Categorical Abstract Algebraic Logic:
Protoalgebraic Classes of Structure Systems

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Abstract

Structure systems were introduced as an abstraction of ordinary first-order structures more suitable for studies relating algebraizability and first-order model theory in a categorical framework. In the present work, the notion of a protoalgebraic class of first-order structures is adapted to cover the case of structure systems. A variety of results, first shown to hold for classes of logical matrix models of protoalgebraic sentential logics, and later abstracted to protoalgebraic classes of first-order models by Elgueta, are now adapted to cover protoalgebraic classes of structure systems. An example of such a result is that an abstract Lyndon class of structure systems is protoalgebraic if and only if its reduced counterpart is closed under subdirect products.

1 Introduction

Perhaps the most important achievement of the theory of Abstract Algebraic Logic, as developed by Czelakowski [7], Blok and Pigozzi [2, 3] and Font and Jansana [18], among others, has been the classification of sentential logics into various steps of an algebraic hierarchy that reflect the degree to which a given logic is amenable to algebraic study techniques. At the bottom of this hierarchy are the protoalgebraic logics, which are generally believed to form the largest class of logics whose metatheory may be analyzed, to a large extent, by using powerful techniques from Universal Algebra.

Many of the results pertaining to the characterization and study of the class of protoalgebraic logics rely on the analysis of structural properties of the classes of their logical matrix models. A matrix model $\mathcal{A} = \langle \mathbf{A}, F\rangle$ consists of an algebra $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}}\rangle$ of the same type $\mathcal{L}$ as the free algebra $\mathbf{Fm}_{\mathcal{L}}(V)$ of formulas of the logic, together with a designated subset F of the carrier A of the algebra, that is called the filter of the logical matrix and, in

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the matrix semantics, is intended to contain the interpretations of the true formulas of the logic.

One of the best known characterizations of the class of protoalgebraic logics [2] states that a logic $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ is protoalgebraic if and only if the Leibniz operator on every $\mathcal{L}$-algebra is monotone on the lattice of $\mathcal{S}$-filters on the algebra. The Leibniz operator [3] on an algebra $\mathbf{A}$ associates with a subset F of the carrier A of the algebra, the largest congruence $\Omega_{\mathbf{A}}(F)$ of $\mathbf{A}$ that is compatible with F. Thus, according to this characterization of protoalgebraicity, $\mathcal{S}$ is protoalgebraic if and only if, for every $\mathcal{L}$-algebra $\mathbf{A}$, and all $\mathcal{S}$-filters F, G on A, if $F \subseteq G$, then $\Omega_{\mathbf{A}}(F) \subseteq \Omega_{\mathbf{A}}(G)$.

In a different direction, it is very well-known (due originally to the work of Bloom [5]) that logical matrix models of a sentential logic may also be perceived as models of a first-order language without equality and with one unary relation symbol, whose terms coincide with the formulas of the sentential logic and whose interpretation of the unary relation symbol coincides with the filter of the logical matrix. Under these identifications, the logical study of matrix models of a sentential logic becomes part of the model theory of equality-free first-order logic with a single unary predicate and, more specifically, of universal Horn logic without equality and with one unary relation symbol. This association led Blok and Pigozzi in [4] to recast many of their previous results on matrix models inside the framework of universal Horn logic without equality. It also inspired Elgueta and his collaborators [13, 14, 15, 16, 9, 17] to study the model theory of equality-free first order logic on its own right. Elgueta's viewpoint is closely related to the Abstract Algebraic Logic viewpoint in that it shows that a variety of results that were first discovered in the context of the model theory of sentential logics, i.e., for matrix models, also hold for arbitrary equality-free first-order models. In a similar direction, but influenced more by results in the model theory of first-order logic, Dellunde and her collaborators [6, 12, 10, 11] complement the work of Elgueta by studying different aspects of the model theory of equality-free first-order logic.

Of particular interest to the developments presented in this paper is the work of Elgueta on formulating a subdirect representation theory for classes of equality-free first-order structures [14]. Since it is well-known that a finitary sentential logic is protoalgebraic if and only if its class of reduced matrix models is closed under subdirect products (see, e.g., Theorem 1.3.7 of [8]), Elgueta spends a good part of [14] exploring a condition analogous to that of protoalgebraicity as applied to equality-free first-order structures. More precisely, having abstracted the notion of the Leibniz operator Ω from matrix models to arbitrary equality-free first-order models in [13], Elgueta defines a class $\mathcal{K}$ of structures to be protoalgebraic in [14], if this generalized version of the Leibniz operator is monotone in $\mathcal{K}$, in the sense that, for any two structures $M, N \in \mathcal{K}$, if N is a filter extension of M, denoted $M \preceq N$, then $\Omega(M) \subseteq \Omega(N)$. Elgueta shows that a wealth of results that previously were known to hold for the class of matrix models of a protoalgebraic logic remain valid for protoalgebraic classes of equality-free first-order structures.

In yet a different, but very related, direction, there has been recently a rapid development of the theory of Categorical Abstract Algebraic Logic, which, by now, includes an abstraction of the operator approach to algebraizability [20], an adaptation of the notion of a model of a sentential logic to π-institutions [21], and a study of algebraic set-functor-based models, suitable for π-institutions, called algebraic systems [22], that generalize universal algebras. Moreover, the notion of the Leibniz operator has been abstracted to the level of logics formalized as π-institutions in [23] and has served to define the class of protoalgebraic π-institutions, a categorical analog of the class of protoalgebraic logics.

These explorations on the categorical side of the theory, coupled with the work of Elgueta and of Dellunde, have led the author to the path of expanding the model theory of equality-free first-order logic to cover, in place of ordinary first-order structures, systems whose underlying algebraic components, rather than being universal algebras, are algebraic systems [26, 27, 28, 29]. These were called structure systems or, simply, systems in [26]. In the present paper the notion of a protoalgebraic class of first-order structures is abstracted to classes of systems. This is done based on an abstraction of the Leibniz operator from the level of first-order structures to the level of systems, carried out in [27]. We prove that a variety of results holding for classes of matrix models of protoalgebraic sentential logics, and shown by Elgueta to also hold for protoalgebraic classes of structures, hold for classes of structure systems as well. This paves the way for abstracting the theory of subdirect representability of Elgueta to a theory of subdirect representability for structure systems [31]. This latter theory also generalizes the theory of subdirect representability for partially ordered algebraic systems, that was developed by the author in [24, 25], based on previous work by Pałasińska and Pigozzi [19].

2 Preliminaries

Recall that a **clone category** is a category $\mathbf{F}$, whose objects are all finite natural numbers, that is isomorphic to the category of natural transformations N on a given functor $\mathrm{SEN} : \mathbf{Sign} \rightarrow \mathbf{Set}$ (see [23] for the definition of a category of natural transformations on a set-valued functor) via an isomorphism that preserves projections, and, as a consequence, also preserves objects. A **(structure system) language** is a triple $\mathcal{L} = \langle \mathbf{F}, R, \rho \rangle$, where $\mathbf{F}$ is a clone category, R is a nonempty set of relation symbols and $\rho : R \rightarrow \omega$ is an arity function.

Let $\mathrm{SEN} : \mathbf{Sign} \rightarrow \mathbf{Set}$ be a functor. An n**-ary relation system** R on SEN is a family $R = \{R_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, such that

- R_Σ is an n-ary relation on $\mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$, and
- $\mathrm{SEN}(f)^n(R_{\Sigma_1}) \subseteq R_{\Sigma_2}$, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$.

An $\mathcal{L}$**-(structure) system** $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$ is a triple consisting of

- a functor $\mathrm{SEN}^{\mathfrak{A}} : \mathbf{Sign}^{\mathfrak{A}} \rightarrow \mathbf{Set}$,
- a category of natural transformations $\mathbf{N}^{\mathfrak{A}}$ on $\mathrm{SEN}^{\mathfrak{A}}$, such that $F : \mathbf{F} \rightarrow \mathbf{N}^{\mathfrak{A}}$ is a surjective functor that preserves all projections $p^{kl} : k \rightarrow 1, k \in \omega, l < k$, and

- $R^{\mathfrak{A}} = \{r^{\mathfrak{A}} : r \in R\}$ a family of relation systems on $\mathrm{SEN}^{\mathfrak{A}}$ indexed by R, such that $r^{\mathfrak{A}}$ is n-ary if $\rho(r) = n$.

Let $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$ be an $\mathcal{L}$-system. A (binary) relation system $\theta = \{\theta_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ on $\mathrm{SEN}^{\mathfrak{A}}$ is said to be an **$\mathbf{N}^{\mathfrak{A}}$-congruence system of** $\mathfrak{A}$ if θ is an $\mathbf{N}^{\mathfrak{A}}$-congruence system on $\mathrm{SEN}^{\mathfrak{A}}$ that is **compatible** with all relation systems of $\mathfrak{A}$, i.e., that satisfies, for all $r \in R^{\mathfrak{A}}$, with $\rho(r) = n$, all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$,

$$\vec{\phi} \in r^{\mathfrak{A}}_\Sigma \quad \text{and} \quad \vec{\phi}\ \theta^n_\Sigma\ \vec{\psi} \quad \text{imply} \quad \vec{\psi} \in r^{\mathfrak{A}}_\Sigma.$$

The collection of all $\mathbf{N}^{\mathfrak{A}}$-congruence systems of $\mathfrak{A}$ will be denoted by $\mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$. It was shown in Proposition 1 of [27] that the partially ordered set $\mathbf{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A}) = \langle \mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A}), \leq\rangle$, of all $\mathbf{N}^{\mathfrak{A}}$-congruence systems of $\mathfrak{A}$ ordered by signature-wise inclusion $\leq$, is a principal ideal of the complete lattice $\mathbf{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathrm{SEN}^{\mathfrak{A}})$ of all $\mathbf{N}^{\mathfrak{A}}$-congruence systems on $\mathrm{SEN}^{\mathfrak{A}}$. Its largest element is called the **Leibniz congruence system** of $\mathfrak{A}$ and is denoted by $\Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$ or, simply, by $\Omega(\mathfrak{A})$.

If $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$ is an $\mathcal{L}$-system, the **reduction** of $\mathfrak{A}$, denoted $\mathfrak{A}^*$, is the quotient of $\mathfrak{A}$ by the Leibniz congruence system $\Omega(\mathfrak{A})$ of $\mathfrak{A}$. Given $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and $\phi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$, we write ϕ^* for $\phi/\Omega_\Sigma(\mathfrak{A}) \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)/\Omega_\Sigma(\mathfrak{A})$.

If $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle, \mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle$ are two $\mathcal{L}$-systems and $\langle F, \alpha\rangle : \mathrm{SEN}^{\mathfrak{A}} \to \mathrm{SEN}^{\mathfrak{B}}$ is a singleton translation, we write $\langle F^*, \alpha^*\rangle$ for the pair $F^* = F$ and $\alpha^* = \{\alpha^*_\Sigma\}_{\Sigma\in|\mathbf{Sign}^{\mathfrak{A}}|}$, defined, for al $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, by

$$\alpha^*_\Sigma(\phi^*) = \alpha_\Sigma(\phi)^*, \quad \text{for all } \phi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma).$$

$\alpha^*_\Sigma : \mathrm{SEN}^{\mathfrak{A}}(\Sigma)/\Omega_\Sigma(\mathfrak{A}) \to \mathrm{SEN}^{\mathfrak{B}}(F(\Sigma))/\Omega_{F(\Sigma)}(\mathfrak{B})$ is not in general a well-defined mapping, but it was shown in Proposition 16 of [27] that, if $\langle F, \alpha\rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$ is a reductive $\mathcal{L}$-morphism, i.e., a surjective strict $\mathcal{L}$-morphism, then $\langle F^*, \alpha^*\rangle : \mathfrak{A}^* \to \mathfrak{B}^*$ is also an $\mathcal{L}$-morphism, such that α^*_Σ is a bijection, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$.

A class $\mathtt{K}$ of $\mathcal{L}$-systems is said to be a **full class** whenever it is closed under expansions and contains an $\mathcal{L}$-system with at least one nonempty relation system. $\mathtt{K}$ is said to be an **abstract class** if it is full and closed under contractions. On the other hand, $\mathtt{K}$ is called a **reduced class** if it contains some nontrivial $\mathcal{L}$-system and all its members are (Leibniz) reduced $\mathcal{L}$-systems.

Let Γ be a set of equality-free $\mathcal{L}$-formulas. Define the **abstract class** $\mathrm{Mod}(\Gamma)$ **of models of** Γ as the class of all $\mathcal{L}$-systems $\mathfrak{A}$, such that $\mathfrak{A} \models \gamma$, for all $\gamma \in \Gamma$. The **reduced class** $\mathrm{Mod}^*(\Gamma)$ **of models of** Γ is the class of all $\mathcal{L}$-systems $\mathfrak{A} \in \mathrm{Mod}(\Gamma)$, that are also reduced. Given a class of $\mathcal{L}$-systems $\mathtt{K}$, the **reduction** $\mathtt{K}^*$ of $\mathtt{K}$ is the class of all isomorphic copies of reductions of $\mathcal{L}$-systems in $\mathtt{K}$. The operator * is called the **reduction operator**. The inverse process of reduction takes a class $\mathtt{K}$ of $\mathcal{L}$-systems into the least abstract class of $\mathcal{L}$-systems that contains $\mathtt{K}$, called the **abstraction of** $\mathtt{K}$.

Lemma 14 of [2] characterizes the abstraction of a class of first-order structures $\mathtt{K}$, in the context of the ordinary model theory of equality-free first-order logic, as the class

$\mathrm{EC}(\mathtt{K})$, where E is the operator of taking expansions and C that of taking contractions. Unfortunately, it has not been possible to derive an analog of this characterization in the present context.

Since, by Proposition 7 of [26], reductive $\mathcal{L}$-morphisms preserve the satisfiability of equality-free $\mathcal{L}$-formulas, $\mathtt{K}$ is equality-free elementarily equivalent to $\mathtt{K}^*$ and, therefore, for every set of equality-free $\mathcal{L}$-formulas Γ, we have that $\mathrm{Mod}^*(\Gamma) = (\mathrm{Mod}(\Gamma))^*$. Moreover, $\mathrm{Mod}(\Gamma)$ is always an abstract class of $\mathcal{L}$-systems, whereas $\mathrm{Mod}^*(\Gamma)$ is a reduced class whenever it contains a nontrivial $\mathcal{L}$-system.

Given an operator O on classes of $\mathcal{L}$-systems, by O^* is denoted the operator sending a class $\mathtt{K}$ to $\mathrm{O}^*(\mathtt{K}) = \mathrm{L}(\mathrm{O}(\mathtt{K})) := (\mathrm{O}(\mathtt{K}))^*$. The L in place of the * for the reduction operator comes from the term **Leibniz reduction**, which is sometimes used in place of the simpler term reduction to make explicit the fact that the quotients are taken with respect to the Leibniz congruence systems of the $\mathcal{L}$-systems in the class $\mathtt{K}$. The following theorem, an analog of the Reduction Operator Lemma, Theorem 4.7 of [13], was proven in [29] and constitutes a basic tool for translating results on abstract classes or full classes to analogous results for their corresponding reduced classes (see also Lemma 1.4 of [14]). Recall from [29] that by S is denoted the operator of taking subsystems of $\mathcal{L}$-systems, by $\mathrm{S_i}$ the operator of taking subsystems with isomorphic functor components and by $\mathrm{S_{ie}}$ the operator of taking elementary subsystems with isomorphic functor components. As expected, by $\mathrm{P}, \mathrm{P_f}, \mathrm{P_u}, \mathrm{P_{sd}}$ are denoted the operators of taking isomorphic copies of direct products, reduced products, ultraproducts and subdirect products, respectively, of systems in a given class.

Theorem 1 (Reduction Operator Lemma, Theorem 7 of [29]) *1.* $\mathrm{LS_i} = \mathrm{LS_iL}$.

2. For all $\mathrm{O} \in \{\mathrm{P}, \mathrm{P_f}, \mathrm{P_u}, \mathrm{P_{sd}}\}$, $\mathrm{LO} = \mathrm{LOL}$.

3. $\mathrm{LS_{ie}} = \mathrm{LS_{ie}L} = \mathrm{S_{ie}L}$.

3 Filter Systems of Structure Systems

It has been shown in [27] that, given two $\mathcal{L}$-systems $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$ and $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}} \rangle, R^{\mathfrak{B}} \rangle$ and a strong $\mathcal{L}$-morphism $\langle F, \alpha \rangle : \mathfrak{A} \rightarrow_s \mathfrak{B}$, the kernel $\mathrm{Ker}(\langle F, \alpha \rangle)$ is a congruence system of $\mathfrak{A}$ and that, conversely, given a congruence system θ of $\mathfrak{A}$, the projection $\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \pi^\theta \rangle : \mathfrak{A} \rightarrow \mathfrak{A}/\theta$ is a strong $\mathcal{L}$-morphism, whose kernel is θ. Thus, in the context of $\mathcal{L}$-systems, congruence systems amount to kernels of strong $\mathcal{L}$-morphisms. We follow Elgueta [14] in providing in this section some extensions of the notion of a congruence system that serve better in handling $\mathcal{L}$-systems than do the congruence systems of [27]. The inspiration for these extensions comes from the approach that views the theory of algebras as the model theory of a universal Horn logic over an equality-free first-order language type with a single binary relation symbol, representing a congruence on the algebra. Elgueta explains that this point of view triggers perceiving, in the context of the model theory of equality-free first-order logic, the set of relations as a form of a generalized congruence on the underlying algebras of the models. Inspired by Elgueta, the same point of view will be taken here in the context of $\mathcal{L}$-systems. More precisely, the set of relation systems will be roughly thought of as a form of a generalized congruence system on the underlying algebraic system of a given $\mathcal{L}$-system.

Let $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$ be an $\mathcal{L}$-system. The **filter system** of $\mathfrak{A}$ is the set $R^{\mathfrak{A}}$ of its relation systems. If $\mathbf{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle\rangle$ is an $\mathcal{L}$-algebraic system, then an **$\mathcal{L}$-filter system** on $\mathbf{A}$ is any set of interpretations in $\mathbf{A}$ of the relation symbols of $\mathcal{L}$ (as relation systems respecting arities). Rather than following the common practice in the theory of logical matrices in Abstract Algebraic Logic of discussing the collection of all $\mathcal{L}$-filter systems on a given $\mathcal{L}$-algebraic system, the collection of all $\mathcal{L}$-systems over the same underlying $\mathcal{L}$-algebraic system will be considered here. These two approaches present certain advantages and disadvantages in different contexts, but they are essentially the same.

Let $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle, \mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle$ be two $\mathcal{L}$-systems. $\mathfrak{B}$ is called a **filter extension** of $\mathfrak{A}$, in symbols $\mathfrak{A} \sqsubseteq \mathfrak{B}$, and $R^{\mathfrak{B}}$ is said to be a **filter system on** $\mathfrak{A}$ if $\mathbf{A} := \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle\rangle = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle\rangle =: \mathbf{B}$ and $r^{\mathfrak{A}} \leq r^{\mathfrak{B}}$, for all $r \in R$. If, in addition, $\mathfrak{B} \in \mathtt{K}$, for some class $\mathtt{K}$ of $\mathcal{L}$-systems, then $\mathfrak{B}$ is called a **$\mathtt{K}$-filter extension** of $\mathfrak{A}$ and $R^{\mathfrak{B}}$ a **$\mathtt{K}$-filter system on** $\mathfrak{A}$. The collection $\mathrm{Fe}_{\mathtt{K}}(\mathfrak{A})$ of all $\mathtt{K}$-filter extensions of $\mathfrak{A}$ equipped with $\sqsubseteq$ is a partially ordered set, denoted by $\mathbf{Fe}_{\mathtt{K}}(\mathfrak{A}) = \langle \mathrm{Fe}_{\mathtt{K}}(\mathfrak{A}), \sqsubseteq\rangle$.

Theorems 2 and 3, that follow, are generalizations in the context of $\mathcal{L}$-systems of Lemmas 2.1 and 2.2 of [14], which are, in turn, generalizations of well-known results on homomorphisms from Universal Algebra.

Theorem 2 (Filter Homomorphism Theorem; Prop. 16 of [27]) *Suppose that* $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$ *and* $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle$ *are $\mathcal{L}$-systems and* $\langle F, \alpha\rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$ *a reductive system morphism. Define* $\langle F^*, \alpha^*\rangle$ *by letting* $F^* = F$ *and* $\alpha^* : \mathrm{SEN}^{\mathfrak{A}}/\Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A}) \to \mathrm{SEN}^{\mathfrak{B}}/\Omega^{\mathbf{N}^{\mathfrak{B}}}(\mathfrak{B}) \circ F^*$ *be given, for all* $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, *by*

$$\alpha^*_\Sigma(\phi^*) = \alpha_\Sigma(\phi)^*, \quad \textit{for all } \phi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma).$$

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, "\langle F,\alpha\rangle"] \arrow[d, "\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \pi^{\mathbf{N}^{\mathfrak{A}}}\rangle"'] & \mathfrak{B} \arrow[d, "\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{B}}}, \pi^{\mathbf{N}^{\mathfrak{B}}}\rangle"] \\
\mathfrak{A}^* \arrow[r, "\langle F^*,\alpha^*\rangle"'] & \mathfrak{B}^*
\end{tikzcd}$$

Then $\langle F^*, \alpha^*\rangle : \mathfrak{A}^* \to \mathfrak{B}^*$ *is an $\mathcal{L}$-morphism, such that* α^*_Σ *is a bijection, for all* $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$.

Theorem 3 (Second Filter Isomorphism Theorem) *Let* $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$, $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle$ *be two $\mathcal{L}$-systems and* $\langle F, \alpha\rangle : \mathfrak{A} \rightarrowtail_s \mathfrak{B}$ *a strong injective $\mathcal{L}$-*

morphism. Then, there exists a strong injective $\mathcal{L}$-morphism $\langle G,\beta\rangle : \mathfrak{A}/\alpha^{-1}(\Omega(\mathfrak{B})) \rightarrowtail_s \mathfrak{B}^*$,

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, "\langle F,\alpha\rangle"] \arrow[dr, "\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \pi^{\alpha^{-1}(\Omega(\mathfrak{B}))}\rangle"'] & \mathfrak{B} \arrow[r, "\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{B}}}, \pi^{\Omega(\mathfrak{B})}\rangle"] & \mathfrak{B}^* \\
& \mathfrak{A}/\alpha^{-1}(\Omega(\mathfrak{B})) \arrow[ur, "\langle G,\beta\rangle"'] &
\end{tikzcd}$$

given by $G=F$ and, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|, \phi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$,

$$\beta_\Sigma(\phi/\alpha_\Sigma^{-1}(\Omega_{F(\Sigma)}(\mathfrak{B}))) = \alpha_\Sigma(\phi)/\Omega_{F(\Sigma)}(\mathfrak{B}),$$

that makes the preceding diagram commute, i.e., such that $\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{B}}}, \pi^{\Omega(\mathfrak{B})}\rangle \circ \langle F,\alpha\rangle = \langle G,\beta\rangle \circ \langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \pi^{\alpha^{-1}(\Omega(\mathfrak{B}))}\rangle$.

Proof:

We first show that $\beta : \mathrm{SEN}^{\mathfrak{A}}/\alpha^{-1}(\Omega(\mathfrak{B})) \to \mathrm{SEN}^{\mathfrak{B}}/\Omega(\mathfrak{B}) \circ F$ is well-defined. Suppose that $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|, \phi,\psi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$, such that $\phi/\alpha_\Sigma^{-1}(\Omega_{F(\Sigma)}(\mathfrak{B})) = \psi/\alpha_\Sigma^{-1}(\Omega_{F(\Sigma)}(\mathfrak{B}))$. Then $\langle\phi,\psi\rangle \in \alpha_\Sigma^{-1}(\Omega_{F(\Sigma)}(\mathfrak{B}))$, whence $\langle\alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\rangle \in \Omega_{F(\Sigma)}(\mathfrak{B})$. Therefore, we obtain that $\alpha_\Sigma(\phi)/\Omega_{F(\Sigma)}(\mathfrak{B}) = \alpha_\Sigma(\psi)/\Omega_{F(\Sigma)}(\mathfrak{B})$, showing that

$$\beta_\Sigma(\phi/\alpha_\Sigma^{-1}(\Omega_{F(\Sigma)}(\mathfrak{B}))) = \beta_\Sigma(\psi/\alpha_\Sigma^{-1}(\Omega_{F(\Sigma)}(\mathfrak{B}))).$$

To see that β is a natural transformation, let $\Sigma_1,\Sigma_2 \in |\mathbf{Sign}^{\mathfrak{A}}|, f \in \mathbf{Sign}^{\mathfrak{A}}(\Sigma_1,\Sigma_2)$ and $\phi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma_1)$. We have

$$\begin{tikzcd}
\mathrm{SEN}^{\mathfrak{A}}(\Sigma_1)/\alpha_{\Sigma_1}^{-1}(\Omega_{F(\Sigma_1)}(\mathfrak{B})) \arrow[r, "\beta_{\Sigma_1}"] \arrow[d, "\mathrm{SEN}^{\mathfrak{A}}(f)/\alpha^{-1}(\Omega(\mathfrak{B}))"'] & \mathrm{SEN}^{\mathfrak{B}}(F(\Sigma_1))/\Omega_{F(\Sigma_1)}(\mathfrak{B}) \arrow[d, "\mathrm{SEN}^{\mathfrak{B}}(F(f))/\Omega(\mathfrak{B})"] \\
\mathrm{SEN}^{\mathfrak{A}}(\Sigma_2)/\alpha_{\Sigma_2}^{-1}(\Omega_{F(\Sigma_2)}(\mathfrak{B})) \arrow[r, "\beta_{\Sigma_2}"'] & \mathrm{SEN}^{\mathfrak{B}}(F(\Sigma_2))/\Omega_{F(\Sigma_2)}(\mathfrak{B})
\end{tikzcd}$$

$$\begin{array}{l}
\beta_{\Sigma_2}(\mathrm{SEN}^{\mathfrak{A}}(f)/\alpha^{-1}(\Omega(\mathfrak{B}))(\phi/\alpha_{\Sigma_1}^{-1}(\Omega_{F(\Sigma_1)}(\mathfrak{B})))) \\
\qquad = \beta_{\Sigma_2}(\mathrm{SEN}^{\mathfrak{A}}(f)(\phi)/\alpha_{\Sigma_2}^{-1}(\Omega_{F(\Sigma_2)}(\mathfrak{B}))) \\
\qquad = \alpha_{\Sigma_2}(\mathrm{SEN}^{\mathfrak{A}}(f)(\phi))/\Omega_{F(\Sigma_2)}(\mathfrak{B}) \\
\qquad = \mathrm{SEN}^{\mathfrak{B}}(F(f))(\alpha_{\Sigma_1}(\phi))/\Omega_{F(\Sigma_2)}(\mathfrak{B}) \\
\qquad = \mathrm{SEN}^{\mathfrak{B}}(F(f))/\Omega(\mathfrak{B})(\alpha_{\Sigma_1}(\phi)/\Omega_{F(\Sigma_1)}(\mathfrak{B})) \\
\qquad = \mathrm{SEN}^{\mathfrak{B}}(F(f))/\Omega(\mathfrak{B})(\beta_{\Sigma_1}(\phi/\alpha_{\Sigma_1}^{-1}(\Omega_{F(\Sigma_1)}(\mathfrak{B})))).
\end{array}$$

To show that $\langle G,\beta\rangle$ is an $\mathcal{L}$-algebraic system morphism, consider σ in $\mathbf{F}$ n-ary, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|, \phi_0,\ldots,\phi_{n-1} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$. Then

$$\begin{array}{rl}
\multicolumn{2}{l}{\beta_\Sigma(\sigma_\Sigma^{\mathfrak{A}/\alpha^{-1}(\Omega(\mathfrak{B}))}(\phi_0/\alpha_\Sigma^{-1}(\Omega_{F(\Sigma)}(\mathfrak{B})),\ldots,\phi_{n-1}/\alpha_\Sigma^{-1}(\Omega_{F(\Sigma)}(\mathfrak{B}))))} \\
= & \beta_\Sigma(\sigma_\Sigma^{\mathfrak{A}}(\phi_0,\ldots,\phi_{n-1})/\alpha_\Sigma^{-1}(\Omega(\mathfrak{B}))) \\
= & \alpha_\Sigma(\sigma_\Sigma^{\mathfrak{A}}(\phi_0,\ldots,\phi_{n-1}))/\Omega_{F(\Sigma)}(\mathfrak{B}) \\
= & \sigma_{F(\Sigma)}^{\mathfrak{B}}(\alpha_\Sigma(\phi_0),\ldots,\alpha_\Sigma(\phi_{n-1}))/\Omega_{F(\Sigma)}(\mathfrak{B}) \\
= & \sigma_{F(\Sigma)}^{\mathfrak{B}/\Omega(\mathfrak{B})}(\alpha_\Sigma(\phi_0)/\Omega_{F(\Sigma)}(\mathfrak{B}),\ldots,\alpha_\Sigma(\phi_{n-1})/\Omega_{F(\Sigma)}(\mathfrak{B})) \\
= & \sigma_{F(\Sigma)}^{\mathfrak{B}^*}(\beta_\Sigma(\phi_0/\alpha_\Sigma^{-1}(\Omega_{F(\Sigma)}(\mathfrak{B}))),\ldots,\beta_\Sigma(\phi_{n-1}/\alpha_\Sigma^{-1}(\Omega_{F(\Sigma)}(\mathfrak{B})))).
\end{array}$$

Next, to see that $\langle G,\beta\rangle$ is an $\mathcal{L}$-morphism and that it is strong, let $r \in R$, with $\rho(r) = n$, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|, \phi_0,\ldots,\phi_{n-1} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$. We have

$$\begin{array}{rl}
\multicolumn{2}{l}{\langle \phi_0/\alpha_\Sigma^{-1}(\Omega_{F(\Sigma)}(\mathfrak{B})),\ldots,\phi_{n-1}/\alpha_\Sigma^{-1}(\Omega_{F(\Sigma)}(\mathfrak{B}))\rangle \in r_\Sigma^{\mathfrak{A}/\alpha^{-1}(\Omega(\mathfrak{B}))}} \\
\text{iff} & \langle \phi_0,\ldots,\phi_{n-1}\rangle \in r_\Sigma^{\mathfrak{A}} \\
\text{iff} & \langle \alpha_\Sigma(\phi_0),\ldots,\alpha_\Sigma(\phi_{n-1})\rangle \in r_{F(\Sigma)}^{\mathfrak{B}} \\
\text{iff} & \langle \alpha_\Sigma(\phi_0)/\Omega_{F(\Sigma)}(\mathfrak{B}),\ldots,\alpha_\Sigma(\phi_{n-1})/\Omega_{F(\Sigma)}(\mathfrak{B})\rangle \in r_{F(\Sigma)}^{\mathfrak{B}^*} \\
\text{iff} & \langle \beta_\Sigma(\phi_0/\alpha_\Sigma^{-1}(\Omega_{F(\Sigma)}(\mathfrak{B}))),\ldots,\beta_\Sigma(\phi_{n-1}/\alpha_\Sigma^{-1}(\Omega_{F(\Sigma)}(\mathfrak{B})))\rangle \in r_{F(\Sigma)}^{\mathfrak{B}^*}.
\end{array}$$

Finally, if $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and $\phi,\psi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$ are such that $\beta_\Sigma(\phi/\alpha_\Sigma^{-1}(\Omega_{F(\Sigma)}(\mathfrak{B}))) = \beta_\Sigma(\psi/\alpha_\Sigma^{-1}(\Omega_{F(\Sigma)}(\mathfrak{B})))$, then we have $\langle \alpha_\Sigma(\phi),\alpha_\Sigma(\psi)\rangle \in \Omega_{F(\Sigma)}(\mathfrak{B})$, whence we obtain that $\langle \phi,\psi\rangle \in \alpha_\Sigma^{-1}(\Omega_{F(\Sigma)}(\mathfrak{B}))$ and, thus, $\phi/\alpha_\Sigma^{-1}(\Omega_{F(\Sigma)}(\mathfrak{B})) = \psi/\alpha_\Sigma^{-1}(\Omega_{F(\Sigma)}(\mathfrak{B}))$, showing that $\langle G,\beta\rangle$ is in fact injective. That the diagram provided in the statement of the theorem commutes is obvious. ∎

4 Structure Systems Over an Algebraic System

Suppose that $\mathbf{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle\rangle$ is an $\mathcal{L}$-algebraic system and $\mathtt{K}$ a class of $\mathcal{L}$-systems. By the collection of $\mathtt{K}$-**systems on** $\mathbf{A}$, in symbols $\mathtt{K}_{\mathbf{A}}$, is meant the collection of all members of $\mathtt{K}$ whose algebraic reduct is $\mathbf{A}$. If $\mathtt{K}$ is the entire class of $\mathcal{L}$-systems, then we write $\mathtt{A}$ in place of $\mathtt{K}_{\mathbf{A}}$. It is clear that $\mathtt{K}_{\mathbf{A}}$ is always a subclass of $\mathtt{A}$, for every class $\mathtt{K}$ of $\mathcal{L}$-systems. The class $\mathtt{A}$, equipped with $\sqsubseteq$, forms an algebraic lattice whose compact elements are the $\mathcal{L}$-systems $\mathfrak{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle, R^{\mathfrak{A}}\rangle$, such that, for all $r \in R$ and all $\Sigma \in |\mathbf{Sign}|$, $r_\Sigma^{\mathfrak{A}}$ is finite. The join and the meet operations in this lattice are given, for all collections $\mathfrak{A}_i = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle, R^i\rangle, i \in I$, of $\mathcal{L}$-systems in $\mathtt{A}$, by

$$\bigvee_{i\in I} \mathfrak{A}_i = \langle \mathbf{A}, \bigcup_{i\in I} R^i\rangle, \quad \bigwedge_{i\in I} \mathfrak{A}_i = \langle \mathbf{A}, \bigcap_{i\in I} R^i\rangle,$$

where, of course, $\bigcup_{i\in I} R^i = \{\bigcup_{i\in I} r^i : r \in R\}$ and $\bigcap_{i\in I} R^i = \{\bigcap_{i\in I} r^i : r \in R\}$, and $\bigcup_{i\in I} r^i$ and $\bigcap_{i\in I} r^i$ denote signature-wise union and intersection, respectively. We also write $\mathbf{K}_{\mathbf{A}} = \langle \mathtt{K}_{\mathbf{A}}, \sqsubseteq\rangle$ and $\mathbf{A} = \langle \mathtt{A}, \sqsubseteq\rangle$ for the corresponding posets.

An indication of the close connection between the order structure of the posets of $\mathtt{K}$-systems on $\mathbf{A}$, for all $\mathcal{L}$-algebraic systems $\mathbf{A}$, and the structural properties of the class $\mathtt{K}$ is given by the following results of [30].

Theorem 4 *Let $\mathtt{K}$ be a class of $\mathcal{L}$-systems.*

1. (Theorem 6 of [30]) *If $\mathtt{K}$ is a full class, then $\mathtt{K}$ is closed under subdirect products if and only if, for every $\mathcal{L}$-algebraic system $\mathbf{A}$, the collection $\mathtt{K}_{\mathbf{A}}$ is closed under arbitrary meets.*

2. (Theorem 10 of [30]) *If $\mathtt{K}$ is an abstract class, that is closed under subsystems and reduced products, then for every $\mathcal{L}$-algebraic system $\mathbf{A}$, the collection $\mathtt{K}_{\mathbf{A}}$ is closed under arbitrary meets and under joins of sets upward directed by $\sqsubseteq$.*

A **Lyndon class** is a class of $\mathcal{L}$-systems which is full and closed under subdirect products. A **quasi-variety** is an abstract class that is closed under subsystems and reduced products.

For each $\mathcal{L}$-algebraic system $\mathbf{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F \rangle\rangle$, if $\mathtt{K}$ is a Lyndon class of $\mathcal{L}$-systems, then $\mathtt{K}$ contains the trivial $\mathcal{L}$-system on $\mathbf{A}$, denoted by $\mathfrak{E}_{\mathbf{A}}$. This is the $\mathcal{L}$-system on $\mathbf{A}$, such that, for all $r \in R$, with $\rho(r) = n$, $r_\Sigma^{\mathfrak{E}_{\mathbf{A}}} = \mathrm{SEN}(\Sigma)^n$, for all $\Sigma \in |\mathbf{Sign}|$. Also, for every $\mathcal{L}$-system $\mathfrak{A}$, $\mathbf{Fe}_{\mathtt{K}}(\mathfrak{A})$ is a complete sublattice of $\mathtt{A}$ whose largest element is $\mathfrak{E}_{\mathbf{A}}$. If, in addition, $\mathtt{K}$ is a quasi-variety, then $\mathbf{Fe}_{\mathtt{K}}(\mathfrak{A})$ is algebraic. The join in this lattice is denoted by $\bigvee^{\mathtt{K}}$ and is given as follows: If $\mathbf{A}$ is the underlying $\mathcal{L}$-algebraic system of $\mathfrak{A}$ and $\mathfrak{A}_i = \langle \mathbf{A}, R^i \rangle \in \mathrm{Fe}_{\mathtt{K}}(\mathfrak{A}), i \in I$, then

$$\bigvee_{i \in I}^{\mathtt{K}} \mathfrak{A}_i = \bigwedge \{\mathfrak{B} \in \mathtt{K}_{\mathbf{A}} : \bigvee_{i \in I} \mathfrak{A}_i \sqsubseteq \mathfrak{B}\}.$$

Thus, if $\mathtt{K}$ is a Lyndon class, there always exists a least $\mathtt{K}$-filter extension of $\mathfrak{A}$, for every $\mathcal{L}$-system $\mathfrak{A}$. This $\mathtt{K}$-filter extension is called the **$\mathtt{K}$-filter extension generated by** $\mathfrak{A}$ and denoted by $\mathrm{Fg}_{\mathtt{K}}(\mathfrak{A})$.

5 The Leibniz Operator

Recall, once more, from [27] that, given an $\mathcal{L}$-system $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$, there always exists a largest $\mathbf{N}^{\mathfrak{A}}$-congruence system $\Omega(\mathfrak{A})$ on $\mathrm{SEN}^{\mathfrak{A}}$ that is compatible with all relation systems in $R^{\mathfrak{A}}$ in the sense that, if $r \in R$, with $\rho(r) = n$, then, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and all $\phi_0, \ldots, \phi_{n-1}, \psi_0, \ldots, \psi_{n-1} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$,

$$\vec{\phi} \in r_\Sigma^{\mathfrak{A}} \quad \text{and} \quad \vec{\phi}\ \Omega_\Sigma(\mathfrak{A})^n\ \vec{\psi} \quad \text{imply} \quad \vec{\psi} \in r_\Sigma^{\mathfrak{A}}.$$

As mentioned previously, $\Omega(\mathfrak{A})$ is called the Leibniz congruence system of $\mathfrak{A}$ and the mapping Ω that assigns to each $\mathcal{L}$-system $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$ its Leibniz congruence system $\Omega(\mathfrak{A})$ is called the **Leibniz operator**. If $\mathtt{K}$ is a class of $\mathcal{L}$-systems and $\mathbf{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle\rangle$ is an $\mathcal{L}$-algebraic system, we write $\Omega_{\mathtt{K},\mathbf{A}}$, or simply $\Omega_{\mathbf{A}}$, to indicate the restriction of Ω to

$\mathtt{K}_{\mathbf{A}}$. $\Omega(\mathfrak{A})$ may also be denoted by $\Omega_{\mathbf{A}}(R^{\mathfrak{A}})$, following the usual notation from the theory of logical matrices in Abstract Algebraic Logic.

A congruence system in the image of $\mathtt{K}_{\mathbf{A}}$ by Ω is called a $\mathtt{K}$-congruence system on $\mathbf{A}$ and the collection of all these congruence systems is denoted by $\mathrm{Con}_{\mathtt{K}}(\mathbf{A})$. Ordered by $\leq$, this collection forms a partially ordered set, which is denoted by $\mathbf{Con}_{\mathtt{K}}(\mathbf{A}) = \langle \mathrm{Con}_{\mathtt{K}}(\mathbf{A}), \leq \rangle$.

Proposition 5 *Suppose that* $\mathtt{K}$ *is an abstract class of* $\mathcal{L}$*-systems,* $\mathbf{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F \rangle \rangle$ *an* $\mathcal{L}$*-algebraic system and* θ *an* $\mathbf{N}$*-congruence system on* $\mathbf{A}$*. Then* $\theta \in \mathrm{Con}_{\mathtt{K}}(\mathbf{A})$ *if and only if, there exists* $\mathfrak{B} = \langle \mathrm{SEN}^\theta, \langle \mathbf{N}^\theta, F^\theta \rangle, R^{\mathfrak{B}} \rangle$*, with* $\mathfrak{B} \in \mathtt{K}$*, and* $\Omega(\mathfrak{B}) = \Delta^{\mathrm{SEN}^\theta}$*, i.e., a reduced* $\mathtt{K}$*-system on* $\mathbf{A}/\theta$*.*

Proof:

Suppose, first, that $\theta \in \mathrm{Con}_{\mathtt{K}}(\mathbf{A})$. Then, there exists $\mathfrak{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F \rangle, R^{\mathfrak{A}} \rangle \in \mathtt{K}$, such that $\Omega(\mathfrak{A}) = \theta$. Thus, the $\mathcal{L}$-system $\mathfrak{A}^\theta = \langle \mathrm{SEN}^\theta, \langle \mathbf{N}^\theta, F^\theta \rangle, R^{\mathfrak{A}^\theta} \rangle$ is well-defined and $\langle \mathrm{I}, \pi^\theta \rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{A}^\theta$ is a reductive $\mathcal{L}$-morphism. Hence, since $\mathfrak{A} \in \mathtt{K}$ and $\mathtt{K}$ is abstract, $\mathfrak{A}^\theta \in \mathtt{K}$ and $\mathfrak{A}^\theta$ is obviously reduced, since $\theta = \Omega(\mathfrak{A})$.

Suppose, conversely, that there exists $\mathfrak{B} = \langle \mathrm{SEN}^\theta, \langle \mathbf{N}^\theta, F^\theta \rangle, R^{\mathfrak{B}} \rangle$, such that $\mathfrak{B} \in \mathtt{K}$ and $\Omega(\mathfrak{B}) = \Delta^{\mathrm{SEN}^\theta}$. Then, by Lemma 5 of [26], $\langle \mathrm{I}, \pi^\theta \rangle : \pi^{\theta^{-1}}(\mathfrak{B}) \twoheadrightarrow_s \mathfrak{B}$ is a reductive $\mathcal{L}$-morphism. Hence $\pi^{\theta^{-1}}(\mathfrak{B}) \in \mathtt{K}$, since $\mathfrak{B} \in \mathtt{K}$ and $\mathtt{K}$ is abstract. Moreover, by Theorem 5 of [27], we have that $\Omega(\pi^{\theta^{-1}}(\mathfrak{B})) = \pi^{\theta^{-1}}(\Omega(\mathfrak{B})) = \pi^{\theta^{-1}}(\Delta^{\mathrm{SEN}^\theta}) = \theta$, whence $\theta \in \mathrm{Con}_{\mathtt{K}}(\mathbf{A})$.

■

6 Filter Congruence Systems and Isomorphism Theorems

Let $\mathtt{K}$ be a class of $\mathcal{L}$-systems and $\mathfrak{A}$ a single $\mathcal{L}$-system. By a $\mathtt{K}$-**filter congruence system** on $\mathfrak{A}$ is meant a pair $\Theta = \langle \mathfrak{B}, \theta \rangle$, where $\mathfrak{B} \in \mathrm{Fe}_{\mathtt{K}}(\mathfrak{A})$ and $\theta \in \mathrm{Con}(\mathfrak{B})$. The collection of all $\mathtt{K}$-filter congruence systems on $\mathfrak{A}$ is denoted by $\mathrm{Fc}_{\mathtt{K}}(\mathfrak{A})$. We write simply $\mathrm{Fc}(\mathfrak{A})$, if $\mathtt{K}$ is the entire class of $\mathcal{L}$-systems. The members of $\mathrm{Fc}(\mathfrak{A})$ are called **filter congruence systems** on $\mathfrak{A}$. Since $\Delta^{\mathrm{SEN}^{\mathfrak{A}}}$ and $\Omega(\mathfrak{A})$ are congruence systems on $\mathfrak{A}$, for every $\mathcal{L}$-system $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$, both $\langle \mathfrak{A}, \Delta^{\mathrm{SEN}^{\mathfrak{A}}} \rangle$ and $\langle \mathfrak{A}, \Omega(\mathfrak{A}) \rangle$ are in $\mathrm{Fc}(\mathfrak{A})$. They are denoted by $\Theta^{\mathfrak{A}}$ and $\Theta^{\mathfrak{A},\Omega}$, respectively, and are called the **trivial filter congruence system** and the **Leibniz filter congruence system** on $\mathfrak{A}$, respectively.

If $\Theta = \langle \mathfrak{B}, \theta \rangle \in \mathrm{Fc}(\mathfrak{A})$, then the **quotient** $\mathcal{L}$**-system** $\mathfrak{A}/\Theta$ is defined as the quotient $\mathcal{L}$-system $\mathfrak{B}/\theta$. In that case, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$,

$$\phi/\Theta_\Sigma := \phi/\theta_\Sigma, \quad \text{for all } \phi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma).$$

If $\langle F, \alpha \rangle : \mathfrak{A} \to \mathfrak{B}$ is an $\mathcal{L}$-morphism and $\Theta = \langle \mathfrak{B}', \theta \rangle \in \mathrm{Fc}(\mathfrak{B})$, the **pre-image of** Θ **by** $\langle F, \alpha \rangle$, written $\alpha^{-1}(\Theta)$, is the pair $\alpha^{-1}(\Theta) = \langle \alpha^{-1}(\mathfrak{B}'), \alpha^{-1}(\theta) \rangle$. It is easy to see that $\alpha^{-1}(\mathfrak{B}')$ is a filter extension of $\mathfrak{A}$ and that $\alpha^{-1}(\theta)$ is a congruence system of $\alpha^{-1}(\mathfrak{B}')$, whence $\alpha^{-1}(\Theta)$ is a well-defined filter congruence system of $\mathfrak{A}$. In fact, if $r \in R$, with $\rho(r) = n$, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and $\phi_0, \ldots, \phi_{n-1}, \psi_0, \ldots, \psi_{n-1} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$ are such that $\vec{\phi} \in r_\Sigma^{\alpha^{-1}(\mathfrak{B}')}$ and

$\vec{\phi}\ \alpha_\Sigma^{-1}(\theta_{F(\Sigma)})^n\ \vec{\psi}$, then we have that $\alpha_\Sigma(\vec{\phi}) \in r^{\mathfrak{B}'}_{F(\Sigma)}$ and $\alpha_\Sigma(\vec{\phi})\ \theta^n_{F(\Sigma)}\ \alpha_\Sigma(\vec{\psi})$, whence we obtain, by the fact that $\theta \in \mathrm{Con}(\mathfrak{B}')$, that $\alpha_\Sigma(\vec{\psi}) \in r^{\mathfrak{B}'}_{F(\Sigma)}$. Hence, $\vec{\psi} \in r_\Sigma^{\alpha^{-1}(\mathfrak{B}')}$.

A version of the Homomorphism Theorem involving filter congruence systems of $\mathcal{L}$-systems will be proven now. This version generalizes Theorem 2.4 of [14].

Theorem 6 (Homomorphism Theorem for $\mathcal{L}$-Systems) *Let $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle, \mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle$ be $\mathcal{L}$-systems, $\langle F, \alpha\rangle : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$ a surjective $\mathcal{L}$-morphism and $\Theta = \langle \mathfrak{B}', \theta\rangle \in \mathrm{Fc}(\mathfrak{B})$. Then, the pair $\langle G, \beta\rangle : \mathfrak{A}/\alpha^{-1}(\Theta) \to \mathfrak{B}/\Theta$, with $G = F$ and, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$,*

$$\beta_\Sigma(\phi/\alpha_\Sigma^{-1}(\Theta_{F(\Sigma)})) = \alpha_\Sigma(\phi)/\Theta_{F(\Sigma)}, \quad \textit{for all } \phi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma),$$

is an $\mathcal{L}$-morphism, such that β_Σ is a bijection, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$.

If, moreover, $F : \mathbf{Sign}^{\mathfrak{A}} \to \mathbf{Sign}^{\mathfrak{B}}$ is an isomorphism, then $\langle G, \beta\rangle : \mathfrak{A}/\alpha^{-1}(\Theta) \to \mathfrak{B}/\Theta$ is an $\mathcal{L}$-isomorphism.

Proof:

Suppose $\Theta = \langle \mathfrak{B}', \theta\rangle \in \mathrm{Fc}(\mathfrak{B})$. The mapping $\langle F, \alpha\rangle : \alpha^{-1}(\mathfrak{B}') \twoheadrightarrow_s \mathfrak{B}'$ is a reductive $\mathcal{L}$-morphism, by Lemma 5 of [26], and the mapping $\langle \mathrm{I}, \pi^\theta\rangle : \mathfrak{B}' \twoheadrightarrow_s \mathfrak{B}'/\theta$ is a reductive $\mathcal{L}$-morphism, by Proposition 8 of [27]. We also have that $\mathrm{Ker}(\langle \mathrm{I}, \pi^\theta\rangle \circ \langle F, \alpha\rangle) = \alpha^{-1}(\theta)$, whence, by the Homomorphism Theorem 10 of [27], we get that, there exists $\langle G, \beta\rangle : \alpha^{-1}(\mathfrak{B}')/\alpha^{-1}(\theta) \to \mathfrak{B}'/\theta$, defined exactly as in the statement of the theorem, such that the following diagram commutes:

$$\begin{tikzcd}
\alpha^{-1}(\mathfrak{B}') \arrow[r, "\langle F,\alpha\rangle"] \arrow[dr, "\langle \mathrm{I},\pi^{\alpha^{-1}(\theta)}\rangle"'] & \mathfrak{B}' \arrow[r, "\langle \mathrm{I},\pi^{\theta}\rangle"] & \mathfrak{B}'/\theta \\
& \alpha^{-1}(\mathfrak{B}')/\alpha^{-1}(\theta) \arrow[ur, "\langle G,\beta\rangle"'] &
\end{tikzcd}$$

Now it suffices to notice that, by definition, $\mathfrak{A}/\alpha^{-1}(\Theta) = \alpha^{-1}(\mathfrak{B}')/\alpha^{-1}(\theta)$ and $\mathfrak{B}/\Theta = \mathfrak{B}'/\theta$. That β_Σ is a bijection, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ is not difficult to see. Indeed, if $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and $\phi, \psi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$ are such that $\beta_\Sigma(\phi/\alpha_\Sigma^{-1}(\theta_{F(\Sigma)})) = \beta_\Sigma(\psi/\alpha_\Sigma^{-1}(\theta_{F(\Sigma)}))$, then we have that $\alpha_\Sigma(\phi)/\theta_{F(\Sigma)} = \alpha_\Sigma(\psi)/\theta_{F(\Sigma)}$, whence $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in \theta_{F(\Sigma)}$, showing that $\langle \phi, \psi\rangle \in \alpha_\Sigma^{-1}(\theta_{F(\Sigma)})$ and, therefore, $\phi/\alpha_\Sigma^{-1}(\theta_{F(\Sigma)}) = \psi/\alpha_\Sigma^{-1}(\theta_{F(\Sigma)})$. ■

For every $\mathcal{L}$-morphism $\langle F, \alpha\rangle : \mathfrak{A} \to \mathfrak{B}$, the pre-image of $\Theta^{\mathfrak{B}} = \langle \mathfrak{B}, \Delta^{\mathrm{SEN}^{\mathfrak{B}}}\rangle$ by $\langle F, \alpha\rangle$ is the pair $\langle \alpha^{-1}(\mathfrak{B}), \mathrm{Ker}(\langle F, \alpha\rangle)\rangle$. It is called the **filter kernel** of $\langle F, \alpha\rangle$ and denoted by $\mathrm{FKer}(\langle F, \alpha\rangle)$. Theorem 6 has the following corollary, an analog for $\mathcal{L}$-systems of Corollary 2.5 of [14].

Corollary 7 *Let $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle, \mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle$ be $\mathcal{L}$-systems and $\langle F, \alpha\rangle : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$ a surjective $\mathcal{L}$-morphism, with $F : \mathbf{Sign}^{\mathfrak{A}} \to \mathbf{Sign}^{\mathfrak{B}}$ an isomorphism. Then $\langle F, \beta\rangle : \mathfrak{A}/\mathrm{FKer}(\langle F, \alpha\rangle) \cong \mathfrak{B}$, where, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, and all $\phi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$,*

$$\beta_\Sigma(\phi/\mathrm{FKer}_\Sigma(\langle F, \alpha\rangle)) = \alpha_\Sigma(\phi).$$

Suppose that $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle, \mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle$ are two $\mathcal{L}$-systems, such that $\mathfrak{A} \subseteq \mathfrak{B}$, i.e., $\mathfrak{A}$ is an $\mathcal{L}$-subsystem of $\mathfrak{B}$, and $\Theta = \langle \mathfrak{B}', \theta\rangle \in \mathrm{Fc}(\mathfrak{B})$. The **restriction of Θ to $\mathfrak{A}$**, in symbols $\Theta\restriction_{\mathfrak{A}}$, is defined to be the pair $\Theta\restriction_{\mathfrak{A}} = \langle \mathfrak{B}'\restriction_{\mathfrak{A}}, \theta \cap \nabla^{\mathrm{SEN}^{\mathfrak{A}}}\rangle$. The following proposition asserts that the pair $\Theta\restriction_{\mathfrak{A}}$ is a filter congruence system on $\mathfrak{A}$.

Proposition 8 *Suppose that* $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle, \mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle$ *are two $\mathcal{L}$-systems, such that* $\mathfrak{A} \subseteq \mathfrak{B}$ *and* $\Theta = \langle \mathfrak{B}', \theta\rangle \in \mathrm{Fc}(\mathfrak{B})$. *Then* $\Theta\restriction_{\mathfrak{A}} \in \mathrm{Fc}(\mathfrak{A})$.

Proof:

Since $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{B} \sqsubseteq \mathfrak{B}'$, we have that $\mathfrak{A} \sqsubseteq \mathfrak{B}'\restriction_{\mathfrak{A}}$. To see that $\theta \cap \nabla^{\mathrm{SEN}^{\mathfrak{A}}}$ is a congruence system of $\mathfrak{B}'\restriction_{\mathfrak{A}}$, suppose that $r \in R$, with $\rho(r) = n$, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$, such that $\vec{\phi} \in r_\Sigma^{\mathfrak{B}'\restriction_{\mathfrak{A}}}$ and $\vec{\phi}\ \theta_\Sigma^n\ \vec{\psi}$. Then $\vec{\phi} \in r_\Sigma^{\mathfrak{B}'}$ and $\vec{\phi}\ \theta_\Sigma^n\ \vec{\psi}$, whence, since θ is a congruence system on $\mathfrak{B}'$, $\vec{\psi} \in r_\Sigma^{\mathfrak{B}'}$, which, combined with $\vec{\psi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$, yields that $\vec{\psi} \in r_\Sigma^{\mathfrak{B}'\restriction_{\mathfrak{A}}}$. Therefore $\Theta\restriction_{\mathfrak{A}}$ is a filter congruence system on $\mathfrak{A}$. ■

The following Isomorphism Theorem for $\mathcal{L}$-systems uses the notion of a restriction of a filter congruence system to a subsystem and generalizes Theorem 2.6 of [14]. Roughly speaking, its first part asserts that, given an $\mathcal{L}$-system $\mathfrak{B}$ and a filter congruence system Θ on $\mathfrak{B}$, together with an $\mathcal{L}$-subsystem $\mathfrak{A}$ of $\mathfrak{B}$, there exists an injective $\mathcal{L}$-system morphism from the quotient of $\mathfrak{A}$ modulo the restriction of Θ to $\mathfrak{A}$ to the quotient of $\mathfrak{B}$ modulo Θ. Its second part presents an analog of this statement for the case of an arbitrary injective $\mathcal{L}$-morphism from $\mathfrak{A}$ to $\mathfrak{B}$ rather than, specifically, for the case of an $\mathcal{L}$-subsystem injection.

Theorem 9 (Second Isomorphism Theorem for $\mathcal{L}$-Systems)

1. *Let* $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$ *and* $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle$ *be two $\mathcal{L}$-systems, such that* $\mathfrak{A} \subseteq \mathfrak{B}$ *and* $\Theta = \langle \mathfrak{B}', \theta\rangle \in \mathrm{Fc}(\mathfrak{B})$. *Then, there exists an injective $\mathcal{L}$-morphism* $\langle J, \alpha\rangle : \mathfrak{A}/\Theta\restriction_{\mathfrak{A}} \to \mathfrak{B}/\Theta$, *where* $J : \mathbf{Sign}^{\mathfrak{A}} \to \mathbf{Sign}^{\mathfrak{B}}$ *is the injection functor and, for all* $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ *and all* $\phi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$,
$$\alpha_\Sigma(\phi/(\Theta\restriction_{\mathfrak{A}})_\Sigma) = \phi/\Theta_\Sigma.$$

2. *More generally, for any monomorphism* $\langle F, \alpha\rangle : \mathfrak{A} \rightarrowtail \mathfrak{B}$ *and any* $\Theta \in \mathrm{Fc}(\mathfrak{B})$, *there exists a monomorphism* $\langle G, \beta\rangle : \mathfrak{A}/\alpha^{-1}(\Theta) \rightarrowtail \mathfrak{B}/\Theta$, *given by* $G = F$ *and, for all* $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|, \phi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$,
$$\beta_\Sigma(\phi/\alpha_\Sigma^{-1}(\Theta_{F(\Sigma)})) = \alpha_\Sigma(\phi)/\Theta_{F(\Sigma)}.$$

Proof:

Let $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$. To see that $\alpha_\Sigma : \mathrm{SEN}^{\mathfrak{A}}(\Sigma)/(\Theta\restriction_{\mathfrak{A}})_\Sigma \to \mathrm{SEN}^{\mathfrak{B}}(\Sigma)/\Theta_\Sigma$ is well-defined, suppose that $\phi, \psi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$, such that $\langle \phi, \psi\rangle \in (\theta\restriction_{\mathfrak{A}})_\Sigma$. Then $\langle \phi, \psi\rangle \in \theta_\Sigma$, whence $\phi/\theta_\Sigma = \psi/\theta_\Sigma$ and, therefore, $\alpha_\Sigma(\phi/(\Theta\restriction_{\mathfrak{A}})_\Sigma) = \alpha_\Sigma(\psi/(\Theta\restriction_{\mathfrak{A}})_\Sigma)$. We leave to the reader the verification that α is a natural transformation.

To see that $\langle J, \alpha\rangle$ is an $\mathcal{L}$-algebraic system morphism, suppose that σ is n-ary in $\mathbf{F}$, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|, \phi_0, \ldots, \phi_{n-1} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$. Then

$$\begin{array}{ll} \alpha_\Sigma(\sigma_\Sigma^{\mathfrak{A}/\Theta\restriction_{\mathfrak{A}}}(\phi_0/(\Theta\restriction_{\mathfrak{A}})_\Sigma, \ldots, \phi_{n-1}/(\Theta\restriction_{\mathfrak{A}})_\Sigma)) & \\ = & \alpha_\Sigma(\sigma_\Sigma^{\mathfrak{A}}(\phi_0, \ldots, \phi_{n-1})/(\Theta\restriction_{\mathfrak{A}})_\Sigma) \\ = & \sigma_\Sigma^{\mathfrak{A}}(\phi_0, \ldots, \phi_{n-1})/\Theta_\Sigma \\ = & \sigma_\Sigma^{\mathfrak{B}}(\phi_0, \ldots, \phi_{n-1})/\Theta_\Sigma \\ = & \sigma_\Sigma^{\mathfrak{B}/\Theta}(\phi_0/\Theta_\Sigma, \ldots, \phi_{n-1}/\Theta_\Sigma) \\ = & \sigma_\Sigma^{\mathfrak{B}/\Theta}(\alpha_\Sigma(\phi_0/(\Theta\restriction_{\mathfrak{A}})_\Sigma), \ldots, \alpha_\Sigma(\phi_{n-1}/(\Theta\restriction_{\mathfrak{A}})_\Sigma)). \end{array}$$

To see that $\langle J, \alpha\rangle$ is an $\mathcal{L}$-system morphism, let $r \in R$, with $\rho(r) = n$, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|, \phi_0, \ldots, \phi_{n-1} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$. Then

$$\begin{array}{ll} \langle \phi_0/(\Theta\restriction_{\mathfrak{A}})_\Sigma, \ldots, \phi_{n-1}/(\Theta\restriction_{\mathfrak{A}})_\Sigma\rangle \in r_\Sigma^{\mathfrak{A}/\Theta\restriction_{\mathfrak{A}}} & \\ \text{iff} & \langle \phi_0, \ldots, \phi_{n-1}\rangle \in r_\Sigma^{\mathfrak{B}'\restriction_{\mathfrak{A}}} \\ \text{iff} & \langle \phi_0, \ldots, \phi_{n-1}\rangle \in r_\Sigma^{\mathfrak{B}'} \\ \text{iff} & \langle \phi_0/\Theta_\Sigma, \ldots, \phi_{n-1}/\Theta_\Sigma\rangle \in r_\Sigma^{\mathfrak{B}/\Theta} \\ \text{iff} & \langle \alpha_\Sigma(\phi_0/(\Theta\restriction_{\mathfrak{A}})_\Sigma), \ldots, \alpha_\Sigma(\phi_{n-1}/(\Theta\restriction_{\mathfrak{A}})_\Sigma)\rangle \in r_\Sigma^{\mathfrak{B}/\Theta}. \end{array}$$

That α_Σ is injective, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, is easy to see.

The second part of the theorem can be proven very similarly and the details will be omitted. ■

Given an $\mathcal{L}$-system $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$, if $\Theta = \langle \mathfrak{B}, \theta\rangle$ and $\Theta' = \langle \mathfrak{B}', \theta'\rangle$ are filter congruences on $\mathfrak{A}$, we write $\Theta \leq \Theta'$ to indicate that $\mathfrak{B} \sqsubseteq \mathfrak{B}'$ and $\theta \leq \theta'$.

In the following proposition, it is shown that, given two filter congruence systems $\Theta = \langle \mathfrak{B}, \theta\rangle, \Theta' = \langle \mathfrak{B}', \theta'\rangle$ of an $\mathcal{L}$-system $\mathfrak{A}$, such that $\Theta \leq \Theta'$, the pair $\langle \mathfrak{B}'/\theta, \theta'/\theta\rangle$ is a filter congruence system on the $\mathcal{L}$-system $\mathfrak{A}/\theta$. This result will pave the way for formulating an analog of the First Isomorphism Theorem for $\mathcal{L}$-systems involving filter congruence systems.

Proposition 10 *Let* $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$ *be an* $\mathcal{L}$*-system and* $\Theta = \langle \mathfrak{B}, \theta\rangle, \Theta' = \langle \mathfrak{B}', \theta'\rangle \in \mathrm{Fc}(\mathfrak{A})$, *such that* $\Theta \leq \Theta'$. *Then* $\Theta'/\Theta = \langle \mathfrak{B}'/\theta, \theta'/\theta\rangle$ *is a filter congruence on* $\mathfrak{A}/\theta$.

Proof:

Since $\theta \leq \theta'$ and $\theta' \in \mathrm{Con}(\mathfrak{B}')$, we get that $\theta \in \mathrm{Con}(\mathfrak{B}')$. Therefore $\mathfrak{B}'/\theta$ is well-defined. Similarly, since $\mathfrak{A} \sqsubseteq \mathfrak{B}$ and $\theta \in \mathrm{Con}(\mathfrak{B})$, we get that $\theta \in \mathrm{Con}(\mathfrak{A})$ and, thus, $\mathfrak{A}/\theta$ is well-defined. Moreover, since $\mathfrak{A} \sqsubseteq \mathfrak{B}'$, we have that $\mathfrak{A}/\theta \sqsubseteq \mathfrak{B}'/\theta$. So it suffices to show that $\theta/\theta' \in \mathrm{Con}(\mathfrak{B}'/\theta)$. This follows from the fact that $\theta' \in \mathrm{Con}(\mathfrak{B}')$ as follows: Let $r \in R$, with $\rho(r) = n$, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|, \phi_0, \ldots, \phi_{n-1}, \psi_0, \ldots, \psi_{n-1} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$, such that $\vec{\phi}/\theta_\Sigma \in r_\Sigma^{\mathfrak{B}'/\theta}$ and $\langle \vec{\phi}/\theta_\Sigma, \vec{\psi}/\theta_\Sigma\rangle \in \theta'_\Sigma/\theta_\Sigma$. Then $\vec{\phi} \in r_\Sigma^{\mathfrak{B}'}$ and $\langle \vec{\phi}, \vec{\psi}\rangle \in \theta'_\Sigma$, whence $\vec{\psi} \in r_\Sigma^{\mathfrak{B}'}$ and, therefore, $\vec{\psi}/\theta_\Sigma \in r_\Sigma^{\mathfrak{B}'/\theta}$. Thus $\theta'/\theta \in \mathrm{Con}(\mathfrak{B}'/\theta)$. ■

Now the analog of the First Isomorphism Theorem for $\mathcal{L}$-systems involving filter congruence systems takes the following form.

Theorem 11 (First Isomorphism Theorem for $\mathcal{L}$-Systems) *Consider an $\mathcal{L}$-system*

$$\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$$

and $\Theta = \langle \mathfrak{B}, \theta \rangle, \Theta' = \langle \mathfrak{B}', \theta' \rangle \in \mathrm{Fc}(\mathfrak{A})$, such that $\Theta \leq \Theta'$. Then, there exists an $\mathcal{L}$-isomorphism $\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \alpha \rangle : (\mathfrak{A}/\Theta)/(\Theta'/\Theta) \to \mathfrak{A}/\Theta'$, where α_Σ is given, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, by $\alpha_\Sigma((\phi/\Theta_\Sigma)/(\Theta'_\Sigma/\Theta_\Sigma)) = \phi/\Theta'_\Sigma$, for all $\phi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$.

Proof:
The pair $\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \beta \rangle : \mathfrak{A}/\Theta \to \mathfrak{A}/\Theta'$, where β_Σ is given, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, by $\alpha_\Sigma(\phi/\Theta_\Sigma) = \phi/\Theta'_\Sigma$, for all $\phi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$, is a surjective $\mathcal{L}$-morphism from $\mathfrak{A}/\Theta$ onto $\mathfrak{A}/\Theta'$ and $\mathrm{FKer}(\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \beta \rangle) = \Theta'/\Theta$. Therefore, the theorem follows directly by Corollary 7. ■

If $\mathtt{K}$ is a Lyndon class of $\mathcal{L}$-systems, then $\mathrm{Fc}_{\mathtt{K}}(\mathfrak{A})$ forms a complete lattice, for every $\mathcal{L}$-system $\mathfrak{A}$. It is denoted by $\mathbf{Fc}_{\mathtt{K}}(\mathfrak{A}) = \langle \mathrm{Fc}_{\mathtt{K}}(\mathfrak{A}), \leq \rangle$. If, in addition, $\mathtt{K}$ is a quasi-variety, then $\mathbf{Fc}_{\mathtt{K}}(\mathfrak{A})$ is algebraic. The join and meet are defined as follows: For every collection $\Theta^i = \langle \mathfrak{A}_i, \theta^i \rangle, i \in I$, of $\mathtt{K}$-filter congruences on $\mathfrak{A}$,

$$\bigvee_{i \in I}^{\mathtt{K}} \Theta^i = \langle \bigvee_{i \in I}^{\mathtt{K}} \mathfrak{A}_i, \bigvee_{i \in I} \theta^i \rangle \quad \text{and} \quad \bigwedge_{i \in I} \Theta^i = \langle \bigwedge_{i \in I} \mathfrak{A}_i, \bigwedge_{i \in I} \theta^i \rangle.$$

If $\mathbf{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle \rangle$ is the underlying $\mathcal{L}$-algebraic system of $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$ and $\theta \in \mathrm{Con}(\mathbf{A})$, then there exists a least $\mathtt{K}$-filter extension $\mathfrak{B}$ of $\mathfrak{A}$ satisfying $\theta \in \mathrm{Con}(\mathfrak{B})$. Such a filter extension of $\mathfrak{A}$ is denoted by $\mathrm{Fg}^{\theta}_{\mathtt{K}}(\mathfrak{A})$ and is called the **$\mathtt{K}$-filter extension of $\mathfrak{A}$ generated by θ**. More generally, if $\mathfrak{A}$ is an $\mathcal{L}$-system and X is a binary relation system on $\mathrm{SEN}^{\mathfrak{A}}$, then there exists a least $\mathtt{K}$-filter congruence system $\langle \mathfrak{B}, \theta \rangle$ on $\mathfrak{A}$, such that $X \leq \theta$. It is denoted by $\mathrm{Fg}_{\mathtt{K}}(\langle \mathfrak{A}, X \rangle)$ and termed the **$\mathtt{K}$-filter congruence system generated by** $\langle \mathfrak{A}, X \rangle$.

7 Protoalgebraic Classes

Let $\mathtt{K}$ be a class of $\mathcal{L}$-systems. $\mathtt{K}$ is called **protoalgebraic** if Ω is $\sqsubseteq$-monotone in $\mathtt{K}$, i.e., if, for all $\mathcal{L}$-systems $\mathfrak{A}, \mathfrak{B} \in \mathtt{K}$,

$$\text{if} \quad \mathfrak{A} \sqsubseteq \mathfrak{B}, \quad \text{then} \quad \Omega(\mathfrak{A}) \leq \Omega(\mathfrak{B}).$$

The following theorem, an analog of Theorem 2.8 of [14] for $\mathcal{L}$-systems, may be traced back to the pioneering work of Blok and Pigozzi [4].

Theorem 12 *Let $\mathtt{K}$ be a class of $\mathcal{L}$-systems. Then the following statements are equivalent:*

1. *$\mathtt{K}$ is protoalgebraic.*

2. *If* $\mathfrak{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle, R^{\mathfrak{A}}\rangle, \mathfrak{B} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle, R^{\mathfrak{B}}\rangle \in \mathtt{K}$ *are such that* $\mathfrak{A} \sqsubseteq \mathfrak{B}$*, then* $\langle \mathrm{I}_{\mathbf{Sign}}, \iota^*\rangle : \mathrm{SEN}^{\Omega(\mathfrak{A})} \to \mathrm{SEN}^{\Omega(\mathfrak{B})}$*, given, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi \in \mathrm{SEN}(\Sigma)$*, by* $\iota^*_\Sigma(\phi/\Omega_\Sigma(\mathfrak{A})) = \phi/\Omega_\Sigma(\mathfrak{B})$*, defines a surjective* $\mathcal{L}$*-morphism* $\langle \mathrm{I}_{\mathbf{Sign}}, \iota^*\rangle : \mathfrak{A}^* \to \mathfrak{B}^*$*, such that* $\mathrm{Ker}(\langle \mathrm{I}_{\mathbf{Sign}}, \iota^*\rangle) = \Omega(\mathfrak{B})/\Omega(\mathfrak{A})$*.*

If, in addition, $\mathtt{K}$ *is full, then 1 and 2 are equivalent to*

3. *For all* $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle, \mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle \in \mathtt{K}$ *and all surjective* $\mathcal{L}$*-morphisms* $\langle F, \alpha\rangle : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$*, the pair* $\langle F^*, \alpha^*\rangle : \mathfrak{A}^* \twoheadrightarrow \mathfrak{B}^*$*, defined by* $F^* = F$ *and, for all* $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ *and all* $\phi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$*, by* $\alpha^*_\Sigma(\phi/\Omega_\Sigma(\mathfrak{A})) = \alpha_\Sigma(\phi)/\Omega_{F(\Sigma)}(\mathfrak{B})$*, is also a surjective* $\mathcal{L}$*-morphism with* $\mathrm{Ker}(\langle F^*, \alpha^*\rangle) = \alpha^{-1}(\Omega(\mathfrak{B}))/\Omega(\mathfrak{A})$*.*

Proof:
That 1 implies 2 follows by Theorem 11, since, by the hypothesis, $\Omega(\mathfrak{A}) \leq \Omega(\mathfrak{B})$.

Suppose, conversely, that 2 holds and let $\mathfrak{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle, R^{\mathfrak{A}}\rangle, \mathfrak{B} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle, R^{\mathfrak{B}}\rangle \in \mathtt{K}$ with $\mathfrak{A} \sqsubseteq \mathfrak{B}$. Then $\langle \mathrm{I}_{\mathbf{Sign}}, \iota^*\rangle : \mathfrak{A}^* \twoheadrightarrow \mathfrak{B}^*$ is a surjective $\mathcal{L}$-morphism, whence, by the fact that ι^* is well-defined, $\Omega(\mathfrak{A}) \leq \Omega(\mathfrak{B})$ and, therefore, $\mathtt{K}$ is protoalgebraic.

Since $3 \to 2$ is obvious by taking $\mathfrak{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle, R^{\mathfrak{A}}\rangle, \mathfrak{B} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle, R^{\mathfrak{B}}\rangle \in \mathtt{K}$, with $\mathfrak{A} \sqsubseteq \mathfrak{B}$, and $\langle F, \alpha\rangle = \langle \mathrm{I}_{\mathbf{Sign}}, \iota\rangle : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$, it suffices now to show that $2 \to 3$ holds under the hypothesis that $\mathtt{K}$ is full.

Suppose, to this end, that $\mathtt{K}$ is full and let $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle, \mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle \in \mathtt{K}$ and $\langle F, \alpha\rangle : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$ a surjective $\mathcal{L}$-morphism. Then $\mathfrak{A} \sqsubseteq \alpha^{-1}(\mathfrak{B}) \in \mathtt{K}$, whence, by 2, $\langle \mathrm{I}_{\mathbf{Sign}}, \iota^*\rangle : \mathfrak{A}^* \twoheadrightarrow (\alpha^{-1}(\mathfrak{B}))^*$ is a surjective $\mathcal{L}$-morphism and, therefore, since, by Proposition 16 of [27], $\langle F^*, \alpha^*\rangle : (\alpha^{-1}(\mathfrak{B}))^* \twoheadrightarrow_s \mathfrak{B}^*$ is a reductive $\mathcal{L}$-morphism, we get that $\langle F^*, \alpha^*\rangle : \mathfrak{A}^* \twoheadrightarrow \mathfrak{B}^*$ is a surjective $\mathcal{L}$-morphism. To see that $\mathrm{Ker}(\langle F^*, \alpha^*\rangle) = \alpha^{-1}(\Omega(\mathfrak{B}))/\Omega(\mathfrak{A})$, suppose that $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and $\phi, \psi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$. Then we have

$$\begin{array}{ll}
\multicolumn{2}{l}{\langle \phi/\Omega_\Sigma(\mathfrak{A}), \psi/\Omega_\Sigma(\mathfrak{A})\rangle \in \mathrm{Ker}(\langle F^*, \alpha^*\rangle)} \\
\text{iff} & \alpha^*_\Sigma(\phi/\Omega_\Sigma(\mathfrak{A})) = \alpha^*_\Sigma(\psi/\Omega_\Sigma(\mathfrak{A})) \\
\text{iff} & \alpha_\Sigma(\phi)/\Omega_{F(\Sigma)}(\mathfrak{B}) = \alpha_\Sigma(\psi)/\Omega_{F(\Sigma)}(\mathfrak{B}) \\
\text{iff} & \langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in \Omega_{F(\Sigma)}(\mathfrak{B}) \\
\text{iff} & \langle \phi, \psi\rangle \in \alpha_\Sigma^{-1}(\Omega_{F(\Sigma)}(\mathfrak{B})) \\
\text{iff} & \langle \phi/\Omega_\Sigma(\mathfrak{A}), \psi/\Omega_\Sigma(\mathfrak{A})\rangle \in \alpha_\Sigma^{-1}(\Omega_{F(\Sigma)}(\mathfrak{B}))/\Omega_\Sigma(\mathfrak{A}).
\end{array}$$

■

Suppose that $\mathfrak{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle, R^{\mathfrak{A}}\rangle$ is an $\mathcal{L}$-system, with $\mathbf{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle\rangle$ its underlying $\mathcal{L}$-algebraic system. Since $\mathrm{Con}(\mathfrak{A})$ is the principal ideal of $\mathrm{Con}(\mathbf{A})$ generated by $\Omega(\mathfrak{A})$, the condition of protoalgebraicity is equivalent to

$$\text{if} \quad \mathfrak{A} \sqsubseteq \mathfrak{B}, \quad \text{then} \quad \mathrm{Con}(\mathfrak{A}) \subseteq \mathrm{Con}(\mathfrak{B}).$$

This condition is termed the **compatibility property**.

8 Protoalgebraicity and the Structure of $\mathrm{Fc}_{\mathtt{K}}(\mathfrak{A})$

Let $\mathtt{K}$ be a class of $\mathcal{L}$-systems and $\mathfrak{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle, R^{\mathfrak{A}}\rangle$ a single $\mathcal{L}$-system, with underlying $\mathcal{L}$-algebraic system $\mathbf{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle\rangle$. If θ is a congruence system on $\mathbf{A}$, then we define

$$\mathrm{Fc}^{\theta}_{\mathtt{K}}(\mathfrak{A}) = \{\langle \mathfrak{B}, \eta\rangle \in \mathrm{Fc}_{\mathtt{K}}(\mathfrak{A}) : \eta = \theta\},$$

$$\mathrm{Fc}^{\downarrow\theta}_{\mathtt{K}}(\mathfrak{A}) = \{\langle \mathfrak{B}, \eta\rangle \in \mathrm{Fc}_{\mathtt{K}}(\mathfrak{A}) : \eta \leq \theta\},$$

$$\mathrm{Fc}^{\uparrow\theta}_{\mathtt{K}}(\mathfrak{A}) = \{\langle \mathfrak{B}, \eta\rangle \in \mathrm{Fc}_{\mathtt{K}}(\mathfrak{A}) : \eta \geq \theta\}.$$

These subsets of $\mathrm{Fc}_{\mathtt{K}}(\mathfrak{A})$ are called, respectively, the θ-**section of** $\mathrm{Fc}_{\mathtt{K}}(\mathfrak{A})$, the θ-**downset of** $\mathrm{Fc}_{\mathtt{K}}(\mathfrak{A})$ and the θ-**upset of** $\mathrm{Fc}_{\mathtt{K}}(\mathfrak{A})$. For simplicity, we write $\mathrm{Fc}^{\Delta}_{\mathtt{K}}(\mathfrak{A})$ for the $\Delta^{\mathrm{SEN}^{\mathfrak{A}}}$-section of $\mathrm{Fc}_{\mathtt{K}}(\mathfrak{A})$. We also define

$$\mathrm{Fc}^{l}_{\mathtt{K}}(\mathfrak{A}) = \{\langle \mathfrak{B}, \eta\rangle \in \mathrm{Fc}_{\mathtt{K}}(\mathfrak{A}) : \eta = \Omega(\mathfrak{B})\}$$

$$\mathrm{Fc}^{\Omega}_{\mathtt{K}}(\mathfrak{A}) = \{\langle \mathfrak{B}, \eta\rangle \in \mathrm{Fc}_{\mathtt{K}}(\mathfrak{A}) : \eta \in \mathrm{Con}_{\mathtt{K}}(\mathbf{A})\},$$

which are called the l-**set of** $\mathrm{Fc}_{\mathtt{K}}(\mathfrak{A})$ and the Ω-**set of** $\mathrm{Fc}_{\mathtt{K}}(\mathfrak{A})$, respectively.

The following theorem, an analog of Theorem 2.9 of [14], gives some of the relations between these subsets of $\mathrm{Fc}_{\mathtt{K}}(\mathfrak{A})$ that follow easily from the definitions involved.

Theorem 13 *Let $\mathtt{K}$ be a class of $\mathcal{L}$-systems. Then the following statements are equivalent:*

1. *$\mathtt{K}$ is protoalgebraic.*
2. *If $\mathfrak{A} \in \mathtt{K}$, then the mapping $\langle \mathfrak{B}, \Omega(\mathfrak{A})\rangle \mapsto \mathfrak{B}$, for all $\langle \mathfrak{B}, \Omega(\mathfrak{A})\rangle \in \mathrm{Fc}^{\Omega(\mathfrak{A})}_{\mathtt{K}}(\mathfrak{A})$, establishes an isomorphism $\mathrm{Fc}^{\Omega(\mathfrak{A})}_{\mathtt{K}}(\mathfrak{A}) \cong \mathrm{Fe}_{\mathtt{K}}(\mathfrak{A})$.*
3. *If $\mathfrak{A} \in \mathtt{K}$ and $\theta \in \mathrm{Con}(\mathfrak{A})$, then the mapping $\langle \mathfrak{B}, \theta\rangle \mapsto \mathfrak{B}$, for all $\langle \mathfrak{B}, \theta\rangle \in \mathrm{Fc}^{\theta}_{\mathtt{K}}(\mathfrak{A})$, establishes an isomorphism $\mathrm{Fc}^{\theta}_{\mathtt{K}}(\mathfrak{A}) \cong \mathrm{Fe}_{\mathtt{K}}(\mathfrak{A})$.*
4. *If $\mathfrak{A} \in \mathtt{K}$, then $\mathrm{Fc}^{\downarrow\Omega(\mathfrak{A})}_{\mathtt{K}}(\mathfrak{A}) = \mathrm{Fe}_{\mathtt{K}}(\mathfrak{A}) \times \mathrm{Con}(\mathfrak{A})$.*
5. *If $\mathfrak{A} \in \mathtt{K}$, then the mapping $\mathfrak{B} \mapsto \langle \mathfrak{B}, \Omega(\mathfrak{B})\rangle$, for all $\mathfrak{B} \in \mathrm{Fe}_{\mathtt{K}}(\mathfrak{A})$, is an embedding of $\mathrm{Fe}_{\mathtt{K}}(\mathfrak{A})$ into $\mathrm{Fc}^{\uparrow\Omega(\mathfrak{A})}_{\mathtt{K}}(\mathfrak{A})$.*
6. *If $\mathfrak{A} \in \mathtt{K}$, then, the mapping $\langle \mathfrak{B}, \Omega(\mathfrak{B})\rangle \mapsto \mathfrak{B}$, for all $\langle \mathfrak{B}, \Omega(\mathfrak{B})\rangle \in \mathrm{Fc}^{l}_{\mathtt{K}}(\mathfrak{A})$, establishes an isomorphism $\mathrm{Fc}^{l}_{\mathtt{K}}(\mathfrak{A}) \cong \mathrm{Fe}_{\mathtt{K}}(\mathfrak{A})$.*

Proof:

If $\mathtt{K}$ is protoalgebraic, all parts follow directly from the relevant definitions. For the converse, it is easy to see, once more based on the relevant definitions, that each of the parts implies the monotonicity of the Leibniz operator on $\mathtt{K}$. ■

Let $\mathtt{K}$ be a protoalgebraic class of $\mathcal{L}$-systems and $\mathfrak{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle, R^{\mathfrak{A}}\rangle$ an $\mathcal{L}$-system. The $\Omega(\mathfrak{A})$-upset of $\mathrm{Fc}_{\mathtt{K}}(\mathfrak{A})$ has no description as simple as that of the corresponding $\Omega(\mathfrak{A})$-downset contained in Part 4 of Theorem 13. If, however, $\mathtt{K}$ is a Lyndon class, the θ-sections of $\mathrm{Fc}_{\mathtt{K}}(\mathfrak{A})$ are still naturally isomorphic to lattices of $\mathtt{K}$-filter extensions, for all congruence systems θ that include $\Omega(\mathfrak{A})$. Actually we have $\mathrm{Fc}^{\theta}_{\mathtt{K}}(\mathfrak{A}) \cong \mathrm{Fe}_{\mathtt{K}}(\mathrm{Fg}^{\theta}_{\mathtt{K}}(\mathfrak{A}))$, for all congruence systems $\theta \geq \Omega(\mathfrak{A})$. The isomorphism sends the pair $\langle \mathfrak{B}, \theta\rangle \in \mathrm{Fc}^{\theta}_{\mathtt{K}}(\mathfrak{A})$ to the $\mathcal{L}$-system $\mathfrak{B} \in \mathrm{Fe}_{\mathtt{K}}(\mathrm{Fg}^{\theta}_{\mathtt{K}}(\mathfrak{A}))$. So, it turns out that, for protoalgebraic classes $\mathtt{K}$ of $\mathcal{L}$-systems, the $\mathtt{K}$-filter congruence systems and the $\mathtt{K}$-filter extensions are, to a large extent, interchangeable.

9 Extensions of the Correspondence Theorem

Let $\mathtt{K}$ be a class of $\mathcal{L}$-systems. As in the case of universal algebras and of equality-free first-order structures, many of the properties of $\mathtt{K}$ depend on the relationship between the poset of $\mathtt{K}$-filter congruence systems (or $\mathtt{K}$-filter extensions) on each $\mathcal{L}$-system in $\mathtt{K}$ and those of its $\mathcal{L}$-morphic images. For universal algebras, this relationship takes the form of the Correspondence Theorem, but it is not as simple for equality-free structures nor for $\mathcal{L}$-systems, because in both of these cases homomorphic images and $\mathcal{L}$-morphic images, respectively, need not be captured uniquely by kernels.

In general, if $\mathtt{K}$ is full and $\langle F, \alpha\rangle : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$ is a surjective $\mathcal{L}$-morphism, then

$$\mathfrak{B}' \mapsto \alpha^{-1}(\mathfrak{B}'), \quad \text{for all } \mathfrak{B}' \in \mathrm{Fe}_{\mathtt{K}}(\mathfrak{B}), \tag{1}$$

is an embedding from $\mathrm{Fe}_{\mathtt{K}}(\mathfrak{B})$ into $\mathrm{Fe}_{\mathtt{K}}(\mathfrak{A})$. If, in addition, $\mathtt{K}$ is a Lyndon class, the poset $\mathbf{Fe}_{\mathtt{K}}(\mathfrak{A})$ is a complete lattice and, if F is an isomorphism, the previous mapping has an order-preserving left inverse, defined on $\mathrm{Fe}_{\mathtt{K}}(\mathfrak{A})$ by

$$\mathfrak{A}' \mapsto \mathfrak{B} \vee^{\mathtt{K}} \mathrm{Fg}_{\mathtt{K}}(\alpha(\mathfrak{A}')) \tag{2}$$

Moreover, in this case, the restriction of the Mapping (2) to the range of the Mapping (1) coincides with $\mathfrak{A}' \mapsto \alpha(\mathfrak{A}')$.

Elgueta [14] showed that the composition of the Mapping (1) with the Mapping (2) is not in general the identity and gave in Lemma 2.10 of [14] a necessary and sufficient condition for the Mapping (2) to be "locally" a right inverse of Mapping (1). An analog of Lemma 2.10 of [14] for $\mathcal{L}$-systems is the following:

Lemma 14 *Let $\mathtt{K}$ be an abstract Lyndon class of $\mathcal{L}$-systems. Let $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle, \mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle \in \mathtt{K}$ and $\langle F, \alpha\rangle : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$ a surjective $\mathcal{L}$-morphism, such that F is an isomorphism. Then, for all $\mathfrak{A}' \in \mathrm{Fe}_{\mathtt{K}}(\mathfrak{A})$,*

$$\alpha^{-1}(\mathfrak{B} \vee^{\mathtt{K}} \mathrm{Fg}_{\mathtt{K}}(\alpha(\mathfrak{A}'))) = \mathfrak{A}' \quad \textit{iff} \quad \alpha^{-1}(\mathfrak{B}) \sqsubseteq \mathfrak{A}' \textit{ and } \mathrm{Ker}(\langle F, \alpha\rangle) \in \mathrm{Con}(\mathfrak{A}').$$

Proof:

Suppose, first, that $\alpha^{-1}(\mathfrak{B} \vee^{\mathtt{K}} \mathrm{Fg}_{\mathtt{K}}(\alpha(\mathfrak{A}'))) = \mathfrak{A}'$. Then, obviously, $\alpha^{-1}(\mathfrak{B}) \sqsubseteq \mathfrak{A}'$. Moreover, by the surjectivity of $\langle F, \alpha\rangle$, $\alpha(\mathfrak{A}') = \mathfrak{B} \vee^{\mathtt{K}} \mathrm{Fg}_{\mathtt{K}}(\alpha(\mathfrak{A}'))$. Therefore, by the hypothesis,

$\mathfrak{A}' = \alpha^{-1}(\alpha(\mathfrak{A}'))$. Hence, we obtain that $\langle F, \alpha\rangle : \mathfrak{A}' \twoheadrightarrow_s \alpha(\mathfrak{A}')$ is a reductive $\mathcal{L}$-morphism and, thus, $\mathrm{Ker}(\langle F, \alpha\rangle) \in \mathrm{Con}(\mathfrak{A}')$.

Suppose, conversely, that $\mathfrak{A}' \in \mathrm{Fe}_{\mathtt{K}}(\mathfrak{A})$, such that $\alpha^{-1}(\mathfrak{B}) \sqsubseteq \mathfrak{A}'$ and $\mathrm{Ker}(\langle F, \alpha\rangle) \in \mathrm{Con}(\mathfrak{A}')$. We show, first, that $\alpha(\mathfrak{A}')$ is a reduction of $\mathfrak{A}'$.

Indeed, if $r \in R$, with $\rho(r) = n$, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$, such that $\alpha_\Sigma(\vec{\phi}) \in r^{\alpha(\mathfrak{A}')}_{F(\Sigma)}$, then, there exists $\vec{\psi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$, such that $\vec{\psi} \in r^{\mathfrak{A}'}_\Sigma$ and $\alpha_\Sigma(\vec{\phi}) = \alpha_\Sigma(\vec{\psi})$. Thus, we get that $\vec{\phi}\ \mathrm{Ker}_\Sigma(\langle F, \alpha\rangle)^n\ \vec{\psi}$. Since $\mathrm{Ker}(\langle F, \alpha\rangle) \in \mathrm{Con}(\mathfrak{A}')$, this shows that $\vec{\phi} \in r^{\mathfrak{A}'}_\Sigma$. Therefore, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and all $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$, $\vec{\phi} \in r^{\mathfrak{A}'}_\Sigma$ if and only if $\alpha_\Sigma(\vec{\phi}) \in r^{\alpha(\mathfrak{A}')}_{F(\Sigma)}$, showing that $\langle F, \alpha\rangle : \mathfrak{A}' \twoheadrightarrow_s \alpha(\mathfrak{A}')$.

Since $\mathtt{K}$ is abstract, we get that $\alpha(\mathfrak{A}') \in \mathtt{K}$, whence $\mathrm{Fg}_{\mathtt{K}}(\alpha(\mathfrak{A}')) = \alpha(\mathfrak{A}')$ and, therefore, $\mathfrak{B} \vee^{\mathtt{K}} \mathrm{Fg}_{\mathtt{K}}(\alpha(\mathfrak{A}')) = \mathfrak{B} \vee^{\mathtt{K}} \alpha(\mathfrak{A}') = \alpha(\mathfrak{A}')$. Thus, $\alpha^{-1}(\mathfrak{B} \vee^{\mathtt{K}} \mathrm{Fg}_{\mathtt{K}}(\alpha(\mathfrak{A}'))) = \mathfrak{A}'$. ∎

We close the section with an analog of Theorem 2.11 of [14] that characterizes those abstract Lyndon classes of $\mathcal{L}$-systems that are protoalgebraic by means of the Correspondence and the Filter Correspondence Properties.

Theorem 15 *Let* $\mathtt{K}$ *be an abstract Lyndon class of* $\mathcal{L}$*-systems. Then, the following statements are equivalent:*

1. $\mathtt{K}$ *is protoalgebraic.*

2. (Correspondence Property) *For all* $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle, \mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle \in \mathtt{K}$ *and all reductive* $\mathcal{L}$*-morphisms* $\langle F, \alpha\rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$*, with* F *an isomorphism, the mapping* $\Theta \mapsto \alpha^{-1}(\Theta)$*, for all* $\Theta \in \mathrm{Fc}_{\mathtt{K}}(\mathfrak{B})$*, is an isomorphism between* $\mathrm{Fc}_{\mathtt{K}}(\mathfrak{B})$ *and* $\mathrm{Fc}_{\mathtt{K}}^{\uparrow \mathrm{Ker}(\langle F, \alpha\rangle)}(\mathfrak{A})$*, whose inverse is* $\Theta \mapsto \alpha(\Theta)$*, for all* $\Theta \in \mathrm{Fc}_{\mathtt{K}}^{\uparrow \mathrm{Ker}(\langle F, \alpha\rangle)}(\mathfrak{A})$.

3. (Filter Correspondence Property) *For all* $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle, \mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle \in \mathtt{K}$ *and all reductive* $\mathcal{L}$*-morphisms* $\langle F, \alpha\rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$*, with* F *an isomorphism, the mapping* $\mathfrak{B}' \mapsto \alpha^{-1}(\mathfrak{B}')$*, for all* $\mathfrak{B}' \in \mathrm{Fe}_{\mathtt{K}}(\mathfrak{B})$*, is an isomorphism between* $\mathrm{Fe}_{\mathtt{K}}(\mathfrak{B})$ *and* $\mathrm{Fe}_{\mathtt{K}}(\mathfrak{A})$*, whose inverse is* $\mathfrak{A}' \mapsto \alpha(\mathfrak{A}')$*, for all* $\mathfrak{A}' \in \mathrm{Fe}_{\mathtt{K}}(\mathfrak{A})$.

Proof:

$1 \to 2$ Suppose that $\mathtt{K}$ is protoalgebraic. Let $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle, \mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle \in \mathtt{K}$ and $\langle F, \alpha\rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$ a reductive $\mathcal{L}$-morphism, with F an isomorphism. Then, the mapping $\Theta \mapsto \alpha^{-1}(\Theta)$, for all $\Theta \in \mathrm{Fe}_{\mathtt{K}}(\mathfrak{B})$, is easily seen to be an embedding from $\mathrm{Fc}_{\mathtt{K}}(\mathfrak{B})$ into $\mathrm{Fc}_{\mathtt{K}}^{\uparrow \mathrm{Ker}(\langle F, \alpha\rangle)}(\mathfrak{A})$. Indeed, if $\Theta = \langle \mathfrak{B}', \eta\rangle \in \mathrm{Fc}_{\mathtt{K}}(\mathfrak{B})$, then $\alpha^{-1}(\mathfrak{B}') \in \mathrm{Fe}_{\mathtt{K}}(\mathfrak{A})$, since $\mathfrak{B}' \in \mathrm{Fe}_{\mathtt{K}}(\mathfrak{A})$ and $\mathtt{K}$ is abstract, and also $\alpha^{-1}(\eta) \in \mathrm{Con}(\alpha^{-1}(\mathfrak{B}'))$, since $\eta \in \mathrm{Con}(\mathfrak{B}')$, such that $\mathrm{Ker}(\langle F, \alpha\rangle) \leq \alpha^{-1}(\eta)$. It will now be shown that the mapping

$$\langle \mathfrak{A}', \theta\rangle \stackrel{\hat{\alpha}}{\mapsto} \langle \mathfrak{B} \vee^{\mathtt{K}} \mathrm{Fg}_{\mathtt{K}}(\alpha(\mathfrak{A}')), \alpha(\theta)\rangle, \quad \text{for all } \langle \mathfrak{A}', \theta\rangle \in \mathrm{Fc}_{\mathtt{K}}^{\uparrow \mathrm{Ker}(\langle F, \alpha\rangle)}(\mathfrak{A}),$$

is its inverse mapping. In fact, since $\langle F, \alpha\rangle$ is strict and $\mathtt{K}$ is protoalgebraic, we have that $\alpha^{-1}(\mathfrak{B}) \sqsubseteq \mathfrak{A}'$ and $\mathrm{Ker}(\langle F, \alpha\rangle) \in \mathrm{Con}(\mathfrak{A}')$. Hence, by Lemma 14, $\alpha^{-1}(\mathfrak{B} \vee^{\mathtt{K}}$

$\mathrm{Fg}_{\mathtt{K}}(\alpha(\mathfrak{A}'))) = \mathfrak{A}'$, whence $\langle F, \alpha\rangle : \mathfrak{A}' \twoheadrightarrow_s \mathfrak{B} \vee^{\mathtt{K}} \mathrm{Fg}_{\mathtt{K}}(\alpha(\mathfrak{A}'))$ is a reductive $\mathcal{L}$-morphism. Thus, by Theorem 5 of [27], $\alpha(\theta) \in \mathrm{Con}(\mathfrak{B}\vee^{\mathtt{K}}\mathrm{Fg}_{\mathtt{K}}(\alpha(\mathfrak{A}')))$, i.e., $\hat{\alpha}$ is well-defined. Since, it has already been shown that $\alpha^{-1}(\mathfrak{B}\vee^{\mathtt{K}}\mathrm{Fg}_{\mathtt{K}}(\alpha(\mathfrak{A}'))) = \mathfrak{A}'$, to see that $\alpha^{-1}(\hat{\alpha}(\Theta)) = \Theta$, it suffices to show that $\alpha^{-1}(\alpha(\theta)) = \theta$. This, however, follows easily from the fact that $\mathrm{Ker}(\langle F, \alpha\rangle) \subseteq \theta$. Hence 2 holds.

$2 \to 1$ Note that, if $\mathfrak{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle, R^{\mathfrak{A}}\rangle, \mathfrak{B} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle, R^{\mathfrak{B}}\rangle$ are two $\mathcal{L}$-systems in $\mathtt{K}$, such that $\mathfrak{A} \sqsubseteq \mathfrak{B}$, then, by considering $\langle I_{\mathbf{Sign}}, \pi^{\Omega(\mathfrak{A})}\rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{A}^*$ in 2 and $\langle \mathfrak{B}, \Omega(\mathfrak{B})\rangle \in \mathrm{Fc}_{\mathtt{K}}(\mathfrak{A})$, we conclude that $\Omega(\mathfrak{B})/\Omega(\mathfrak{A}) \in \mathrm{Con}(\mathfrak{B}/\Omega(\mathfrak{A}))$, which shows that $\Omega(\mathfrak{A}) \leq \Omega(\mathfrak{B})$, i.e., that $\mathtt{K}$ is protoalgebraic.

$2 \to 3$ This follows by taking into account the direction $2 \to 1$ and applying Theorem 13, which shows that $\mathrm{Fe}_{\mathtt{K}}(\mathfrak{B}) \cong \mathrm{Fc}^{\Delta}_{\mathtt{K}}(\mathfrak{B})$ and $\mathrm{Fe}_{\mathtt{K}}(\mathfrak{A}) \cong \mathrm{Fc}^{\mathrm{Ker}(\langle F,\alpha\rangle)}_{\mathtt{K}}(\mathfrak{A})$. Then the hypothesis 2 does the rest, since it shows that $\mathrm{Fc}^{\Delta}_{\mathtt{K}}(\mathfrak{B}) \cong \mathrm{Fc}^{\mathrm{Ker}(\langle F,\alpha\rangle)}_{\mathtt{K}}(\mathfrak{A})$.

$3 \to 1$ Suppose, finally, that $\mathtt{K}$ satisfies the Filter Correspondence Property and let $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle, \mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle \in \mathtt{K}$, such that $\mathfrak{A} \sqsubseteq \mathfrak{B}$. Let $\langle I_{\mathbf{Sign}}, \pi^{\Omega(\mathfrak{A})}\rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{A}^*$ be the projection $\mathcal{L}$-morphism. Then, since $\mathfrak{B} \in \mathrm{Fe}_{\mathtt{K}}(\mathfrak{A})$, we get that $\pi^{\Omega(\mathfrak{A})^{-1}}(\mathfrak{A}^* \vee^{\mathtt{K}} \mathrm{Fg}_{\mathtt{K}}(\pi^{\Omega(\mathfrak{A})}(\mathfrak{B}))) = \mathfrak{B}$, whence, by Lemma 14, we get that $\mathrm{Ker}(\langle I_{\mathbf{Sign}}, \pi^{\Omega(\mathfrak{A})}\rangle) \in \mathrm{Con}(\mathfrak{B})$. But $\mathrm{Ker}(\langle I_{\mathbf{Sign}}, \pi^{\Omega(\mathfrak{A})}\rangle) = \Omega(\mathfrak{A})$, whence $\Omega(\mathfrak{A}) \leq \Omega(\mathfrak{B})$ and, therefore, $\mathtt{K}$ is protoalgebraic.

■

As Elgueta points out in [14], the equivalence between the Filter Correspondence Property and protoalgebraicity was first established by Blok and Pigozzi in Theorem 2.4 of [2] and Theorem 7.6 of [4] for the special case of matrix models of sentential logics. Elgueta also points out that the Filter Correspondence Property readily implies the Correspondence Theorem of Universal Algebra (see the remarks after Theorem 2.11 of [14]).

10 Other Characterizations of Protoalgebraicity

If $\mathtt{K}$ is a Lyndon class of $\mathcal{L}$-systems, then the $\sqsubseteq$-monotonicity of Ω turns out to be equivalent to the fact that $\Omega_{\mathtt{K},\mathbf{A}}$ is a meet complete homomorphism, for every $\mathcal{L}$-algebraic system $\mathbf{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle\rangle$, i.e., to the property that, for every collection $\mathfrak{A}_i = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle, R^i\rangle, i \in I$, of $\mathtt{K}$-systems on $\mathbf{A}$,

$$\Omega(\bigcap_{i\in I} \mathfrak{A}_i) = \bigcap_{i\in I} \Omega(\mathfrak{A}_i).$$

Suppose, now, that $\mathfrak{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle, R^{\mathfrak{A}}\rangle$ is an $\mathcal{L}$-system. Let $r \in R$, with $\rho(r) = n$, and $\Sigma \in |\mathbf{Sign}|, \vec{\phi} \in \mathrm{SEN}(\Sigma)^n$. Set

$$\mathrm{Fg}^{\mathfrak{A}}_{\mathtt{K}}[r; \langle \Sigma, \vec{\phi}\rangle] = \bigwedge\{\mathfrak{B} \in \mathrm{Fe}_{\mathtt{K}}(\mathfrak{A}) : \vec{\phi} \in r^{\mathfrak{B}}_{\Sigma}\}.$$

The closure of $\mathtt{K}$ under subdirect products guarantees that $\mathrm{Fg}^{\mathfrak{A}}_{\mathtt{K}}[r;\langle\Sigma,\vec{\phi}\rangle]$ is still a member of $\mathtt{K}$. Then, the following analog of Theorem 2.12 of [14] holds. It characterizes protoalgebraic Lyndon classes of $\mathcal{L}$-systems in terms of a condition that involves the filter extensions that are generated by all relation symbols and all tuples of elements over every signature of all $\mathcal{L}$-systems in the class.

Theorem 16 *Let* $\mathtt{K}$ *be a Lyndon class of* $\mathcal{L}$*-systems. Then* $\mathtt{K}$ *is protoalgebraic if and only if, for all* $\mathfrak{A}\in\mathtt{K}$*, all* $r\in R$*, with* $\rho(r)=n$*, and all* $\Sigma\in|\mathbf{Sign}|,\vec{\phi}\in\mathrm{SEN}(\Sigma)^n$*,*

$$\vec{\phi}\ \Omega(\mathfrak{A})^n\ \vec{\psi}\quad\textit{implies}\quad \mathrm{Fg}^{\mathfrak{A}}_{\mathtt{K}}[r;\langle\Sigma,\vec{\phi}\rangle]=\mathrm{Fg}^{\mathfrak{A}}_{\mathtt{K}}[r;\langle\Sigma,\vec{\psi}\rangle].$$

Proof:

Suppose, first, that $\mathtt{K}$ is protoalgebraic and let $\mathfrak{A}=\langle\mathrm{SEN},\langle\mathbf{N},F\rangle,R^{\mathfrak{A}}\rangle\in\mathtt{K}$ and $r\in R$, with $\rho(r)=n$. Let $\Sigma\in|\mathbf{Sign}|,\vec{\phi},\vec{\psi}\in\mathrm{SEN}(\Sigma)^n$, such that $\vec{\phi}\ \Omega(\mathfrak{A})^n\ \vec{\psi}$. Let $\vec{u}^i=\langle\psi_0,\ldots,\psi_i,\phi_{i+1},\ldots,\phi_{n-1}\rangle$, for all $i<n$. Then, we have

$$\begin{array}{lll}\phi_i\ \Omega(\mathfrak{A})\ \psi_i & \text{implies} & (\forall\mathfrak{B}\in\mathrm{Fe}_{\mathtt{K}}(\mathfrak{A}))(\phi_i\ \Omega(\mathfrak{B})\ \psi_i)\\ & \text{implies} & (\forall\mathfrak{B}\in\mathrm{Fe}_{\mathtt{K}}(\mathfrak{A}))(\mathfrak{B}\models_\Sigma(\forall\vec{z})(r(z_0,\ldots,z_{i-1},x,z_{i+1},\ldots,z_{n-1})\leftrightarrow\\ & & \qquad r(z_0,\ldots,z_{i-1},y,z_{i+1},\ldots,z_{n-1}))[\phi_i,\psi_i])\\ & \text{implies} & (\forall\mathfrak{B}\in\mathrm{Fe}_{\mathtt{K}}(\mathfrak{A}))(\vec{u}^{i-1}\in r^{\mathfrak{B}}_\Sigma\quad\text{iff}\quad\vec{u}^i\in r^{\mathfrak{B}}_\Sigma)\\ & \text{implies} & \mathrm{Fg}^{\mathfrak{A}}_{\mathtt{K}}[r;\langle\Sigma,\vec{u}^{i-1}\rangle]=\mathrm{Fg}^{\mathfrak{A}}_{\mathtt{K}}[r;\langle\Sigma,\vec{u}^i\rangle].\end{array}$$

Thus, since $\vec{\phi}=\vec{u}^0$ and $\vec{\psi}=\vec{u}^{n-1}$, we get the desired equality.

Suppose, conversely, that $\mathfrak{A}=\langle\mathrm{SEN},\langle\mathbf{N},F\rangle,R^{\mathfrak{A}}\rangle,\mathfrak{B}=\langle\mathrm{SEN},\langle\mathbf{N},F\rangle,R^{\mathfrak{B}}\rangle\in\mathtt{K}$, such that $\mathfrak{A}\sqsubseteq\mathfrak{B}$. To see that $\Omega(\mathfrak{A})\leq\Omega(\mathfrak{B})$, it suffices to show that $\Omega(\mathfrak{A})$ is a congruence system of $\mathfrak{B}$. To this end, let $r\in R$, with $\rho(r)=n$, $\Sigma\in|\mathbf{Sign}|$ and $\vec{\phi},\vec{\psi}\in\mathrm{SEN}(\Sigma)^n$, such that $\vec{\phi}\in r^{\mathfrak{B}}_\Sigma$ and $\vec{\phi}\ \Omega(\mathfrak{A})^n\ \vec{\psi}$. Then, we have, by hypothesis, $\mathrm{Fg}^{\mathfrak{A}}_{\mathtt{K}}[r;\langle\Sigma,\vec{\phi}\rangle]=\mathrm{Fg}^{\mathfrak{A}}_{\mathtt{K}}[r;\langle\Sigma,\vec{\psi}\rangle]$ and, also, by the minimality of $\mathrm{Fg}^{\mathfrak{A}}_{\mathtt{K}}[r;\langle\Sigma,\vec{\phi}\rangle]$, that $\mathrm{Fg}^{\mathfrak{A}}_{\mathtt{K}}[r;\langle\Sigma,\vec{\phi}\rangle]\sqsubseteq\mathfrak{B}$. Therefore, we get that $\vec{\psi}\in r^{\mathfrak{B}}_\Sigma$ and, hence, $\Omega(\mathfrak{A})$ is a congruence system of $\mathfrak{B}$. ■

We conclude the final section of the paper by presenting an analog of Theorem 2.13 of [14] for $\mathcal{L}$-systems. This theorem was first proven by Blok and Pigozzi (Theorem 9.3 of [4]) in the context of matrix models of sentential logics. Theorem 17 characterizes protoalgebraic abstract Lyndon classes in terms of the closedness of their reduced counterparts under subdirect products. The proof of the theorem in the present setting requires Lemma 3.1 of [31].

Theorem 17 *Let* $\mathtt{K}$ *be an abstract Lyndon class of* $\mathcal{L}$*-systems. Then* $\mathtt{K}$ *is protoalgebraic if and only if* $\mathtt{K}^*$ *is closed under subdirect products.*

Proof:

Assume, first, that $\mathtt{K}$ is protoalgebraic. Let $\langle F,\alpha\rangle:\mathfrak{A}\rightarrowtail_{\mathrm{sd}}\prod_{i\in I}\mathfrak{B}_i$, with $\mathfrak{B}_i\in\mathtt{K}^*$, for all $i\in I$. Since $\mathtt{K}$ is an abstract Lyndon class, we get that $\mathfrak{A}\in\mathtt{K}$, whence, it suffices to show that $\mathfrak{A}$ is reduced. Let $\langle F^i,\alpha^i\rangle=\langle P^i,\pi^i\rangle\circ\langle F,\alpha\rangle$, for all $i\in I$, where $\langle P^i,\pi^i\rangle:\prod_{i\in I}\mathfrak{B}_i\to\mathfrak{B}_i$ is the projection $\mathcal{L}$-morphism, for all $i\in I$.

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, "\langle F,\alpha\rangle"] \arrow[dr, "\langle F^i,\alpha^i\rangle"'] & \prod_{i\in I}\mathfrak{B}_i \arrow[d, "\langle P^i,\pi^i\rangle"] \\
& \mathfrak{B}_i
\end{tikzcd}$$

Then, by Part 3 of Lemma 3.1 of [31], $\mathfrak{A} = \bigcap_{i\in I} {\alpha^i}^{-1}(\mathfrak{B}_i)$ and $\bigcap_{i\in I} \mathrm{Ker}(\langle F^i,\alpha^i\rangle) = \Delta^{\mathrm{SEN}^{\mathfrak{A}}}$. Therefore, by the $\sqsubseteq$-monotonicity of Ω and by Lemma 4 of [27], we get that

$$\Omega(\mathfrak{A}) = \Omega(\bigcap_{i\in I} {\alpha^i}^{-1}(\mathfrak{B}_i)) = \bigcap_{i\in I} {\alpha^i}^{-1}(\Omega(\mathfrak{B}_i)) = \bigcap_{i\in I} \mathrm{Ker}(\langle F^i,\alpha^i\rangle) = \Delta^{\mathrm{SEN}^{\mathfrak{A}}},$$

which proves that $\mathfrak{A}$ is reduced and, as a consequence, that $\mathtt{K}^*$ is closed under subdirect products.

Suppose, conversely, that $\mathtt{K}^*$ is closed under subdirect products and let $\mathfrak{A} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle, R^{\mathfrak{A}}\rangle, \mathfrak{B} = \langle \mathrm{SEN}, \langle \mathbf{N}, F\rangle, R^{\mathfrak{B}}\rangle \in \mathtt{K}$, such that $\mathfrak{A} \sqsubseteq \mathfrak{B}$. Define $\langle F,\alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}/\Omega(\mathfrak{A}) \times \mathrm{SEN}/\Omega(\mathfrak{B})$ by setting $F(\Sigma) = \langle \Sigma, \Sigma\rangle$, for all $\Sigma \in |\mathbf{Sign}|$, and similarly for morphisms, and, for all $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$,

$$\alpha_\Sigma(\phi) = \langle \phi/\Omega_\Sigma(\mathfrak{A}), \phi/\Omega_\Sigma(\mathfrak{B})\rangle.$$

Since $\mathfrak{A} \sqsubseteq \mathfrak{B}$, $\langle F,\alpha\rangle : \mathfrak{A} \to_s \mathfrak{A}^* \times \mathfrak{B}^*$. Moreover, $\mathrm{Ker}(\langle F,\alpha\rangle) = \Omega(\mathfrak{A}) \cap \Omega(\mathfrak{B})$, whence, by the Homomorphism Theorem 10 of [27], we obtain $\langle F,\alpha\rangle : \mathfrak{A}/(\Omega(\mathfrak{A}) \cap \Omega(\mathfrak{B})) \rightarrowtail_{\mathrm{sd}} \mathfrak{A}^* \times \mathfrak{B}^*$. Hence, since $\mathtt{K}^*$ is closed under subdirect products, $\mathfrak{A}/(\Omega(\mathfrak{A}) \cap \Omega(\mathfrak{B})) \in \mathtt{K}^*$. This shows that $\Omega(\mathfrak{A}) = \Omega(\mathfrak{A}) \cap \Omega(\mathfrak{B})$, i.e., that $\Omega(\mathfrak{A}) \leq \Omega(\mathfrak{B})$, and that $\mathtt{K}$ is protoalgebraic. ■

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