations on SEN, is said to have the N-**uniterm deduction-detachment property (N-uDDP) relative to a binary term** $\Rightarrow$ if $\Rightarrow$ is an N-deduction-detachment term for $\mathcal{I}$ and it is said to have the N-**uniterm deduction-detachment property** if it has the N-uniterm deduction detachment property relative to some binary term.

In the next proposition it is shown that, if a π-institution has the N-deduction-detachment property relative to two different binary natural transformations $\Rightarrow$ and $\Rightarrow'$, then, for every signature Σ and all Σ-sentences ϕ, ψ of $\mathcal{I}$, the Σ-sentences $\phi \Rightarrow_\Sigma \psi$ and $\phi \Rightarrow'_\Sigma \psi$ are interderivable in $\mathcal{I}$, whence, if $\mathcal{I}$ happens to be N-selfextensional, they are also indistinguishable modulo the Tarski N-congruence system $\widetilde{\Omega}^N(\mathcal{I})$.

Proposition 4 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, with N a category of natural transformations on* SEN. *If $\Rightarrow, \Rightarrow'$ are N-deduction-detachment terms for $\mathcal{I}$, then, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $C_\Sigma(\phi \Rightarrow_\Sigma \psi) = C_\Sigma(\phi \Rightarrow'_\Sigma \psi)$. Moreover, if $\mathcal{I}$ is N-selfextensional, then, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $\langle \phi \Rightarrow_\Sigma \psi, \phi \Rightarrow'_\Sigma \psi\rangle \in \widetilde{\Omega}^N_\Sigma(\mathcal{I})$.*

Proof:

Indeed, we obviously have $\phi \Rightarrow_\Sigma \psi \in C_\Sigma(\phi \Rightarrow_\Sigma \psi)$, whence, since $\Rightarrow$ is an N-deduction-detachment term for $\mathcal{I}$, we get that $\psi \in C_\Sigma(\phi, \phi \Rightarrow_\Sigma \psi)$ and, therefore, since $\Rightarrow'$ is also an

N-deduction-detachment term for $\mathcal{I}$, $\phi \Rightarrow'_\Sigma \psi \in C_\Sigma(\phi \Rightarrow_\Sigma \psi)$. Now the conclusion follows by symmetry. The last part of the statement is obvious, since if $\mathcal{I}$ is N-selfextensional, then $\widetilde{\Omega}^N(\mathcal{I}) = \Lambda(\mathcal{I})$. ■

Suppose that SEN : **Sign** $\rightarrow$ **Set** is a set-valued functor and N a category of natural transformations on SEN. A class K of N-algebraic systems is said to be N-**Hilbert-based relative to the binary term** $\Rightarrow$, if $\Rightarrow$: SEN$^2 \rightarrow$ SEN is a binary natural transformation in N, such that the following equations are valid in K:

(H1) $x \Rightarrow x \approx y \Rightarrow y$
(H2) $(x \Rightarrow x) \Rightarrow x \approx x$
(H3) $x \Rightarrow (y \Rightarrow z) \approx (x \Rightarrow y) \Rightarrow (x \Rightarrow z)$
(H4) $(x \Rightarrow y) \Rightarrow ((y \Rightarrow x) \Rightarrow y) \approx (y \Rightarrow x) \Rightarrow ((x \Rightarrow y) \Rightarrow x)$

This is tantamount to saying that, for all $\mathbf{A} = \langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in$ K and all $\Sigma' \in |\mathbf{Sign}'|$, the ordinary universal algebra $\langle \mathrm{SEN}'(\Sigma'), \Rightarrow_{\Sigma'}\rangle$ is a Hilbert algebra. (H1)-(H4) are known as the **Hilbert equations**. A class of N-algebraic systems K will be said to be N-**Hilbert-based** if it is N-Hilbert-based relative to some binary natural transformation.

Let again SEN : **Sign** $\rightarrow$ **Set** be a functor and N a category of natural transformations on SEN. A class Q of N-algebraic systems is said to be N-**pointed** if there exists a natural transformation σ : SEN$^n \rightarrow$ SEN in N, such that, for all $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in$ Q, all $\Sigma' \in |\mathbf{Sign}'|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}'(\Sigma')^n$,

$$\sigma'_{\Sigma'}(\vec{\phi}) = \sigma'_{\Sigma'}(\vec{\psi}),$$

i.e., if and only if Q $\models \sigma(\vec{x}) \approx \sigma(\vec{y})$, for disjoint vectors of variables $\vec{x}, \vec{y}$ of appropriate length. Such an n-ary natural transformation will be referred to as an N-**constant term** since it really behaves like a constant in Q.

It is shown next that every N-Hilbert-based class of N-algebraic systems relative to the binary term $\Rightarrow$ is N-pointed with N-constant term the unary term $1(x) := x \Rightarrow x$.

Proposition 5 *Let* SEN : **Sign** $\rightarrow$ **Set** *be a functor, with N a category of natural transformations on* SEN*, and* K *an N-Hilbert-based class of N-algebraic systems relative to the binary term $\Rightarrow$. Then* K *is N-pointed relative to* $1(x) := x \Rightarrow x$.

Proof:
It suffices to show that, for all $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in$ K, all $\Sigma' \in |\mathbf{Sign}'|$ and all $\phi, \psi \in \mathrm{SEN}'(\Sigma')$, $\phi \Rightarrow'_{\Sigma'} \phi = \psi \Rightarrow'_{\Sigma'} \psi$. But this holds by the hypothesis that K is N-Hilbert-based relative to $\Rightarrow$, since, then, K $\models x \Rightarrow x \approx y \Rightarrow y$. ■

The common value $1'_{\Sigma'}(\phi)$, for all $\phi \in \mathrm{SEN}'(\Sigma')$, will be denoted simply by $1'_{\Sigma'}$. With this in mind, we may proceed to define the binary relation system $\leq' = \{\leq'_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}'|}$. Under the same hypotheses as above, i.e., given a functor SEN, with a category N of natural transformations on SEN, an N-Hilbert-based class K of N-algebraic systems and

$\langle \mathrm{SEN}', \langle N', F' \rangle\rangle \in \mathtt{K}$, we define the binary relation system $\leq' = \{\leq'_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}'|}$ on SEN', for all $\Sigma' \in |\mathbf{Sign}'|$ and all $\phi, \psi \in \mathrm{SEN}'(\Sigma')$, by

$$\phi \leq'_{\Sigma'} \psi \quad \text{iff} \quad \phi \Rightarrow'_{\Sigma'} \psi = 1'_{\Sigma'}.$$

Moreover, given an axiom family $F = \{F_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}'|}$ on SEN', F is said to be an $\Rightarrow$-**implicative filter family on** SEN' if, for all $\Sigma' \in |\mathbf{Sign}'|$,

1. $1'_{\Sigma'} \in F_{\Sigma'}$ and
2. for all $\phi, \psi \in \mathrm{SEN}'(\Sigma')$, $\phi \in F_{\Sigma'}$ and $\phi \Rightarrow'_{\Sigma'} \psi \in F_{\Sigma'}$ imply that $\psi \in F_{\Sigma'}$.

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ be a finitary π-institution, with N a category of natural transformations on SEN. $\mathcal{I}$ is said to be N-**Hilbert-based relative to a binary term** $\Rightarrow$ **and a class of** N-**algebraic systems** $\mathtt{K}$, which is N-Hilbert-based relative to $\Rightarrow$ if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi_0, \ldots, \phi_n, \phi \in \mathrm{SEN}(\Sigma)$,

$$\phi \in C_\Sigma(\phi_0, \ldots, \phi_n)$$

if and only if, for all $\langle \mathrm{SEN}', \langle N', F' \rangle\rangle \in \mathtt{K}$ and all surjective (N, N')-epimorphic translations $\langle F, \alpha \rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}'$, $\alpha_\Sigma(\phi_0 \Rightarrow_\Sigma (\cdots \Rightarrow_\Sigma (\phi_{n-1} \Rightarrow_\Sigma (\phi_n \Rightarrow_\Sigma \phi)) \cdots)) = 1'_{F(\Sigma)}$ and

$$\phi \in C_\Sigma(\emptyset)$$

if and only if, for all $\langle \mathrm{SEN}', \langle N', F' \rangle\rangle \in \mathtt{K}$ and all surjective (N, N')-epimorphic translations $\langle F, \alpha \rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}'$, $\alpha_\Sigma(\phi) = 1'_{F(\Sigma)}$.

Note that, if $\mathcal{I}$ is N-Hilbert-based relative to $\Rightarrow$ and $\mathtt{K}$, then it is N-Hilbert-based relative to $\Rightarrow$ and the class $\mathrm{cor}^N(\mathtt{K})$ of all those N-algebraic systems $\langle \mathrm{SEN}', \langle N', F' \rangle\rangle$ in $\mathtt{K}$ for which there exists at least one surjective (N, N')-epimorphic translation $\langle F, \alpha \rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}'$. This class will be called the N-**core of** $\mathtt{K}$.

In the condition $\alpha_\Sigma(\phi_0 \Rightarrow_\Sigma (\cdots \Rightarrow_\Sigma (\phi_{n-1} \Rightarrow_\Sigma (\phi_n \Rightarrow_\Sigma \phi)) \cdots)) = 1'_{F(\Sigma)}$ the order of the ϕ_i's is not important because permuting them in any way will not change the value of the resulting expression in $\mathrm{SEN}(\Sigma)$. Also, from now on, when the expression "$\mathcal{I}$ is N-Hilbert-based relative to $\Rightarrow$ and $\mathtt{K}$" is used, it will always be assumed that $\mathtt{K}$ is N-Hilbert-based relative to $\Rightarrow$.

Finally, a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, with N a category of natural transformations on SEN, is said to be N-**Hilbert-based** if it is N-Hilbert-based relative to some binary natural transformation $\Rightarrow$ and to some N-Hilbert-based class $\mathtt{K}$ of N-algebraic systems relative to $\Rightarrow$.

The next proposition, an analog of Proposition 5 of [13], shows that, if the π-institution $\mathcal{I}$ is N-Hilbert-based relative to $\Rightarrow$ and $\mathtt{K}$ and also relative to $\Rightarrow'$ and $\mathtt{K}'$, then the varieties of N-algebraic systems that are generated by $\mathrm{cor}^N(\mathtt{K})$ and $\mathrm{cor}^N(\mathtt{K}')$ in the sense of [21, 24] coincide.

Proposition 6 *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ *be a* π*-institution, with* N *a category of natural transformations on* SEN*. If* $\mathcal{I}$ *is* N*-Hilbert-based relative to* $\Rightarrow$ *and* $\mathtt{K}$ *and also relative to* $\Rightarrow'$ *and* $\mathtt{K}'$*, then* $\mathbf{V}^N(\mathrm{cor}^N(\mathtt{K})) = \mathbf{V}^N(\mathrm{cor}^N(\mathtt{K}'))$.

Proof:

Suppose that $\sigma, \tau : \mathrm{SEN}^n \to \mathrm{SEN}$ is an N-equation, such that $\mathtt{K} \models \sigma \approx \tau$. This means that, for every $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}$, all $\Sigma' \in |\mathbf{Sign}'|$ and all $\vec{\phi} \in \mathrm{SEN}'(\Sigma')^n$, we have that $\sigma'_{\Sigma'}(\vec{\phi}) = \tau'_{\Sigma'}(\vec{\phi})$. Therefore, for all $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}$, all surjective $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^n$, $\sigma'_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})) = \tau'_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi}))$, which is equivalent to $\alpha_\Sigma(\sigma_\Sigma(\vec{\phi})) = \alpha_\Sigma(\tau_\Sigma(\vec{\phi}))$. Since $\mathtt{K}$ is N-Hilbert-based relative to $\Rightarrow$, this yields that $\alpha_\Sigma(\sigma_\Sigma(\vec{\phi}) \Rightarrow_\Sigma \tau_\Sigma(\vec{\phi})) = \alpha_\Sigma(\tau_\Sigma(\vec{\phi}) \Rightarrow_\Sigma \sigma_\Sigma(\vec{\phi})) = 1'_{F(\Sigma)}$. Therefore, since $\mathcal{I}$ is N-Hilbert-based relative to $\Rightarrow$ and $\mathtt{K}$, we obtain that $C_\Sigma(\sigma_\Sigma(\vec{\phi})) = C_\Sigma(\tau_\Sigma(\vec{\phi}))$. Reversing all the steps in the above derivation, but using $\Rightarrow'$ and $\mathtt{K}'$ in place of $\Rightarrow$ and $\mathtt{K}$, respectively, we obtain that, for all $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}'$, all $\Sigma' \in |\mathbf{Sign}'|$ and all $\vec{\phi} \in \mathrm{SEN}'(\Sigma')^n$, we have that $\sigma'_{\Sigma'}(\vec{\phi}) = \tau'_{\Sigma'}(\vec{\phi})$, i.e., that $\mathtt{K}' \models \sigma \approx \tau$.

The converse statement, i.e., that every N-identity of $\mathtt{K}'$ is also an N-identity of $\mathtt{K}$ is obtained by a symmetric reasoning. ■

Proposition 6 shows that, if $\mathcal{I}$ is N-Hilbert-based, then, there exists essentially only one variety relative to which $\mathcal{I}$ is N-Hilbert-based. It is the variety of N-algebraic systems that is generated by $\mathrm{cor}^N(\mathtt{K})$, for any class $\mathtt{K}$ of N-algebraic systems relative to which $\mathcal{I}$ is N-Hilbert based. It will be denoted by $\mathbf{V}^N(\mathcal{I})$. Moreover, since, if $\mathcal{I}$ is N-Hilbert-based relative to $\mathtt{K}$, we have that, for all $\sigma, \tau : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, $C_\Sigma(\sigma_\Sigma(\vec{\phi})) = C_\Sigma(\tau_\Sigma(\vec{\phi}))$, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^n$, if and only if $\mathrm{cor}^N(\mathtt{K}) \models \sigma \approx \tau$, we also obtain that

$$C_\Sigma(\sigma_\Sigma(\vec{\phi})) = C_\Sigma(\tau_\Sigma(\vec{\phi})), \text{ for all } \Sigma \in |\mathbf{Sign}|, \vec{\phi} \in \mathrm{SEN}(\Sigma)^n, \quad \text{iff} \quad \mathbf{V}^N(\mathcal{I}) \models \sigma \approx \tau. \qquad (3)$$

The following proposition is an analog of Proposition 7 of [13] for π-institutions. More precisely, Proposition 7 shows that, if $\mathcal{I}$ is an N-Hilbert-based π-institution relative to $\Rightarrow$, then it is N-selfextensional, $\Rightarrow$ is an N-deduction-detachment term for $\mathcal{I}$ and, also, that $\mathbf{V}^N(\mathcal{I}) = \mathrm{K}^N_{\mathcal{I}}$.

Proposition 7 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be an N-Hilbert-based π-institution relative to $\Rightarrow$. Then*

1. *$\Rightarrow$ is an N-deduction-detachment term for $\mathcal{I}$;*
2. *$\mathcal{I}$ is N-selfextensional;*
3. *$\mathbf{V}^N(\mathcal{I}) = \mathrm{K}^N_{\mathcal{I}}$, showing that $\mathcal{I}$ is N-Hilbert-based relative to its intrinsic N-variety $\mathtt{K}^N_{\mathcal{I}}$.*

Proof:

Suppose that $\mathcal{I}$ is an N-Hilbert-based π-institution relative to $\Rightarrow$ and the variety $\mathtt{K}$.

1. Let $\Sigma \in |\mathbf{Sign}|$ and $\Gamma \cup \{\phi, \psi\} \subseteq \mathrm{SEN}(\Sigma)$. If $\psi \in C_\Sigma(\Gamma, \phi)$, then, by finitarity, there exist $\phi_0, \ldots, \phi_{n-1} \in \Gamma$, such that $\psi \in C_\Sigma(\phi_0, \ldots, \phi_{n-1}, \phi)$ or $\psi \in C_\Sigma(\phi)$. Hence, since $\mathcal{I}$ is N-Hilbert-based relative to $\Rightarrow$ and $\mathtt{K}$, we get that, for all $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}$ and all surjective $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, $\alpha_\Sigma(\phi_0 \Rightarrow_\Sigma (\cdots \Rightarrow_\Sigma (\phi \Rightarrow_\Sigma \psi) \cdots)) = 1'_{F(\Sigma)}$

or $\alpha_\Sigma(\phi \Rightarrow_\Sigma \psi) = 1'_{F(\Sigma)}$. Therefore, we obtain that $\phi \Rightarrow_\Sigma \psi \in C_\Sigma(\phi_0,\ldots,\phi_{n-1}) \subseteq C_\Sigma(\Gamma)$ or $\phi \Rightarrow_\Sigma \psi \in C_\Sigma(\emptyset) \subseteq C_\Sigma(\Gamma)$.

If, conversely, $\phi \Rightarrow_\Sigma \psi \in C_\Sigma(\Gamma)$, then, either $\phi \Rightarrow_\Sigma \psi \in C_\Sigma(\emptyset)$ or, there exist $\phi_0,\ldots,\phi_{n-1} \in \Gamma$, such that $\phi \Rightarrow_\Sigma \psi \in C_\Sigma(\phi_0,\ldots,\phi_{n-1})$. Thus, we get that, for all $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}$ and all surjective $\langle F, \alpha\rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}'$, $\alpha_\Sigma(\phi \Rightarrow_\Sigma \psi) = 1'_{F(\Sigma)}$ or $\alpha_\Sigma(\phi_0 \Rightarrow_\Sigma (\cdots \Rightarrow_\Sigma (\phi_{n-1} \Rightarrow_\Sigma (\phi \Rightarrow_\Sigma \psi))\cdots)) = 1'_{F(\Sigma)}$. Therefore $\psi \in C_\Sigma(\phi) \subseteq C_\Sigma(\Gamma,\phi)$ or $\psi \in C_\Sigma(\phi_0,\ldots,\phi_{n-1},\phi) \subseteq C_\Sigma(\Gamma,\phi)$.

This concludes the proof that $\Rightarrow$ is an N-deduction-detachment term for $\mathcal{I}$.

2. For this part it suffices to show that $\Lambda(\mathcal{I})$ is an N-congruence system of $\mathcal{I}$. To this end, suppose that $\tau : \mathrm{SEN}^n \to \mathrm{SEN}$ is in N, $\Sigma \in |\mathbf{Sign}|$ and $\phi_i, \psi_i \in \mathrm{SEN}(\Sigma), i < n$, such that $\langle \phi_i, \psi_i\rangle \in \Lambda_\Sigma(\mathcal{I})$, for all $i < n$. This shows that $C_\Sigma(\phi_i) = C_\Sigma(\psi_i)$, for all $i < n$. Therefore, since $\mathcal{I}$ is N-Hilbert-based relative to $\Rightarrow$, we get, by Part 1, that for all $i < n$, $\phi_i \Rightarrow_\Sigma \psi_i, \psi_i \Rightarrow_\Sigma \phi_i \in C_\Sigma(\emptyset)$. Therefore, again by the N-Hilbert-based property of $\mathcal{I}$, we get that $\tau_\Sigma(\phi_0,\ldots,\phi_{n-1}) \Rightarrow_\Sigma \tau_\Sigma(\psi_0,\ldots,\psi_{n-1}) \in C_\Sigma(\{\phi_i \Rightarrow_\Sigma \psi_i, \psi_i \Rightarrow_\Sigma \phi_i : i < n\}) \subseteq C_\Sigma(\emptyset)$, whence, again by Part 1, we obtain that $\tau_\Sigma(\psi_0,\ldots,\psi_{n-1}) \in C_\Sigma(\tau_\Sigma(\phi_0,\ldots,\phi_{n-1}))$. Now, by symmetry, we get that $C_\Sigma(\tau_\Sigma(\phi_0,\ldots,\phi_{n-1})) = C_\Sigma(\tau_\Sigma(\psi_0,\ldots,\psi_{n-1}))$, which shows that
$$\langle \tau_\Sigma(\phi_0,\ldots,\phi_{n-1}), \tau_\Sigma(\psi_0,\ldots,\psi_{n-1})\rangle \in \Lambda_\Sigma(\mathcal{I})$$
and, hence, $\Lambda(\mathcal{I})$ is an N-congruence system on SEN.

3. This part follows from the fact that, by 2, $\mathcal{I}$ is N-selfextensional and by putting together the Equivalences (2) and the Equivalence (3).

■

Corollary 8 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN, *is an N-Hilbert-based π-institution relative to $\Rightarrow$ and also relative to $\Rightarrow'$. Then, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $C_\Sigma(\phi \Rightarrow_\Sigma \psi) = C_\Sigma(\phi \Rightarrow'_\Sigma \psi)$ and $\langle \phi \Rightarrow_\Sigma \psi, \phi \Rightarrow'_\Sigma \psi\rangle \in \widetilde{\Omega}^N_\Sigma(\mathcal{I})$.*

Proof:

By Part 1 of Proposition 7, we have that both $\Rightarrow$ and $\Rightarrow'$ are N-deduction-detachment terms for $\mathcal{I}$, whence we get $\phi \Rightarrow_\Sigma \psi \in C_\Sigma(\phi \Rightarrow_\Sigma \psi)$ implies $\psi \in C_\Sigma(\phi, \phi \Rightarrow_\Sigma \psi)$ and, hence, $\phi \Rightarrow'_\Sigma \psi \in C_\Sigma(\phi \Rightarrow_\Sigma \psi)$. Therefore, by symmetry, we obtain that $C_\Sigma(\phi \Rightarrow_\Sigma \psi) = C_\Sigma(\phi \Rightarrow'_\Sigma \psi)$.

The previous part shows that $\langle \phi \Rightarrow_\Sigma \psi, \phi \Rightarrow'_\Sigma \psi\rangle \in \Lambda_\Sigma(\mathcal{I})$, whence, since, by Part 2 of Proposition 7, $\mathcal{I}$ is N-selfextensional, we obtain that $\langle \phi \Rightarrow_\Sigma \psi, \phi \Rightarrow'_\Sigma \psi\rangle \in \widetilde{\Omega}^N_\Sigma(\mathcal{I})$. ■

The following theorem, an analog of Theorem 9 of [13], characterizes N-Hilbert-based π-institutions as those that are at the same time N-selfextensional and possess the N-uniterm deduction-detachment property.

Theorem 9 *A finitary π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN, *is N-selfextensional and has the N-uniterm deduction-detachment property if and only if it is N-Hilbert-based.*

Proof:
If $\mathcal{I}$ is N-Hilbert-based, then it is finitary, by definition. Moreover, Proposition 7 shows that, if $\mathcal{I}$ is N-Hilbert-based, then it is N-selfextensional and has the N-uniterm deduction-detachment property.

Suppose, conversely, that $\mathcal{I}$ is a finitary N-selfextensional π-institution, that has the N-uniterm deduction-detachment property relative to the binary natural transformation $\Rightarrow$. Consider the N-algebraic system $\langle \mathrm{SEN}^N, \langle \overline{N}, \overline{F}\rangle\rangle$. It is not difficult to see that $\{\langle \mathrm{SEN}^N, \langle \overline{N}, \overline{F}\rangle\rangle\}$ is N-Hilbert-based relative to $\Rightarrow$. Let $\Sigma \in |\mathbf{Sign}|, \phi_0, \ldots, \phi_{n-1}, \phi \in \mathrm{SEN}(\Sigma)$. Then we have

$$\begin{array}{lll}
\phi \in C_\Sigma(\phi_0, \ldots, \phi_{n-1}) & \text{iff} & \phi_0 \Rightarrow_\Sigma (\cdots \Rightarrow_\Sigma (\phi_{n-1} \Rightarrow_\Sigma \phi) \cdots) \in C_\Sigma(\emptyset) \\
& & \text{(by the } N\text{-uniterm DDT)} \\
& \text{iff} & C_\Sigma(\phi_0 \Rightarrow_\Sigma (\cdots \Rightarrow_\Sigma (\phi_{n-1} \Rightarrow_\Sigma \phi) \cdots)) = C_\Sigma(\phi \Rightarrow_\Sigma \phi) \\
& & \text{(since } \phi \Rightarrow_\Sigma \phi \in \mathrm{Thm}_\Sigma) \\
& \text{iff} & \langle \phi_0 \Rightarrow_\Sigma (\cdots \Rightarrow_\Sigma (\phi_{n-1} \Rightarrow_\Sigma \phi) \cdots), \phi \Rightarrow_\Sigma \phi\rangle \in \widetilde{\Omega}^N_\Sigma(\mathcal{I}) \\
& & \text{(since } \Lambda(\mathcal{I}) = \widetilde{\Omega}^N(\mathcal{I}) \text{ by } N\text{-selfextensionality)} \\
& \text{iff} & \text{for all surjective } \langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}^N, \\
& & \alpha_\Sigma(\phi_0 \Rightarrow_\Sigma (\cdots \Rightarrow_\Sigma (\phi_{n-1} \Rightarrow_\Sigma \phi) \cdots)) = 1^N_{F(\Sigma)}
\end{array}$$

and, also,

$$\begin{array}{lll}
\phi \in C_\Sigma(\emptyset) & \text{iff} & C_\Sigma(\phi) = C_\Sigma(\phi \Rightarrow_\Sigma \phi) \\
& \text{iff} & \langle \phi, \phi \Rightarrow_\Sigma \phi\rangle \in \widetilde{\Omega}^N_\Sigma(\mathcal{I}) \\
& \text{iff} & \text{for all surjective } \langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}^N, \\
& & \alpha_\Sigma(\phi) = 1^N_{F(\Sigma)}.
\end{array}$$

The above equivalences, together with the finitarity of $\mathcal{I}$, show that $\mathcal{I}$ is N-Hilbert-based relative to $\Rightarrow$ and the variety $\mathtt{K}^N_{\mathcal{I}} := \mathbf{V}^N(\langle \mathrm{SEN}^N, \langle \overline{N}, \overline{F}\rangle\rangle)$. ■

Suppose, next, that $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is a set-valued functor, with N a category of natural transformations on SEN. Let $\mathtt{K}$ be an N-Hilbert-based variety of N-algebraic systems relative to $\Rightarrow$, such that $\mathtt{K} = \mathbf{V}^N(\mathrm{cor}^N(\mathtt{K}))$. The finitary π-institution $\mathcal{I}^{\Rightarrow,\mathtt{K}} = \langle \mathbf{Sign}, \mathrm{SEN}, C^{\Rightarrow,\mathtt{K}}\rangle$ is defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi_0, \ldots, \phi_n, \phi \in \mathrm{SEN}(\Sigma)$, by

$$\begin{array}{lll}
\phi \in C^{\Rightarrow,\mathtt{K}}_\Sigma(\phi_0, \ldots, \phi_n) & \text{iff} & \text{for all } \langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K} \\
& & \text{and surjective } \langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}', \\
& & \alpha_\Sigma(\phi_0 \Rightarrow_\Sigma (\cdots \Rightarrow_\Sigma (\phi_n \Rightarrow_\Sigma \phi) \cdots)) = 1'_{F(\Sigma)}
\end{array}$$

and, also,

$$\begin{array}{lll}
\phi \in C^{\Rightarrow,\mathtt{K}}_\Sigma(\emptyset) & \text{iff} & \text{for all } \langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K} \\
& & \text{and surjective } \langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}', \\
& & \alpha_\Sigma(\phi) = 1'_{F(\Sigma)}.
\end{array}$$

It follows directly from the definitions involved:

Proposition 10 *Let* SEN : **Sign** $\to$ **Set** *be a functor and* N *a category of natural transformations on* SEN. *Then, for every* N*-Hilbert-based variety* $\mathtt{K}$ *of* N*-algebraic systems relative to* $\Rightarrow$, *such that* $\mathtt{K} = \mathbf{V}^N(\mathrm{cor}^N(\mathtt{K}))$, *the* π*-institution* $\mathcal{I}^{\Rightarrow,\mathtt{K}}$ *is* N*-Hilbert-based relative to* $\Rightarrow$ *and* $\mathtt{K}$ *and* $\mathbf{V}^N(\mathcal{I}^{\Rightarrow,\mathtt{K}}) = \mathtt{K}$.

Let, again, SEN : **Sign** $\to$ **Set** be a set-valued functor, with N a category of natural transformations on SEN. Fix a binary natural transformation $\Rightarrow$ in N. From the results proven so far it will follow that, if SEN is symmetrically N-rule based, then there is a bijection between the N-Hilbert-based π-institutions on SEN relative to $\Rightarrow$ and the N-Hilbert-based varieties $\mathtt{K}$ of N-algebraic systems relative to $\Rightarrow$, such that $\mathtt{K} = \mathbf{V}^N(\mathrm{cor}^N(\mathtt{K}))$. This bijection, generalizing the corresponding bijection established in [13] between deductive systems and ordinary universal algebraic varieties, is a dual isomorphism if one takes into account the extension ordering of the π-institutions on SEN and the subvariety ordering on the class of varieties of N-algebraic systems.

An N-Hilbert-based π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is completely determined by its Frege relation, i.e., by all pairs of Σ-sentences that are Σ-interderivable. Moreover, as the following proposition shows, the extension relation between N-Hilbert-based π-institutions on the same functor corresponds to the inclusion relation between the corresponding Frege relations. This results forms an analog at the present level of Proposition 11 of [13].

Proposition 11 *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle, \mathcal{I}' = \langle \mathbf{Sign}, \mathrm{SEN}, C'\rangle$, *with* N *a category of natural transformations on* SEN, *are two* N*-Hilbert-based* π*-institutions. Then*

$$\Lambda(\mathcal{I}) \leq \Lambda(\mathcal{I}') \quad \textit{if and only if} \quad C \leq C'.$$

as a consequence, if $\Lambda(\mathcal{I}) = \Lambda(\mathcal{I}')$, *then* $\mathcal{I} = \mathcal{I}'$.

Proof:

Suppose, first, that $C \leq C'$. Let $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi\rangle \in \Lambda_\Sigma(\mathcal{I})$. Then $C_\Sigma(\phi) = C_\Sigma(\psi)$, whence $\psi \in C_\Sigma(\phi) \subseteq C'_\Sigma(\phi)$. Therefore, by symmetry, $C'_\Sigma(\phi) = C'_\Sigma(\psi)$, which shows that $\langle \phi, \psi\rangle \in \Lambda_\Sigma(\mathcal{I}')$. Thus, we get that $\Lambda(\mathcal{I}) \leq \Lambda(\mathcal{I}')$.

Suppose, conversely, that $\Lambda(\mathcal{I}) \leq \Lambda(\mathcal{I}')$. Let $\Sigma \in |\mathbf{Sign}|, \phi_0, \ldots, \phi_{n-1}, \phi \in \mathrm{SEN}(\Sigma)$. We have

$$\begin{array}{lll}
\phi \in C_\Sigma(\phi_0, \ldots, \phi_{n-1}) & \text{iff} & \phi_0 \Rightarrow_\Sigma (\cdots \Rightarrow_\Sigma (\phi_{n-1} \Rightarrow_\Sigma \phi)\cdots) \in C_\Sigma(\emptyset)\\
& \text{iff} & C_\Sigma(\phi_0 \Rightarrow_\Sigma (\cdots \Rightarrow_\Sigma (\phi_{n-1} \Rightarrow_\Sigma \phi)\cdots)) = C_\Sigma(\phi \Rightarrow_\Sigma \phi)\\
& \text{implies} & C'_\Sigma(\phi_0 \Rightarrow_\Sigma (\cdots \Rightarrow_\Sigma (\phi_{n-1} \Rightarrow_\Sigma \phi)\cdots)) = C'_\Sigma(\phi \Rightarrow_\Sigma \phi)\\
& \text{iff} & \phi_0 \Rightarrow_\Sigma (\cdots \Rightarrow_\Sigma (\phi_{n-1} \Rightarrow_\Sigma \phi)\cdots) \in C'_\Sigma(\emptyset)\\
& \text{iff} & \phi \in C'_\Sigma(\phi_0, \ldots, \phi_{n-1}),
\end{array}$$

and, also,

$$\begin{array}{lll}
\phi \in C_\Sigma(\emptyset) & \text{iff} & C_\Sigma(\phi) = C_\Sigma(\phi \Rightarrow_\Sigma \phi)\\
& \text{implies} & C'_\Sigma(\phi) = C'_\Sigma(\phi \Rightarrow_\Sigma \phi)\\
& \text{iff} & \phi \in C'_\Sigma(\emptyset).
\end{array}$$

Therefore $\mathcal{I}'$ is in fact an extension of $\mathcal{I}$. ■

The following theorem, an analog of Theorem 12 of [13] is one of the main theorems of the paper. It asserts a dual correspondence between N-Hilbert-based π-institutions on an underlying functor SEN, that is symmetrically N-rule based, ordered under extension, and varieties $\mathtt{K}$ of N-algebraic systems, such that $\mathtt{K} = \mathbf{V}^N(\mathrm{cor}^N(\mathtt{K}))$, ordered under the subvariety relation. To formulate the statement of the theorem, given a functor SEN : **Sign** $\to$ **Set**, with N a category of natural transformations on SEN, and $\Rightarrow$: $\mathrm{SEN}^2 \to \mathrm{SEN}$ in N, let us denote by $\mathtt{K}^N_{\Rightarrow}$ the variety of N-algebraic systems that is axiomatized by the Hilbert equations (H1)-(H4).

Theorem 12 *Let* SEN : **Sign** $\to$ **Set***, with* N *a category of natural transformations on* SEN*, be a symmetrically* N*-rule based functor and* $\Rightarrow$: $\mathrm{SEN}^2 \to \mathrm{SEN}$ *a binary natural transformation in* N*. Then, there exists a dual isomorphism between the collection of all* N*-Hilbert-based* π*-institutions on* SEN *relative to* $\Rightarrow$*, ordered by extension, and the collection of all subvarieties* $\mathtt{K}$ *of the variety* $\mathtt{K}^N_{\Rightarrow}$*, such that* $\mathtt{K} = \mathbf{V}^N(\mathrm{cor}^N(\mathtt{K}))$*, ordered by inclusion. The isomorphism sends* $\mathcal{I}$ *to* $\mathtt{K}^N_{\mathcal{I}}$*.*

Proof:

Recall from the Equivalences (2) that, for an N-selfextensional π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, the Frege relation $\Lambda(\mathcal{I})$ determines exactly the equations that hold in the variety $\mathtt{K}^N_{\mathcal{I}}$, since

$$\mathrm{K}^N_{\mathcal{I}} \models \sigma \approx \tau \quad \text{iff,} \quad \text{for all } \Sigma \in |\mathbf{Sign}|, \vec{\phi} \in \mathrm{SEN}(\Sigma)^n,\ C_\Sigma(\sigma_\Sigma(\vec{\phi})) = C_\Sigma(\tau_\Sigma(\vec{\phi})).$$

This fact, together with the symmetric N-rule basedness of SEN, implies that, if $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle, \mathcal{I}' = \langle \mathbf{Sign}, \mathrm{SEN}, C'\rangle$ are N-Hilbert-based π-institutions relative to $\Rightarrow$ and $\mathtt{K}^N_{\mathcal{I}} = \mathtt{K}^N_{\mathcal{I}'}$, then $\Lambda(\mathcal{I}) = \Lambda(\mathcal{I}')$.

To see this, suppose that $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi\rangle \in \Lambda_\Sigma(\mathcal{I})$. Then $C_\Sigma(\phi) = C_\Sigma(\psi)$. Thus, by the symmetric N-rule basedness of $\mathcal{I}$, we get that, there exist natural transformations $\sigma^{\langle \Sigma,\phi\rangle}, \sigma^{\langle \Sigma,\psi\rangle} : \mathrm{SEN}^k \to \mathrm{SEN}$ in N and $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, such that $\sigma^{\langle \Sigma,\phi\rangle}_\Sigma(\vec{\chi}) = \phi$, $\sigma^{\langle \Sigma,\psi\rangle}_\Sigma(\vec{\chi}) = \psi$ and both $\langle \{\sigma^{\langle \Sigma,\phi\rangle}\}, \sigma^{\langle \Sigma,\psi\rangle}\rangle$ and $\langle \{\sigma^{\langle \Sigma,\psi\rangle}\}, \sigma^{\langle \Sigma,\phi\rangle}\rangle$ are N-rules of $\mathcal{I}$. This implies, by the displayed condition above, that $\mathtt{K}^N_{\mathcal{I}} \models \sigma^{\langle \Sigma,\phi\rangle} \approx \sigma^{\langle \Sigma,\psi\rangle}$, whence, since $\mathtt{K}^N_{\mathcal{I}} = \mathtt{K}^N_{\mathcal{I}'}$, we get that $\mathtt{K}^N_{\mathcal{I}'} \models \sigma^{\langle \Sigma,\phi\rangle} \approx \sigma^{\langle \Sigma,\psi\rangle}$. Now, by reversing the steps in the preceding deduction, we obtain that $C'_\Sigma(\phi) = C'_\Sigma(\psi)$, i.e., that $\langle \phi, \psi\rangle \in \Lambda_\Sigma(\mathcal{I}')$. By symmetry, we now get that $\Lambda(\mathcal{I}) = \Lambda(\mathcal{I}')$.

Thus, by Proposition 11, we now get that $\mathcal{I} = \mathcal{I}'$. Therefore the function $\mathcal{I} \mapsto \mathtt{K}^N_{\mathcal{I}}$ is injective. It is onto, since, by Proposition 10, every N-Hilbert-based variety $\mathtt{K}$ of N-algebraic systems, such that $\mathtt{K} = \mathbf{V}^N(\mathrm{cor}^N(\mathtt{K}))$, defines an N-Hilbert-based π-institution $\mathcal{I}^{\Rightarrow,\mathtt{K}}$, such that, by Proposition 7, $\mathrm{K}^N_{\mathcal{I}^{\Rightarrow,\mathtt{K}}} = \mathbf{V}^N(\mathcal{I}^{\Rightarrow,\mathtt{K}}) = \mathtt{K}$. Finally, by Proposition 11, it follows that $\mathcal{I}'$ is an extension of $\mathcal{I}$ if and only if $\mathrm{K}^N_{\mathcal{I}}$ is a subvariety of $\mathtt{K}^N_{\mathcal{I}'}$, whence the function $\mathcal{I} \mapsto \mathtt{K}^N_{\mathcal{I}}$ is indeed an order reversing isomorphism. ■

It will be shown next that, for any N-selfextensional π-institution $\mathcal{I}$ with the N-uniterm deduction-detachment property, the class $\mathrm{Alg}^N(\mathcal{I})$ of all $(\mathcal{I}, N)$-algebraic systems coincides with the intrinsic N-variety $\mathtt{K}^N_{\mathcal{I}}$ of $\mathcal{I}$.

First, we prove that the closure systems on the min models via surjective logical morphisms of an N-Hilbert-based π-institution that are based on N-algebraic systems in the intrinsic N-variety of the π-institution consist exactly of the implicative filter families of these N-algebraic systems.

To do this, the characterization of the closure systems of min models of finitary π-institutions via surjective logical morphisms that was provided in Lemma 2.1 of [17] will be used. The reader is encouraged to recall this result before studying the proof of Lemma 13.

Lemma 13 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, is an N-Hilbert-based π-institution relative to $\Rightarrow$. Then, for all N-algebraic systems $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathrm{K}^N_{\mathcal{I}}$, all surjective (N, N')-epimorphic translations $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, and all $\Sigma' \in |\mathbf{Sign}'|$, the Σ'-theories of the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on* SEN$'$ *coincide with the Σ'-components of the $\Rightarrow$-implicative filter families of $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle$.*

Proof:
Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is an N-Hilbert-based π-institution relative to $\Rightarrow$, $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}^N_{\mathcal{I}}$ and $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$ a surjective (N, N')-epimorphic translation. In addition, let $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ be the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on SEN$'$.

Assume, first, that $F_{\Sigma'}$ is a Σ'-theory of $\mathcal{I}'$. Since, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$, $\phi \Rightarrow_\Sigma \phi \in C_\Sigma(\emptyset)$ and $\psi \in C_\Sigma(\phi, \phi \Rightarrow_\Sigma \psi)$, and $\langle F, \alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$, we obtain that $1'_{\Sigma'} \in F_{\Sigma'}$ and that, if $\phi', \phi' \Rightarrow'_{\Sigma'} \psi' \in F_{\Sigma'}$, then $\psi' \in F_{\Sigma'}$. Thus, F_Σ is indeed a Σ'-component of an $\Rightarrow$-implicative filter family of $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle$.

Suppose, conversely, that $F_{\Sigma'}$ is the Σ'-component of an $\Rightarrow$-implicative filter family of $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle$. It suffices to show that $C'_{\Sigma'}(F_{\Sigma'}) \subseteq F_{\Sigma'}$. Let $\phi' \in C'_{\Sigma'}(F_{\Sigma'})$. Then, there exist $\phi'_0, \ldots, \phi'_{n-1} \in F_{\Sigma'}$, such that $\phi' \in C'_{\Sigma'}(\phi'_0, \ldots, \phi'_{n-1})$. Thus, there exists $\Sigma \in |\mathbf{Sign}|, \phi_0, \ldots, \phi_{n-1}, \phi \in \mathrm{SEN}(\Sigma)$, such that $F(\Sigma) = \Sigma'$ and $\alpha_\Sigma(\phi_0) = \phi'_0, \ldots, \alpha_\Sigma(\phi_{n-1}) = \phi'_{n-1}, \alpha_\Sigma(\phi) = \phi'$. Therefore $\alpha_\Sigma(\phi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\phi_0), \ldots, \alpha_\Sigma(\phi_{n-1}))$. Hence, using the notation of Lemma 2.1 of [17], we get that $\alpha_\Sigma(\phi) \in \bigcup_{n\geq 0} X^n_{F(\Sigma)}(\alpha_\Sigma(\phi_0), \ldots, \alpha_\Sigma(\phi_{n-1}))$. So to show that $\alpha_\Sigma(\phi) \in F_{F(\Sigma)}$, it suffices to show, by induction on $n \geq 0$, that

$$X^n_{F(\Sigma)}(\alpha_\Sigma(\phi_0), \ldots, \alpha_\Sigma(\phi_{n-1})) \subseteq F_{F(\Sigma)}.$$

For $n = 0$, we have that $X^0_{F(\Sigma)}(\alpha_\Sigma(\phi_0), \ldots, \alpha_\Sigma(\phi_{n-1})) = \{\alpha_\Sigma(\phi_0), \ldots, \alpha_\Sigma(\phi_{n-1})\} \subseteq F_{F(\Sigma)}$. As the induction hypothesis, suppose that $X^k_{F(\Sigma)}(\alpha_\Sigma(\phi_0), \ldots, \alpha_\Sigma(\phi_{n-1})) \subseteq F_{F(\Sigma)}$. For the induction step, let $\chi' \in X^{k+1}_{F(\Sigma)}(\alpha_\Sigma(\phi_0), \ldots, \alpha_\Sigma(\phi_{n-1}))$. Then, there exist, by definition, $\chi_0, \ldots, \chi_{m-1}, \chi \in \mathrm{SEN}(\Sigma)$, such that $\alpha_\Sigma(\chi) = \chi'$, $\alpha_\Sigma(\chi_i) \in X^k_{F(\Sigma)}(\alpha_\Sigma(\phi_0), \ldots, \alpha_\Sigma(\phi_{n-1}))$, for all $i < m$, and $\chi \in C_\Sigma(\chi_0, \ldots, \chi_{m-1})$. Now, since $\mathcal{I}$ is N-Hilbert-based relative to $\Rightarrow$ and $\mathtt{K}^N_{\mathcal{I}}$, we obtain that $\alpha_\Sigma(\chi_0 \Rightarrow_\Sigma (\cdots \Rightarrow_\Sigma (\chi_{m-1} \Rightarrow_\Sigma \chi) \cdots)) = 1'_{F(\Sigma)}$. Thus, since, by the inductive hypothesis, $\alpha_\Sigma(\chi_0), \ldots, \alpha_\Sigma(\chi_{m-1}) \in X^k_{F(\Sigma)}(\alpha_\Sigma(\phi_0), \ldots, \alpha_\Sigma(\phi_{n-1})) \subseteq F_{F(\Sigma)}$, we get, taking into account that $F_{F(\Sigma)}$ is the Σ'-component of an $\Rightarrow$-implicative filter family of

$\langle \mathrm{SEN}', \langle N', F' \rangle \rangle$, that $\chi' = \alpha_\Sigma(\chi) \in F_{F(\Sigma)}$. This concludes the inductive step and the proof that $C'_{\Sigma'}(F_{\Sigma'}) \subseteq F_{\Sigma'}$. ■

In the following lemma, an analog of Lemma 15 of [13] for π-institutions, it is shown that the Frege equivalence system of every min model via a surjective logical morphism of an N-Hilbert-based π-institution $\mathcal{I}$ on an N-algebraic system belonging to $\mathtt{K}^N_{\mathcal{I}}$ is the identity equivalence system. It follows, as a consequence, that every such model is reduced and has the congruence property.

Lemma 14 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, with N a category of natural transformations on* SEN*, be an N-Hilbert-based π-institution. Then for every N-algebraic system $\langle \mathrm{SEN}', \langle N', F' \rangle \rangle \in \mathtt{K}^N_{\mathcal{I}}$ and for every surjective (N, N')-epimorphic translation $\langle F, \alpha \rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, the Frege equivalence system of the $\langle F, \alpha \rangle$-min (N, N')-model $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C' \rangle$ of $\mathcal{I}$ on* SEN' *is the identity equivalence system $\Delta^{\mathrm{SEN}'}$, whence $\mathcal{I}'$ is N'-reduced and has the N'-congruence property.*

Proof:

By the surjectivity of $\langle F, \alpha \rangle$, it suffices to show that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\alpha_\Sigma(\phi) \neq \alpha_\Sigma(\psi)$, we have that $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \notin \Lambda_{F(\Sigma)}(\mathcal{I}')$. To this end consider the sets

$$T^{\alpha_\Sigma(\phi)} = \{\chi \in \mathrm{SEN}'(F(\Sigma)) : \alpha_\Sigma(\phi) \Rightarrow'_{F(\Sigma)} \chi = 1'_{F(\Sigma)}\}$$

and

$$T^{\alpha_\Sigma(\psi)} = \{\chi \in \mathrm{SEN}'(F(\Sigma)) : \alpha_\Sigma(\psi) \Rightarrow'_{F(\Sigma)} \chi = 1'_{F(\Sigma)}\}.$$

Since $\langle \mathrm{SEN}', \langle N', F' \rangle \rangle \in \mathtt{K}^N_{\mathcal{I}}$ and $\mathcal{I}$ is N-Hilbert-based, both $T^{\alpha_\Sigma(\phi)}$ and $T^{\alpha_\Sigma(\psi)}$ are $F(\Sigma)$-components of $\Rightarrow$-implicative filter families of $\langle \mathrm{SEN}', \langle N', F' \rangle \rangle$. Therefore, by Lemma 13, they are both $F(\Sigma)$-theories of $\mathcal{I}'$. If $T^{\alpha_\Sigma(\phi)} = T^{\alpha_\Sigma(\psi)}$, then we would have $\alpha_\Sigma(\phi) \Rightarrow'_{F(\Sigma)} \alpha_\Sigma(\psi) = \alpha_\Sigma(\psi) \Rightarrow'_{F(\Sigma)} \alpha_\Sigma(\phi) = 1'_{F(\Sigma)}$, whence it would follow that $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$, contrary to the hypothesis. Therefore, we must have $\alpha_\Sigma(\phi) \notin T^{\alpha_\Sigma(\psi)}$ or $\alpha_\Sigma(\psi) \notin T^{\alpha_\Sigma(\phi)}$. This shows that $C'_{F(\Sigma)}(\alpha_\Sigma(\phi)) \neq C'_{F(\Sigma)}(\alpha_\Sigma(\psi))$ and, hence, that $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi) \rangle \notin \Lambda_{F(\Sigma)}(\mathcal{I}')$, as required.

That $\mathcal{I}'$ is N'-reduced follows now from the fact that $\widetilde{\Omega}^{N'}(\mathcal{I}') \leq \Lambda(\mathcal{I}') = \Delta^{\mathrm{SEN}'}$, which also shows that $\mathcal{I}'$ has the N'-congruence property. ■

As another of the significant theorems of the paper, it is proven that, given an N-Hilbert-based π-institution $\mathcal{I}$, the classes of N-algebraic systems $\mathrm{Alg}^N(\mathcal{I})$, $\mathtt{K}^N_{\mathcal{I}}$ and $\mathbf{V}^N(\mathcal{I})$ coincide, which shows that $\mathrm{Alg}^N(\mathcal{I})$ is a variety of N-algebraic systems and that $\mathcal{I}$ is N-Hilbert-based relative to the class $\mathrm{Alg}^N(\mathcal{I})$. This result forms an analog at the level of π-institutions of Theorem 16 of [13].

Theorem 15 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, with N a category of natural transformations on* SEN*, be an N-Hilbert-based π-institution. Then*

1. $\mathrm{Alg}^N(\mathcal{I}) = \mathtt{K}^N_{\mathcal{I}} = \mathbf{V}^N(\mathcal{I})$*,*

2. $\mathrm{Alg}^N(\mathcal{I})$ *is a variety,*

3. $\mathcal{I}$ *is N-Hilbert-based relative to* $\mathrm{Alg}^N(\mathcal{I})$*.*

Proof: By Proposition 2, we have that $\mathrm{Alg}^N(\mathcal{I}) \subseteq \mathtt{K}^N_{\mathcal{I}}$. By Lemma 14, we get that $\mathtt{K}^N_{\mathcal{I}} \subseteq \mathrm{Alg}^N(\mathcal{I})$. The third equality follows from Part 3 of Proposition 7. Now Part 2 follows from Part 1 and the fact that $\mathtt{K}^N_{\mathcal{I}}$ is a variety and Part 3 follows from Part 1 and Part 3 of Proposition 7. ∎

Combining Theorem 15 with Theorem 12, we obtain the following reformulation of Theorem 12, an analog of Theorem 13 of [13].

Theorem 16 *Let* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$*, with N a category of natural transformations on* SEN*, be a symmetrically N-rule based functor and* $\Rightarrow: \mathrm{SEN}^2 \to \mathrm{SEN}$ *a binary natural transformation in N. Then, the map* $\mathcal{I} \mapsto \mathrm{Alg}^N(\mathcal{I})$ *is a dual isomorphism between the collection of all N-Hilbert-based π-institutions on* SEN *relative to $\Rightarrow$, ordered by extension, and the collection of all subvarieties* $\mathtt{K}$ *of the variety* $\mathtt{K}^N_{\Rightarrow}$*, such that* $\mathtt{K} = \mathbf{V}^N(\mathrm{cor}^N(\mathtt{K}))$*, ordered by inclusion.*

In closing this section, we show that every N-selfextensional π-institution $\mathcal{I}$ with the N-uniterm deduction-detachment property is fully N-selfextensional. This result was first established for deductive systems in [9] via the use of Gentzen systems and later proven in [13] without using Gentzen systems. Our proof, of course, follows that of [13], which provided the overall inspiration for the present work.

Theorem 17 *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$*, with N a category of natural transformations on* SEN*, be an N-selfextensional π-institution. If $\mathcal{I}$ has the N-uniterm deduction-detachment property, then $\mathcal{I}$ is fully N-selfextensional.*

Proof:
Suppose that $\mathcal{I}$ is N-selfextensional with the N-uniterm deduction-detachment theorem relative to the binary natural transformation $\Rightarrow$. By Theorem 9 and the discussion following its proof, we get that $\mathcal{I}$ is N-Hilbert-based relative to $\Rightarrow$ and $\mathtt{K}^N_{\mathcal{I}}$. Thus, by Theorem 15, $\mathtt{K}^N_{\mathcal{I}} = \mathrm{Alg}^N(\mathcal{I})$. Thus, by Lemma 14, if $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathrm{Alg}^N(\mathcal{I})$ and $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$ is a surjective (N, N')-epimorphic translation and $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on SEN', we have that $\Lambda(\mathcal{I}') = \Delta^{\mathrm{SEN}'}$ is an N'-congruence system on SEN'. If $\mathcal{I}''$ is an (N, N'')-full model of $\mathcal{I}$ via a surjective (N, N'')-logical morphism $\langle G, \beta\rangle : \mathcal{I} \rangle\!\!-^{se} \mathcal{I}''$, then its reduction $\mathcal{I}''^{\overline{N''}}$ is the $\langle G, \pi^{N'}_{G}\beta\rangle$-min $(N, \overline{N''})$-model of $\mathcal{I}$ on $\mathrm{SEN}''^{\overline{N''}}$, whence, by the discussion above $\mathcal{I}''^{\overline{N''}}$ has the $\overline{N''}$-congruence property and, therefore, by Proposition 3.7 of [17], $\mathcal{I}''$ also has the N''-congruence property. Therefore $\mathcal{I}$ is fully N-selfextensional. ∎

5 Fregean and Fully Fregean Protoalgebraic π-Institutions

In this section a few results concerning Fregean and fully Fregean π-institutions will be provided that abstract to the π-institution level corresponding results that are known to hold for deductive systems. This study will lead to some additional results on Fregean π-institutions with implication along the lines of the results on self-extensional π-institutions with implication that were presented in the previous section. To start with, the reader is invited to recall from the Introduction the definitions of an N-Fregean and an N-fully Fregean π-institutions.

In the first result of the section, it will be shown that bilogical morphisms forward preserve the Fregean property. Moreover, if they have isomorphic functor components, then they also preserve the Fregean property in the opposite direction.

Lemma 18 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle, \mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$, with N, N' categories of natural transformations on $\mathrm{SEN}, \mathrm{SEN}'$, respectively, are two π-institutions and $\langle F, \alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}'$ an (N, N')-bilogical morphism.*

1. *If $\mathcal{I}$ is N-Fregean, then $\mathcal{I}'$ is N'-Fregean.*
2. *If $F : \mathbf{Sign} \to \mathbf{Sign}'$ is an isomorphism and $\mathcal{I}'$ is N'-Fregean, then $\mathcal{I}$ is N-Fregean.*

Proof:

1. Suppose that $\mathcal{I}$ is N-Fregean. Let T be a theory family of $\mathcal{I}'$, $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, $\Sigma \in |\mathbf{Sign}|, \phi_0, \ldots, \phi_{n-1}, \psi_0, \ldots, \psi_{n-1} \in \mathrm{SEN}(\Sigma)$, such that $\langle \alpha_\Sigma(\phi_i), \alpha_\Sigma(\psi_i)\rangle \in \Lambda^{\mathcal{I}'}_{F(\Sigma)}(T)$, for all $i < n$. Then, we have that, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,
$$C'_{F(\Sigma')}(T_{F(\Sigma')}, \mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi_i))) = C'_{F(\Sigma')}(T_{F(\Sigma')}, \mathrm{SEN}'(F(f))(\alpha_\Sigma(\psi_i))),$$
for all $i < n$. This is equivalent to, for all $i < n$,
$$C'_{F(\Sigma')}(T_{F(\Sigma')}, \alpha_\Sigma(\mathrm{SEN}(f)(\phi_i))) = C'_{F(\Sigma')}(T_{F(\Sigma')}, \alpha_\Sigma(\mathrm{SEN}(f)(\psi_i))).$$
Therefore, since $\langle F, \alpha\rangle$ is an (N, N')-bilogical morphism, we obtain that
$$C_{\Sigma'}(\alpha^{-1}_{\Sigma'}(T_{F(\Sigma')}), \mathrm{SEN}(f)(\phi_i)) = C_{\Sigma'}(\alpha^{-1}_{\Sigma'}(T_{F(\Sigma')}), \mathrm{SEN}(f)(\psi_i)),$$
for all $i < n$. Hence $\langle \phi_i, \psi_i\rangle \in \Lambda^{\mathcal{I}}_\Sigma(\alpha^{-1}(T))$. But $\mathcal{I}$ is N-Fregean, whence
$$\langle \sigma_\Sigma(\phi_0, \ldots, \phi_{n-1}), \sigma_\Sigma(\psi_0, \ldots, \psi_{n-1})\rangle \in \Lambda^{\mathcal{I}}_\Sigma(\alpha^{-1}(T)).$$
Thus, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,
$$C_{\Sigma'}(\alpha^{-1}_{\Sigma'}(T_{F(\Sigma')}), \mathrm{SEN}(f)(\sigma_\Sigma(\phi_0, \ldots, \phi_{n-1}))) = C_{\Sigma'}(\alpha^{-1}_{\Sigma'}(T_{F(\Sigma')}), \mathrm{SEN}(f)(\sigma_\Sigma(\psi_0, \ldots, \psi_{n-1}))).$$

And, again, since $\langle F,\alpha\rangle$ is an (N,N')-bilogical morphism, we get that

$$\begin{array}{l}C'_{F(\Sigma')}(T_{F(\Sigma')},\alpha_{\Sigma'}(\mathrm{SEN}(f)(\sigma_\Sigma(\phi_0,\ldots,\phi_{n-1})))) = \\ \qquad\qquad C'_{F(\Sigma')}(T_{F(\Sigma')},\alpha_{\Sigma'}(\mathrm{SEN}(f)(\sigma_\Sigma(\psi_0,\ldots,\psi_{n-1})))),\end{array}$$

i.e., because of the commutativity of the two rectangles,

$$\begin{tikzcd}
\mathrm{SEN}(\Sigma) \arrow[r, "\alpha_\Sigma"] \arrow[d, "\mathrm{SEN}(f)"'] & \mathrm{SEN}'(F(\Sigma)) \arrow[d, "\mathrm{SEN}'(F(f))"] \\
\mathrm{SEN}(\Sigma') \arrow[r, "\alpha_{\Sigma'}"'] & \mathrm{SEN}'(F(\Sigma'))
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{SEN}(\Sigma)^n \arrow[r, "\alpha^n_\Sigma"] \arrow[d, "\sigma_\Sigma"'] & \mathrm{SEN}'(F(\Sigma))^n \arrow[d, "\sigma'_{F(\Sigma)}"] \\
\mathrm{SEN}(\Sigma) \arrow[r, "\alpha_\Sigma"'] & \mathrm{SEN}'(F(\Sigma))
\end{tikzcd}$$

that

$$\begin{array}{l}C'_{F(\Sigma')}(T_{F(\Sigma')},\mathrm{SEN}'(F(f))(\sigma'_{F(\Sigma)}(\alpha_\Sigma(\phi_0),\ldots,\alpha_\Sigma(\phi_{n-1})))) = \\ \qquad\qquad C'_{F(\Sigma')}(T_{F(\Sigma')},\mathrm{SEN}'(F(f))(\sigma'_{F(\Sigma)}(\alpha_\Sigma(\psi_0),\ldots,\alpha_\Sigma(\psi_{n-1})))).\end{array}$$

Therefore $\langle\sigma'_{F(\Sigma)}(\alpha_\Sigma(\phi_0),\ldots,\alpha_\Sigma(\phi_{n-1})),\sigma'_{F(\Sigma)}(\alpha_\Sigma(\psi_0),\ldots,\alpha_\Sigma(\psi_{n-1}))\rangle \in \Lambda^{\mathcal{I}'}_{F(\Sigma)}(T)$, showing that $\mathcal{I}'$ is N'-Fregean.

2. Suppose that $\mathcal{I}'$ is N'-Fregean and let T be a theory family of $\mathcal{I}$, $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, $\Sigma \in |\mathbf{Sign}|, \phi_0,\ldots,\phi_{n-1},\psi_0,\ldots,\psi_{n-1} \in \mathrm{SEN}(\Sigma)$, such that $\langle\phi_i,\psi_i\rangle \in \Lambda^{\mathcal{I}}_\Sigma(T)$, for all $i<n$. Then, we have that, for all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma,\Sigma')$, $C_{\Sigma'}(T_{\Sigma'},\mathrm{SEN}(f)(\phi_i)) = C_{\Sigma'}(T_{\Sigma'},\mathrm{SEN}(f)(\psi_i))$, for all $i<n$. Therefore, since $\langle F,\alpha\rangle$ is an (N,N')-bilogical morphism, we obtain that $C'_{F(\Sigma')}(\alpha_{\Sigma'}(T_{\Sigma'}),\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi_i))) = C'_{F(\Sigma')}(\alpha_{\Sigma'}(T_{\Sigma'}),\alpha_{\Sigma'}(\mathrm{SEN}(f)(\psi_i)))$, for all $i<n$. This is equivalent to, for all $i<n$,

$$C'_{F(\Sigma')}(\alpha_{\Sigma'}(T_{\Sigma'}),\mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi_i))) = C'_{F(\Sigma')}(\alpha_{\Sigma'}(T_{\Sigma'}),\mathrm{SEN}'(F(f))(\alpha_\Sigma(\psi_i))).$$

Hence, $\langle\alpha_\Sigma(\phi_i),\alpha_\Sigma(\psi_i)\rangle \in \Lambda^{\mathcal{I}'}_{F(\Sigma)}(\alpha(T))$, for all $i<n$. Thus, since $\mathcal{I}'$ is N'-Fregean, we obtain that

$$\langle\sigma'_{F(\Sigma)}(\alpha_\Sigma(\phi_0),\ldots,\alpha_\Sigma(\phi_{n-1})),\sigma'_{F(\Sigma)}(\alpha_\Sigma(\psi_0),\ldots,\alpha_\Sigma(\psi_{n-1}))\rangle \in \Lambda^{\mathcal{I}'}_{F(\Sigma)}(\alpha(T)),$$

i.e., that $\langle\alpha_\Sigma(\sigma_\Sigma(\phi_0,\ldots,\phi_{n-1})),\alpha_\Sigma(\sigma_\Sigma(\psi_0,\ldots,\psi_{n-1}))\rangle \in \Lambda^{\mathcal{I}'}_{F(\Sigma)}(\alpha(T))$. This shows that, for all $\Sigma \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma,\Sigma')$,

$$\begin{array}{l}C'_{F(\Sigma')}(\alpha_{\Sigma'}(T_{\Sigma'}),\mathrm{SEN}'(F(f))(\alpha_\Sigma(\sigma_\Sigma(\phi_0,\ldots,\phi_{n-1})))) = \\ \qquad\qquad C'_{F(\Sigma')}(\alpha_{\Sigma'}(T_{\Sigma'}),\mathrm{SEN}'(F(f))(\alpha_\Sigma(\sigma_\Sigma(\psi_0,\ldots,\psi_{n-1})))),\end{array}$$

which is equivalent to

$$\begin{array}{l}C'_{F(\Sigma')}(\alpha_{\Sigma'}(T_{\Sigma'}),\alpha_{\Sigma'}(\mathrm{SEN}(f)(\sigma_\Sigma(\phi_0,\ldots,\phi_{n-1})))) = \\ \qquad\qquad C'_{F(\Sigma')}(\alpha_{\Sigma'}(T_{\Sigma'}),\alpha_{\Sigma'}(\mathrm{SEN}(f)(\sigma_\Sigma(\psi_0,\ldots,\psi_{n-1})))).\end{array}$$

This yields, since $\langle F,\alpha\rangle$ is an (N,N')-bilogical morphism, that

$$C_{\Sigma'}(T_{\Sigma'},\mathrm{SEN}(f)(\sigma_\Sigma(\phi_0,\ldots,\phi_{n-1})))=C_{\Sigma'}(T_{\Sigma'},\mathrm{SEN}(f)(\sigma_\Sigma(\psi_0,\ldots,\psi_{n-1}))).$$

Thus, we obtain that $\langle\sigma_\Sigma(\phi_0,\ldots,\phi_{n-1}),\sigma_\Sigma(\psi_0,\ \ldots,\psi_{n-1})\rangle\in\Lambda^{\mathcal{I}}_\Sigma(T)$, showing that $\mathcal{I}$ is N-Fregean.

■

By combining the two Parts of Lemma 18 we immediately obtain the following corollary to the effect that a π-institution $\mathcal{I}$ is N-Fregean if and only if its N-Tarski reduction $\mathcal{I}^N$ is $\overline{N}$-Fregean.

Corollary 19 *Suppose that $\mathcal{I}=\langle\mathbf{Sign},\mathrm{SEN},C\rangle$, with N a category of natural transformations on* SEN *is a π-institution. Then $\mathcal{I}$ is N-Fregean if and only if its reduction $\mathcal{I}^N$ is $\overline{N}$-Fregean.*

Next, it is shown that, given a π-institution $\mathcal{I}=\langle\mathbf{Sign},\mathrm{SEN},C\rangle$, with N a category of natural transformations on SEN, the π-institution $\mathcal{I}^T=\langle\mathbf{Sign},\mathrm{SEN},C^T\rangle$ is N-Fregean, for all theory systems $T\in\mathrm{ThSys}(\mathcal{I})$, provided that $\mathcal{I}$ is N-Fregean.

Lemma 20 *Suppose that $\mathcal{I}=\langle\mathbf{Sign},\mathrm{SEN},C\rangle$ is a π-institution, with N a category of natural transformations on* SEN*. If $\mathcal{I}$ is N-Fregean, then $\mathcal{I}^T=\langle\mathbf{Sign},\mathrm{SEN},C^T\rangle$ is also N-Fregean, for every theory system T of $\mathcal{I}$.*

Proof:
Suppose that $\mathcal{I}$ is N-Fregean. Let T' be a theory family of $\mathcal{I}^T$, $\sigma:\mathrm{SEN}^n\to\mathrm{SEN}$ in N, $\Sigma\in|\mathbf{Sign}|,\phi_0,\ldots,\phi_{n-1},\psi_0,\ldots,\psi_{n-1}\in\mathrm{SEN}(\Sigma)$, such that $\langle\phi_i,\psi_i\rangle\in\Lambda^{\mathcal{I}^T}_\Sigma(T')$, for all $i<n$. Therefore, for all $i<n$, we get that $C^T_{\Sigma'}(T'_{\Sigma'},\mathrm{SEN}(f)(\phi_i))=C^T_{\Sigma'}(T'_{\Sigma'},\mathrm{SEN}(f)(\psi_i))$, for all $\Sigma'\in|\mathbf{Sign}|,f\in\mathbf{Sign}(\Sigma,\Sigma')$. Thus, since $T\leq T'$, we get that $C_{\Sigma'}(T'_{\Sigma'},\mathrm{SEN}(f)(\phi_i))=C_{\Sigma'}(T'_{\Sigma'},\mathrm{SEN}(f)(\psi_i))$, for all $i<n$, whence, we get that $\langle\phi_i,\psi_i\rangle\in\Lambda^{\mathcal{I}}_\Sigma(T')$, for all $i<n$. But $\mathcal{I}$ is, by hypothesis, N-Fregean, whence $\langle\sigma_\Sigma(\phi_0,\ldots,\phi_{n-1}),\sigma_\Sigma(\psi_0,\ldots,\psi_{n-1})\rangle\in\Lambda^{\mathcal{I}}_\Sigma(T')$. Therefore, for all $\Sigma'\in|\mathbf{Sign}|$ and all $f\in\mathbf{Sign}(\Sigma,\Sigma')$,

$$C_{\Sigma'}(T'_{\Sigma'},\mathrm{SEN}(f)(\sigma_\Sigma(\phi_0,\ldots,\phi_{n-1})))=C_{\Sigma'}(T'_{\Sigma'},\mathrm{SEN}(f)(\sigma_\Sigma(\psi_0,\ldots,\psi_{n-1}))).$$

This shows that

$$\langle\sigma_\Sigma(\phi_0,\ldots,\phi_{n-1}),\sigma_\Sigma(\psi_0,\ldots,\psi_{n-1})\rangle\in\Lambda^{\mathcal{I}^T}_\Sigma(T'),$$

whence $\mathcal{I}^T$ is in fact N-Fregean. ■

Czelakowski and Pigozzi prove in Corollary 80 of [6] that every protoalgebraic and Fregean deductive system is fully Fregean. Here, the analog of this result is shown to hold for finitary, N-rule based and N-protoalgebraic π-institutions. For definitions and results pertaining to N-protoalgebraic π-institutions see [18, 19]. Here, we will use specifically, Corollary 4.20 of [18], an analog of the well-known Correspondence Property of protoalgebraic deductive systems for finitary N-rule based N-protoalgebraic π-institutions.

Proposition 21 (Corollary 4.20 of [18]) *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be a finitary, N-rule based π-institution. $\mathcal{I}$ has the family N-correspondence property if and only if it is N-protoalgebraic.*

Based on this result, it may now be shown that every finitary, N-rule based and N-protoalgebraic π-institution that is N-Fregean is fully N-Fregean, an analog in the context of π-institutions of Corollary 80 of [6].

Theorem 22 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be a finitary, N-rule based, N-protoalgebraic π-institution. If $\mathcal{I}$ is N-Fregean, then $\mathcal{I}$ is fully N-Fregean.*

Proof:
Assume that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is a finitary, N-rule based, N-protoalgebraic π-institution, that is N-Fregean. To show that it is fully N-Fregean, it suffices, by Corollary 19, to show that, given a surjective (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}'$, the $\langle F, \alpha\rangle$-min (N, N')-model $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ of $\mathcal{I}$ on SEN' is N'-Fregean. Suppose, to this end, that $\langle F, \alpha\rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}'$ is a surjective (N, N')-epimorphic translation and let $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ be the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on SEN'. Then, since $\mathcal{I}$ is finitary, N-rule based and N-protoalgebraic, it has, by Proposition 21, the family N-correspondence property. Therefore, denoting by Thm' the theorem system of $\mathcal{I}'$, $\langle F, \alpha\rangle : \mathcal{I}^{\alpha^{-1}(\mathrm{Thm}')} \vdash^{se} \mathcal{I}'$ is an (N, N')-bilogical morphism. By Lemma 20, we have that $\mathcal{I}^{\alpha^{-1}(\mathrm{Thm}')}$ is N-Fregean, whence, by Part 1 of Lemma 18, we get that $\mathcal{I}'$ is N'-Fregean. Therefore, $\mathcal{I}$ is fully N-Fregean. ■

6 Fregean π-Institutions with the DDT

In this section some of the results that were obtained in the previous section for selfextensional π-institutions with the uniterm deduction-detachment property are adapted to obtain some results relating to Fregean π-institutions with the uniterm deduction-detachment property. We note that Fregean deductive systems and many of their properties have been extensively studied by Czelakowski and Pigozzi in [6, 7]. Many of the results obtained in [6, 7] are adapted by the author to the level of logics formalized as π-institutions in [26, 27].

The first result of this section characterizes those π-institution $\mathcal{I}$ with an N-deduction-detachment term $\Rightarrow$ that are N-Fregean as those in which the set $\{x \Rightarrow y, y \Rightarrow x\}$ is an N-equivalence system for $\mathcal{S}$ in the sense of [23].

Proposition 23 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN *and $\Rightarrow$ an N-deduction-detachment term for $\mathcal{I}$. Then $\mathcal{I}$ is N-Fregean if and only if the set $\{x \Rightarrow y, y \Rightarrow x\}$ is an N-equivalence system for $\mathcal{I}$.*

Proof:

Suppose, first, that the set $E = \{x \Rightarrow y, y \Rightarrow x\}$ is an N-equivalence system for $\mathcal{I}$. Given a theory family T of $\mathcal{I}$, recall from [19] that the notation $E(T) = \{E_\Sigma(T)\}_{\Sigma \in |\mathbf{Sign}|}$ is used to denote the relation system on SEN, defined, for all $\Sigma \in |\mathbf{Sign}|$ by

$$\begin{array}{rcl} E_\Sigma(T) & = & \{\langle\phi,\psi\rangle : \mathrm{SEN}(f)(\phi) \Rightarrow_{\Sigma'} \mathrm{SEN}(f)(\psi), \mathrm{SEN}(f)(\psi) \Rightarrow_{\Sigma'} \mathrm{SEN}(f)(\phi) \in T_{\Sigma'}, \\ & & \text{for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma,\Sigma')\}. \end{array}$$

Then, by Theorem 5 of [23], we have that, for all theory families T of $\mathcal{I}$ and all $\Sigma \in |\mathbf{Sign}|$, $E_\Sigma(T) = \Omega^N_\Sigma(T)$. Moreover, since $\Rightarrow$ is an N-deduction-detachment term for $\mathcal{I}$, we have that

$$\begin{array}{rcl} E_\Sigma(T) & = & \{\langle\phi,\psi\rangle : \mathrm{SEN}(f)(\phi) \Rightarrow_{\Sigma'} \mathrm{SEN}(f)(\psi), \mathrm{SEN}(f)(\psi) \Rightarrow_{\Sigma'} \mathrm{SEN}(f)(\phi) \in T_{\Sigma'}, \\ & & \text{for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma,\Sigma')\} \\ & = & \{\langle\phi,\psi\rangle : \mathrm{SEN}(f)(\psi) \in C_{\Sigma'}(T_{\Sigma'}, \mathrm{SEN}(f)(\phi)), \\ & & \mathrm{SEN}(f)(\phi) \in C_{\Sigma'}(T_{\Sigma'}, \mathrm{SEN}(f)(\psi)), \text{ for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma,\Sigma')\} \\ & = & \{\langle\phi,\psi\rangle : C_{\Sigma'}(T_{\Sigma'}, \mathrm{SEN}(f)(\phi)) = C_{\Sigma'}(T_{\Sigma'}, \mathrm{SEN}(f)(\psi)), \\ & & \text{for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma,\Sigma')\} \\ & = & \Lambda^{\mathcal{I}}_\Sigma(T). \end{array}$$

This shows that, for every theory family T of $\mathcal{I}$, $\Lambda_{\mathcal{I}}(T) = \Omega^N(T)$ and, therefore $\mathcal{I}$ is N-Fregean.

Suppose, conversely, that $\Rightarrow$ is an N-deduction-detachment term for $\mathcal{I}$ and that $\mathcal{I}$ is N-Fregean. Then, for every theory family T of $\mathcal{I}$, we have, for all $\Sigma \in |\mathbf{Sign}|$,

$$\begin{array}{rcl} \Omega^N_\Sigma(T) & = & \Lambda^{\mathcal{I}}_\Sigma(T) \quad \text{(by the Fregean Property)} \\ & = & \{\langle\phi,\psi\rangle \in \mathrm{SEN}(\Sigma)^2 : C_{\Sigma'}(T_{\Sigma'}, \mathrm{SEN}(f)(\phi)) = C_{\Sigma'}(T_{\Sigma'}, \mathrm{SEN}(f)(\psi)), \\ & & \text{for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma,\Sigma')\} \quad \text{(by definition)} \\ & = & \{\langle\phi,\psi\rangle \in \mathrm{SEN}(\Sigma)^2 : \mathrm{SEN}(f)(\phi) \Rightarrow_{\Sigma'} \mathrm{SEN}(f)(\psi), \\ & & \qquad \mathrm{SEN}(f)(\psi) \Rightarrow_{\Sigma'} \mathrm{SEN}(f)(\phi) \in T_{\Sigma'} \\ & & \text{for all } \Sigma' \in \mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma,\Sigma')\} \quad \text{(by the DDP)} \\ & = & E_\Sigma(T), \end{array}$$

which shows that $\{x \Rightarrow y, y \Rightarrow x\}$ defines the Leibniz N-congruence systems of $\mathcal{I}$ and, therefore, by Theorem 5 of [23], it is an N-equivalence system for $\mathcal{I}$. ∎

Given a π-institution $\mathcal{I} = \langle\mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, if $\mathcal{I}$ is N-selfextensional and has an N-deduction-detachment term $\Rightarrow$, then, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi_0,\ldots,\phi_n,\phi \in \mathrm{SEN}(\Sigma)$ and all permutations π of $\{0,1,\ldots,n\}$ we have that

$$C_\Sigma(\phi_0 \Rightarrow_\Sigma (\cdots \Rightarrow_\Sigma (\phi_n \Rightarrow_\Sigma \phi)\cdots)) = C_\Sigma(\phi_{\pi(0)} \Rightarrow_\Sigma (\cdots \Rightarrow_\Sigma (\phi_{\pi(n)} \Rightarrow_\Sigma \phi)\cdots)).$$

Whenever this is the case, and following [13], it makes sense to introduce, given a sequence of Σ-sentences $\vec{\phi} = \langle\phi_0,\ldots,\phi_{n-1}\rangle$ and a Σ-sentence ψ, the notation $\vec{\phi} \Rightarrow_\Sigma \psi$ to denote the longer Σ-sentence $\phi_0 \Rightarrow_\Sigma (\cdots \Rightarrow_\Sigma (\phi_{n-1} \Rightarrow_\Sigma \psi)\cdots)$.

With this notation in mind, the following partial analog of Proposition 22 of [13] may be formulated. It provides a sufficient syntactic condition for a finitary, N-rule based and N-selfextensional π-institution having the N-deduction-detachment property to be N-Fregean. It is only a partial analog of Proposition 22 of [13] because this condition, in the deductive system framework, is not only sufficient but actually characterizes completely Fregean deductive systems among all selfextensional deductive systems with the deduction-detachment property. We will elaborate more on the reasons why the condition does not seem to be necessary in the framework of π-institutions after the proof of Proposition 24.

Proposition 24 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, is a finitary, N-rule based and N-selfextensional π-institution with an N-deduction-detachment term $\Rightarrow$. If, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, all $k \in \omega$ and all different variables $z_0, \ldots, z_{k-1}, x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1}$, the N-quasi-equations*

$$\begin{aligned}\textstyle\bigwedge_{i<n}[\vec{z} \Rightarrow (x_i \Rightarrow y_i) \approx 1] \wedge \bigwedge_{i<n}[\vec{z} \Rightarrow (y_i \Rightarrow x_i) \approx 1] \longrightarrow& \\ [\vec{z} \Rightarrow (\sigma(x_0, \ldots, x_{n-1}) \Rightarrow \sigma(y_0, \ldots, y_{n-1})) \approx 1]&\end{aligned}$$

are N-quasi-identities of $\mathrm{Alg}^N(\mathcal{I})$, then $\mathcal{I}$ is N-Fregean.

Proof:
Let $\mathcal{I}$ be a finitary, N-rule based and N-selfextensional π-institution, that has an N-deduction-detachment term $\Rightarrow$. Then, by Proposition 3.2 of [19], $\mathcal{I}$ is N-protoalgebraic.

Suppose that the displayed N-quasi-equations are indeed N-quasi-identities of $\mathrm{Alg}^N(\mathcal{I})$. Let $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathrm{Alg}^N(\mathcal{I})$, so that there exists a surjective (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, such that the $\langle F, \alpha\rangle$-min (N, N')-model $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ of $\mathcal{I}$ on SEN' is N'-reduced. Let $X' = \{X'_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}'|}$ be a finite axiom family of $\mathcal{I}'$. It suffices to show that $\Lambda_{\mathcal{I}'}(X')$ is an N-congruence system of $\mathcal{I}'$. To this end, suppose that $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ is in N, $\Sigma \in |\mathbf{Sign}|$ and $\phi_0, \ldots, \phi_{n-1}, \psi_0, \ldots, \psi_{n-1} \in \mathrm{SEN}(\Sigma)$, such that $\langle \alpha_\Sigma(\phi_i), \alpha_\Sigma(\psi_i)\rangle \in \Lambda^{\mathcal{I}'}_{F(\Sigma)}(X')$, for all $i < n$. Then we have that, for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$C'_{F(\Sigma')}(X'_{F(\Sigma')}, \mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi_i))) = C'_{F(\Sigma')}(X'_{F(\Sigma')}, \mathrm{SEN}'(F(f))(\alpha_\Sigma(\psi_i))),$$

for all $i < n$. Assuming, without loss of generality that, for all $\Sigma' \in |\mathbf{Sign}'|$, $1'_{\Sigma'} \in X'_{\Sigma'}$ and that, $\vec{X}'_{\Sigma'}$ is a sequence of all the elements of $X'_{\Sigma'}$ of the same length as the cardinality of $X'_{\Sigma'}$, $\Sigma' \in |\mathbf{Sign}'|$, we get, for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\vec{X}'_{F(\Sigma')} \Rightarrow'_{F(\Sigma')} (\mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi_i)) \Rightarrow'_\Sigma \mathrm{SEN}'(F(f))(\alpha_\Sigma(\psi_i))) \in C'_{F(\Sigma')}(1'_{F(\Sigma')})$$

and

$$\vec{X}'_{F(\Sigma')} \Rightarrow'_{F(\Sigma')} (\mathrm{SEN}'(F(f))(\alpha_\Sigma(\psi_i)) \Rightarrow'_{\Sigma'} \mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi_i))) \in C'_{F(\Sigma')}(1'_{F(\Sigma')}),$$

for all $i < n$. Hence, we obtain that, for every $i < n$,

$$\vec{X}'_{F(\Sigma')} \Rightarrow'_{F(\Sigma')} (\mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi_i)) \Rightarrow'_{\Sigma'} \mathrm{SEN}'(F(f))(\alpha_\Sigma(\psi_i))) = 1'_{F(\Sigma')}$$

and
$$\vec{X}'_{F(\Sigma')} \Rightarrow'_{F(\Sigma')} (\mathrm{SEN}'(F(f))(\alpha_\Sigma(\psi_i)) \Rightarrow'_{\Sigma'} \mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi_i))) = 1'_{F(\Sigma')}.$$
Therefore, since by the hypothesis, the displayed N-quasi-equations are N-quasi-identities of $\mathrm{Alg}^N(\mathcal{I})$, we have that
$$\begin{aligned}\vec{X}'_{F(\Sigma')} \Rightarrow_{F(\Sigma')} (\mathrm{SEN}'(F(f))(\sigma'_{F(\Sigma)}(\alpha_\Sigma(\phi_0),\ldots,\alpha_\Sigma(\phi_{n-1}))) \Rightarrow'_{F(\Sigma')}&\\ \mathrm{SEN}'(F(f))(\sigma'_{F(\Sigma)}(\alpha_\Sigma(\psi_0),\ldots,\alpha_\Sigma(\psi_{n-1})))) = 1'_{F(\Sigma')}&\end{aligned}$$
and, similarly,
$$\begin{aligned}\vec{X}'_{F(\Sigma')} \Rightarrow_{F(\Sigma')} (\mathrm{SEN}'(F(f))(\sigma'_{F(\Sigma)}(\alpha_\Sigma(\psi_0),\ldots,\alpha_\Sigma(\psi_{n-1}))) \Rightarrow'_{F(\Sigma')}&\\ \mathrm{SEN}'(F(f))(\sigma'_{F(\Sigma)}(\alpha_\Sigma(\phi_0),\ldots,\alpha_\Sigma(\phi_{n-1})))) = 1'_{F(\Sigma')}.&\end{aligned}$$
Thus, we obtain that, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma,\Sigma')$,
$$\begin{aligned}C'_{F(\Sigma')}(X'_{F(\Sigma')}, \mathrm{SEN}'(F(f))(\sigma_{F(\Sigma)}(\alpha_\Sigma(\phi_0),\ldots,\alpha_\Sigma(\phi_{n-1})))) =&\\ C'_{F(\Sigma')}(X'_{F(\Sigma')}, \mathrm{SEN}'(F(f))(\sigma_{F(\Sigma)}(\alpha_\Sigma(\psi_0),\ldots,\alpha_\Sigma(\psi_{n-1})))),&\end{aligned}$$
i.e., that $\langle \sigma_{F(\Sigma)}(\alpha_\Sigma(\phi_0),\ldots,\alpha_\Sigma(\phi_{n-1})), \sigma_{F(\Sigma)}(\alpha_\Sigma(\psi_0),\ldots,\alpha_\Sigma(\psi_{n-1}))\rangle \in \Lambda^{\mathcal{I}'}_{F(\Sigma)}(X')$. ■

We indicate why the N-Fregean property for $\mathcal{I}$ is unlikely to imply that the displayed N-quasi-equations hold in $\mathrm{Alg}^N(\mathcal{I})$. Assume that $\mathcal{I}$ is a finitary, N-rule based and N-Fregean π-institution, that has an N-deduction-detachment term $\Rightarrow$. Since it is N-protoalgebraic, it is, by Theorem 22, also fully N-Fregean. Let $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathrm{Alg}^N(\mathcal{I})$. Then, there exists a surjective (N,N')-epimorphic translation $\langle F,\alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, such that the $\langle F,\alpha\rangle$-min (N,N')-model $\mathcal{I}' = \langle \mathbf{Sign}, \mathrm{SEN}', C'\rangle$ of $\mathcal{I}$ on SEN' is N'-Fregean. Assume, for the sake of attempting to prove that the displayed N-quasi-equations are valid in $\mathrm{Alg}^N(\mathcal{I})$, that $\Sigma \in |\mathbf{Sign}'|$ and $\chi_0,\ldots,\chi_{k-1},\phi_0,\ldots,\phi_{n-1},\psi_0,\ldots,\psi_{n-1} \in \mathrm{SEN}'(\Sigma)$, such that $\vec{\chi} \Rightarrow'_\Sigma (\phi_i \Rightarrow'_\Sigma \psi_i) = 1'_\Sigma$ and $\vec{\chi} \Rightarrow'_\Sigma (\psi_i \Rightarrow'_\Sigma \phi_i) = 1'_\Sigma$. Then, to obtain that, for all $\Sigma' \in |\mathbf{Sign}'|$ and all $f \in \mathbf{Sign}'(\Sigma,\Sigma')$, $C'_{\Sigma'}(X_{\Sigma'}, \mathrm{SEN}'(f)(\phi_i)) = C'_{\Sigma'}(X_{\Sigma'}, \mathrm{SEN}'(f)(\psi_i))$, for all $i < n$, and, as a consequence, that $\langle \phi_i,\psi_i\rangle \in \Lambda^{\mathcal{I}'}_\Sigma(X)$, with the goal of being able to use the N'-Fregean property of $\mathcal{I}'$, we must set $X = \{X_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}'|}$, with
$$X_{\Sigma'} = \bigcup\{\mathrm{SEN}'(f)(\chi_0),\ldots,\mathrm{SEN}'(f)(\chi_{k-1}) : f \in \mathbf{Sign}'(\Sigma,\Sigma')\}$$
and this set may not be finite any more. If we do this, then, we will indeed have
$$C'_{\Sigma'}(X_{\Sigma'}, \mathrm{SEN}'(f)(\phi_i)) = C'_{\Sigma'}(X_{\Sigma'}, \mathrm{SEN}'(f)(\psi_i)),$$
for all $i < n$, and, as a consequence, that $\langle \phi_i,\psi_i\rangle \in \Lambda^{\mathcal{I}'}_\Sigma(X)$. Thus, since $\mathcal{I}'$ is N'-Fregean, we obtain that
$$\langle \sigma'_\Sigma(\phi_0,\ldots,\phi_{n-1}), \sigma'_\Sigma(\psi_0,\ldots,\psi_{n-1})\rangle \in \Lambda^{\mathcal{I}'}_\Sigma(X),$$
i.e., that, for all $\Sigma' \in |\mathbf{Sign}'|$ and all $f \in \mathbf{Sign}'(\Sigma,\Sigma')$,
$$C'_{\Sigma'}(X_{\Sigma'}, \mathrm{SEN}'(f)(\sigma'_\Sigma(\phi_0,\ldots,\phi_{n-1}))) = C'_{\Sigma'}(X_{\Sigma'}, \mathrm{SEN}'(f)(\sigma'_\Sigma(\psi_0,\ldots,\psi_{n-1}))).$$

This shows that $\sigma'_\Sigma(\psi_0,\ldots,\psi_{n-1}) \in C'_\Sigma(X_\Sigma, \sigma'_\Sigma(\phi_0,\ldots,\phi_{n-1}))$. Now the next key step of the proof that applies in the case of deductive systems does not necessarily carry over in this case! The last inclusion *does not necessarily imply* that $\vec{\chi} \Rightarrow'_\Sigma (\sigma'_\Sigma(\phi_0,\ldots,\phi_{n-1}) \Rightarrow'_\Sigma \sigma'_\Sigma(\psi_0,\ldots,\psi_{n-1}))) \in C'_\Sigma(1'_\Sigma)$, because the collection (or set if **Sign** is locally small) X_Σ may be much larger than the set $\{\chi_0,\ldots,\chi_{k-1}\}$.

Lemma 25 *Suppose that* SEN : **Sign** $\to$ **Set** *is a functor, with N a category of natural transformations on* SEN *and $\Rightarrow$ a binary natural transformation in N. Let $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle$ be an N-algebraic system that satisfies the Hilbert equations relative to $\Rightarrow$. The N-quasi-equations*

$$\begin{aligned}\textstyle\bigwedge_{i<n}[\vec{z} \Rightarrow (x_i \Rightarrow y_i) \approx 1] \wedge \bigwedge_{i<n}[\vec{z} \Rightarrow (y_i \Rightarrow x_i) \approx 1] \longrightarrow &\\ [\vec{z} \Rightarrow (\sigma(x_0,\ldots,x_{n-1}) \Rightarrow \sigma(y_0,\ldots,y_{n-1})) \approx 1]&\end{aligned} \tag{4}$$

are valid in $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle$ if and only if, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, and for X any sequence of all elements in $\{x_i \Rightarrow y_i, y_i \Rightarrow x_i : i < n\}$, the N-equations

$$\vec{X} \Rightarrow (\sigma(x_0,\ldots,x_{n-1}) \Rightarrow \sigma(y_0,\ldots,y_{n-1})) \approx 1 \tag{5}$$

are valid in $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle$.

Proof:

Suppose, first, that $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle$ satisfies the N-quasi-equations (4). Since it also satisfies all Hilbert equations, by hypothesis, the equations $\vec{X} \Rightarrow (x_i \Rightarrow y_i) \approx 1$ and $\vec{X} \Rightarrow (y_i \Rightarrow x_i) \approx 1$ are valid in $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, for all $i < n$. Thus, using the N-quasi-equations (4), we obtain that both $\vec{X} \Rightarrow (\sigma(x_0,\ldots,x_{n-1}) \Rightarrow \sigma(y_0,\ldots,y_{n-1})) \approx 1$ and $\vec{X} \Rightarrow (\sigma(y_0,\ldots,y_{n-1}) \Rightarrow \sigma(x_0,\ldots,x_{n-1})) \approx 1$ are valid in $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle$ and these are exactly the N-equations in (5).

Suppose, conversely, that the N-equations in (5) are valid in $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle$. Then, the N-equations $\vec{X} \Rightarrow (\sigma(y_0,\ldots,y_{n-1}) \Rightarrow \sigma(x_0,\ldots,x_{n-1})) \approx 1$ are also valid in $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle$. Now, if $z_0,\ldots,z_k$ are different from all $x_i, y_i, i < n$, and, for all $\Sigma \in |\mathbf{Sign}'|$ and all $\chi_0,\ldots,\chi_{n-1},\phi_0,\ldots,\phi_{n-1},\psi_0,\ldots,\psi_{n-1} \in \mathrm{SEN}'(\Sigma)$, $\vec{\chi} \Rightarrow'_\Sigma (\phi_i \Rightarrow'_\Sigma \psi_i) = 1'_\Sigma$ and $\vec{\chi} \Rightarrow'_\Sigma (\psi_i \Rightarrow'_\Sigma \phi_i) = 1'_\Sigma$, for all $i < n$, we get that

$$\vec{\chi} \Rightarrow'_\Sigma (\sigma'_\Sigma(\phi_0,\ldots,\phi_{n-1}) \Rightarrow'_\Sigma \sigma'_\Sigma(\psi_0,\ldots,\psi_{n-1})) = 1'_\Sigma.$$

Therefore, the N-quasi-equations (4) are valid in $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle$. ■

By combining the dual isomorphism Theorem 12 with Proposition 24 and Lemma 25, we obtain the following weak version (only one of the two directions) of Theorem 25 of [13] in the context of π-institutions.

Corollary 26 *Let* SEN : **Sign** $\to$ **Set**, *with N a category of natural transformations on* SEN, *be a symmetrically N-rule based functor and $\Rightarrow: \mathrm{SEN}^2 \to \mathrm{SEN}$ a binary natural transformation in N. If an N-Hilbert-based class* K *of N-algebraic systems relative to $\Rightarrow$, such* *that* $\mathtt{K} = \mathbf{V}^N(\mathrm{cor}^N(\mathtt{K}))$, *is a subvariety of the variety axiomatized by the Hilbert equations and, for all* $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$, *the equations*

$$\vec{X} \Rightarrow (\sigma(x_0, \dots, x_{n-1}) \Rightarrow \sigma(y_0, \dots, y_{n-1})) \approx 1 \tag{6}$$

where X *is any sequence of all elements in* $\{x_i \Rightarrow y_i, y_i \Rightarrow x_i : i < n\}$, *then* $\mathtt{K}$ *is the variety* $\mathrm{Alg}^N(\mathcal{I})$ *of a finitary* N*-rule based and* N*-Fregean* π*-institution* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, *having the* N*-deduction-detachment property relative to* $\Rightarrow$.

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