International Journal on Semantic Web and Information Systems
Volume, number, pages

Package-based Description Logics: Syntax, Semantics and Complexity

Jie Bao[1], George Voutsadakis[2], Giora Slutzki[2], and Vasant Honavar[2]
[1]Rensselaer Polytechnic Institute, Troy, NY 12180, USA. {baojie@cs.rpi.edu}
[2]Iowa State University, Ames, IA 50011-1040, USA. {gvoutsad, slutzki,honavar}@cs.iastate.edu

Draft of September 23, 2008

Abstract

We present the syntax and semantics of a family of modular ontology languages, Package-based Description Logics (P-DL), to support context-specific reuse of knowledge from multiple ontologies. In particular, we discussed a P-DL $\mathcal{SHOIQP}$ that allows the "importing" of concept, role and nominal names between multiple ontology modules (each of which can be viewed as a $\mathcal{SHOIQ}$ ontology). $\mathcal{SHOIQP}$ supports contextualized interpretation, i.e., interpretation from the *point of view* of a specific package. We establish the necessary and sufficient constraints on domain relations (i.e., the relations between individuals in different local domains) to preserve the satisfiability of concept formulae, monotonicity of inference, and transitive reuse of knowledge. We further discuss the support for restricted inter-module role mappings and negated roles in P-DL and show that the P-DL $\mathcal{ALCH}^{+}\mathcal{IO}(\neg)\mathcal{P}$ is decidable.

Introduction

The success of the world wide web can be partially attributed to the *network effect*: The absence of central control on the content and the organization of the web allows thousands of independent actors to contribute resources (web pages) that are interlinked to form the web. Ongoing efforts to extend the current web into a *semantic web* are aimed at enriching the web with machine interpretable content and interoperable resources and services (Berners-Lee et al., 2001). Realizing the full potential of the semantic web requires the large-scale adoption and use of ontology-based approaches to sharing of information and resources. Constructing large ontologies typically requires collaboration among multiple individuals or groups with expertise in specific areas, with each participant contributing only a

Some parts of this paper without proofs were previously published in (Bao et al., 2007, 2008a,b).

part of the ontology. Therefore, instead of a single, centralized ontology, in most application domains it is natural to have multiple distributed ontologies covering parts of the domain. Such ontologies represent the *local* knowledge of the ontology designers, i.e., knowledge that is applicable in a *context*. Because no single ontology can meet the needs of all users under every conceivable scenario, there is an urgent need for theoretically sound, yet practical, approaches that allow knowledge from multiple autonomously developed ontologies to be adapted and reused in user, context, or application-specific scenarios.

Ontologies on the semantic web need to satisfy two apparently conflicting objectives (Bouquet et al., 2003):

• *Sharing* and *reuse* of knowledge across autonomously developed ontologies. An ontology may reuse another ontology by direct *importing* of selected terms in the other ontology (e.g., by referring to their URLs), or by using *mappings* between ontologies.

• The *contextuality* of knowledge or accommodation of the *local points of view*. For example, an assertion of the form "everything has the property that..." is usually made within an implicit local context which is often omitted from the statement. In fact, such a statement should be understood as "everything *in this domain* has the property that...". However, when reusing an existing ontology, the contextual nature of assertions is often neglected, leading to unintended inferences.

OWL adopts an importing mechanism to support integration of ontology modules. However, the importing mechanism in OWL, implemented by the `owl:imports` construct, in its current form, suffers from several serious drawbacks: (a) It directly introduces both terms and axioms of the imported ontologies into the importing ontology, and thus fails to support contextual reuse; (b) It provides no support for partial reuse of an ontology module.

Consequently, there have been several efforts aimed at developing formalisms that allow *reuse* of knowledge from multiple ontologies via *contextualized interpretations* in multiple local domains instead of a single shared global interpretation domain. Contextualized reuse of knowledge requires the interactions between local interpretations to be controlled. Examples of such modular ontology languages include: Distributed Description Logics (DDL) (Borgida & Serafini, 2003), $\mathcal{E}$-Connections (Grau et al., 2004) and Semantic Importing (Pan et al., 2006).

An alternative approach to knowledge reuse is based on the notion of *conservative extension* (Ghilardi et al., 2006; Grau et al., 2007, 2006; Grau & Kutz, 2007), which allows ontology modules to be interpreted using standard semantics by requiring that they share the same global interpretation domain. To avoid undesired effects from combining ontology modules, this approach requires that such a combination be a conservative extension of component modules. More precisely, if O is the union of a set of ontology modules $\{O_1, ..., O_n\}$, then we say O is a conservative extension of O_i if $O \models \alpha \Leftrightarrow O_i \models \alpha$, for any α in the language of O_i. This guarantees that combining knowledge from several ontology modules does not alter the consequences of knowledge contained in any component module. Thus, a combination of ontology modules cannot induce a new concept inclusion relation between concepts expressible in any of the component modules.

Current approaches to knowledge reuse have several limitations. To preserve contextuality, existing modular ontology languages offer only limited ways to connect ontology modules and, hence, limited ability to reuse knowledge across modules. For instance, DDL does not allow concept construction using foreign roles or concepts. $\mathcal{E}$-Connections, on the other hand, does not allow concept subsumptions across ontology modules or the use of foreign roles. Finally, Semantic Importing, in its current form, only allows each component module to be in $\mathcal{ALC}$. None of the existing approaches supports knowledge reuse in a setting where each ontology module uses a representation language that is as expressive as OWL-DL, i.e., $\mathcal{SHOIN}(D)$.

Furthermore, some of the existing modular ontology languages suffer from reasoning difficulties that can be traced back to the absence of natural ways to restrict the relations between individuals in different local domains. For example, DDL does not support the transitivity of inter-module concept subsumptions (known as *bridge rules*) in general. Moreover, in DDL a concept that is declared as being more specific than two disjoint concepts in another module may still be satisfiable (the inter-module satisfiability problem) (Bao et al., 2006c; Grau et al., 2004). Undisciplined use of generalized links in $\mathcal{E}$-Connections has also been shown to lead to reasoning difficulties (Bao et al., 2006b).

Conservative extensions (Grau et al., 2007, 2006; Grau & Kutz, 2007), in their current form, require a single global interpretation domain and, consequently, prevent different modules from interpreting axioms within their own local contexts. Hence, the designers of different ontology modules have to anticipate all possible contexts in which knowledge from a specific module might be reused. As a result, several modeling scenarios that would, otherwise, be quite useful in practice, such as the refinement of relations between existing concepts in an ontology module and the general reuse of nominals (Lutz et al., 2007), are precluded.

Against this background, this chapter, building on previous work of a majority of the authors (Bao et al., 2006c), develops a formalism that can support *contextual* reuse of knowledge from multiple ontology modules. The resulting modular ontology language, Package-based Description Logic (P-DL) $\mathcal{SHOIQP}$:

- Allows each ontology module to use a subset of $\mathcal{SHOIQ}$ (Horrocks & Sattler, 2005), i.e., $\mathcal{ALC}$ augmented with transitive roles, role inclusion, role inversion, qualified number restriction and nominal concepts and, hence, covers a significant fragment of OWL-DL.
- Supports more flexible modeling scenarios than those supported by existing approaches through a mechanism of *semantic importing* of names (including concept, role and nominal names) across ontology modules[1].
- Contextualizes the interpretation of reused knowledge. Locality of axioms in ontology modules is obtained "for free" by its *contextualized semantics*, thereby freeing ontology engineers from the burden of ensuring the reusability of an ontology module in contexts that are hard to foresee when constructing the module. A natural consequence of contextualized interpretation is that inferences are always drawn *from the point of view* of a *witness* module. Thus, different modules might infer different consequences, based on the knowledge that they import from other modules.
- Ensures that the results of reasoning are always the same as those obtained by a standard reasoner over an integrated ontology resulting from combining the relevant knowledge in a context-specific manner. Thus, unlike in the case of DDL and Semantic

[1]Note that importing in OWL, implemented by the `owl:imports` is essentially syntactic in nature. The difference between syntactic importing and semantic importing is best illustrated by an analogy with the writing of scientific articles: Knowledge reuse via `owl:imports` is analogous to *cut and paste* from a source article. In contrast, semantic importing is akin to knowledge reuse by means of *citation* of a source article.

Importing of Pan et al., P-DL ensures the *monotonicity* of inference in the distributed setting.

- Avoids several of the known reasoning difficulties of the existing approaches, e.g., lack of support for transitive reusability and nonpreservation of concept unsatisfiability.

Semantic Importing

This section introduces the syntax and semantics of the proposed language $\mathcal{SHOIQP}$. We will use a simple example shown in Figure 1 to illustrate some of the basic features of the P-DL syntax.

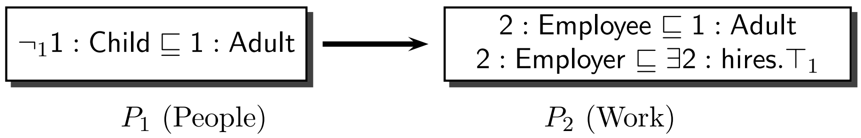


Figure 1. Semantic Importing

Syntax

Packages.

Informally, a package in $\mathcal{SHOIQP}$ can be viewed as a $\mathcal{SHOIQ}$ TBox and RBox. For example, in Figure 1 there are two packages, package P_1 describes the domain of People and P_2 describes the domain of Work.

We define the *signature* $\mathsf{Sig}(P_i)$ of a package P_i as the set of names used in P_i. $\mathsf{Sig}(P_i)$ is the disjoint union of the set of concept names NC_i, the set of role names NR_i and the set of nominal names NI_i used in package P_i. The set of roles in P_i is defined as $\overline{\mathsf{NR}}_i = \mathsf{NR}_i \cup \{R^-|R \in \mathsf{NR}_i\}$ where R^- is the *inverse* of the role name R.

The signature $\mathsf{Sig}(P_i)$ of package P_i is divided into two disjoint parts: its local signature $\mathsf{Loc}(P_i)$ and its external signature $\mathsf{Ext}(P_i)$. Thus, in the example shown in Figure 1, $\mathsf{Sig}(P_2) = \{\mathsf{Employee}, \mathsf{Adult}, \mathsf{Employer}, \mathsf{hires}\}$; $\mathsf{Loc}(P_2) = \{\mathsf{Employee}, \mathsf{Employer}, \mathsf{hires}\}$; and $\mathsf{Ext}(P_2) = \{\mathsf{Adult}\}$.

For all $t \in \mathsf{Loc}(P_i)$, P_i (and only P_i) is the *home package* of t, denoted by $P_i = \mathsf{Home}(t)$, and t is called an *i-name* (more specifically, an *i-concept name*, an *i-role name*, or an *i-nominal name*). We will use "$i : X$" to denote an i-name X and may drop the prefix when it is clear from the context. We use *i-role* to refer to an i-role name or its *inverse*. In the example shown in Figure 1, the home package of the terms Child and Adult is P_1 (People); and that of Employee, Employer and hires is P_2 (Work).

A role name $R \in \mathsf{NR}_i$ may be declared to be *transitive* in P_i using an axiom $\mathsf{Trans}_i(R)$. If R is declared transitive, R^- is also said to be *transitive*. We use $\mathsf{Tr}_i(R)$ to denote a role R being transitive in P_i.

A *role inclusion* axiom in P_i is an expression of the form $R \sqsubseteq S$, where R and S are i-roles. The *role hierarchy* for P_i is the set of all role inclusion axioms in P_i. The RBox $\mathcal{R}_i$ consists of the role hierarchy $\mathbf{R}_i$ for P_i and the set of role transitivity declarations $\mathsf{Trans}_i(R)$. For a role hierarchy $\mathbf{R}_i$, if $R \sqsubseteq S \in \mathbf{R}_i$, then R is called a *sub-role* of S and S is called a *super-role* of R *w.r.t.* $\mathbf{R}_i$. An i-role is called *locally simple* if it neither is transitive nor has any transitive sub-role in P_i.

The set of $\mathcal{SHOIQP}$ concepts in P_i is defined inductively by the following grammar:

$$C := A|o|\neg_k C|C \sqcap C|C \sqcup C|\forall R.C|\exists R.C|(\leq nS.C)|(\geq nS.C)$$

where $A \in \mathsf{NC}_i$, $o \in \mathsf{NI}_i$, n is a non-negative integer, $R \in \overline{\mathsf{NR}}_i$, and $S \in \overline{\mathsf{NR}}_i$ is a locally simple role; $\neg_k C$ denotes the *contextualized negation* of concept C *w.r.t.* P_k. For any k and k-concept name C, $\top_k = \neg_k C \sqcup C$, and $\bot = \neg_k C \sqcap C$. Thus, there is no universal top ($\top$) concept or global negation ($\neg$). Instead, we have for each package P_k, a contextualized top $\top_k$ and a contextualized negation $\neg_k$. This allows a logical formula in P-DL (including $\mathcal{SHOIQP}$) to be interpreted within the context of a specific package. Thus, in the example shown in Figure 1, $\neg_1 1:\mathsf{Child}$ in P_1 describes only the individuals *in the domain of People* that are not children (that is, *not* $1:\mathsf{Child}$).

A *general concept inclusion* (GCI) *axiom* in P_i is an expression of the form $C \sqsubseteq D$, where C, D are concepts in P_i. The TBox $\mathcal{T}_i$ of P_i is the set of GCIs in P_i. Thus, formally, a *package* P_i is a pair $P_i := \langle \mathcal{T}_i, \mathcal{R}_i \rangle$. A $\mathcal{SHOIQP}$ ontology Σ is a set of packages $\{P_i\}$. We assume that every name used in a $\mathcal{SHOIQP}$ ontology Σ has a home package in Σ.

Semantic Importing between Packages.

If a concept, role or nominal name $t \in \mathsf{Loc}(P_j) \cap \mathsf{Ext}(P_i)$, $i \neq j$, we say that P_i *imports* t and denote it as $P_j \xrightarrow{t} P_i$. We require that transitivity of roles be preserved under importing. Thus, if $P_j \xrightarrow{R} P_i$ where R is a j-role name, then $\mathsf{Trans}_i(R)$ iff $\mathsf{Trans}_j(R)$. If any local name of P_j is imported into P_i, we say that P_i imports P_j and denote it by $P_j \mapsto P_i$. In the example shown in Figure 1, P_2 imports P_1.

The *importing transitive closure* of a package P_i, denoted by P_i^+, is the set of all packages that are directly or indirectly imported by P_i. That is, P_i^+ is the smallest subset of $\{P_i\}$, such that

- $\forall j \neq i, P_j \mapsto P_i \Rightarrow P_j \in P_i^+$
- $\forall k \neq j \neq i, (P_k \mapsto P_j) \wedge (P_j \in P_i^+) \Rightarrow P_k \in P_i^+$

Let $P_i^* = \{P_i\} \cup P_i^+$. A $\mathcal{SHOIQP}$ ontology $\Sigma = \{P_i\}$ has an *acyclic importing relation* if, for all i, $P_i \notin P_i^+$; otherwise, it has a *cyclic importing relation*. The importing relation in the example in Figure 1 is acyclic.

We denote a Package-based Description Logic (P-DL) by adding the letter $\mathcal{P}$ to the notation for the corresponding DL. For example, $\mathcal{ALCP}$ is the package extension of the DL $\mathcal{ALC}$. We denote by $\mathcal{P}_\mathcal{C}$ a restricted type of P-DL that only allows importing of concept names. $\mathcal{P}^-$ denotes a P-DL with acyclic importing. In particular, $\mathcal{ALCP}_\mathcal{C}^-$ was studied in (Bao et al., 2006a), $\mathcal{ALCP}_\mathcal{C}$ was studied in (Bao et al., 2006d) and $\mathcal{SHOIQP}$ was studied in (Bao et al., 2007). The example in Figure 1 is in $\mathcal{ALCP}_\mathcal{C}^-$.

Syntax Restrictions on Semantic Importing.

Restrictions on Negations. We require that $\neg_k C$ (hence also $\top_k$) can appear in P_i, $i \neq k$, only if $P_k \mapsto P_i$. Intuitively, this means that k-negation can appear only in P_k or any package that directly imports P_k.

Restrictions on Imported Role Names. We require that *an imported role should not be used in role inclusion axioms*. This restriction is imposed because of two reasons. First, decidability requires that a role that is used in number restrictions be "globally" simple, i.e., that it has no transitive sub-role across any importing chain[2] (Horrocks et al., 1999). In practice, it is useful to restrict the use of imported roles in such a way that a role is globally simple iff it is locally simple. Second, a reduction of $\mathcal{SHOIQP}$ without such a restriction to an integrated ontology may require some features that are beyond the expressivity of $\mathcal{SHOIQ}$, such as role intersection. The decidability of $\mathcal{SHOIQP}$ with unrestricted use of imported role names still remains an open problem.[3]

$\mathcal{SHOIQP}$ Examples.

The semantic importing approach described here can model a broad range of scenarios that can also be modeled using existing approaches.

Example 1 Inter-module concept and role inclusions*. Suppose we have a* People *ontology* P_1*:*

$$\begin{aligned}
\neg_1 1:\mathsf{Man} &\sqsubseteq 1:\mathsf{Woman}\\
1:\mathsf{Man} &\sqsubseteq 1:\mathsf{People}\\
1:\mathsf{Woman} &\sqsubseteq 1:\mathsf{People}\\
1:\mathsf{Boy}\sqcup 1:\mathsf{Girl} &\sqsubseteq 1:\mathsf{Child}\\
1:\mathsf{Husband} &\sqsubseteq 1:\mathsf{Man}\sqcap\exists 1:\mathsf{marriedTo}.1:\mathsf{Woman}
\end{aligned}$$

Suppose the Work *ontology* P_2 *imports some of the knowledge from the* People *ontology:*

$$2:\mathsf{Employee} \sqsubseteq 1:\mathsf{People} \tag{1}$$

$$2:\mathsf{Employer} \equiv \exists 2:\mathsf{hires}.1:\mathsf{People} \tag{2}$$

$$1:\mathsf{Child} \sqsubseteq \neg_2 2:\mathsf{Employee} \tag{3}$$

$$2:\mathsf{EqualOpportunityEmployer} \sqsubseteq \exists 2:\mathsf{hires}.1:\mathsf{Man}\sqcap\exists 2:\mathsf{hires}.1:\mathsf{Woman} \tag{4}$$

Axiom (1) *models inter-module concept inclusion. This example also illustrates that the semantic importing approach can realize concept specialization* (*Axiom* (1)) *and generalization* (*Axiom* (3))*.*

Example 2 Use of foreign roles or foreign concepts to construct local concepts*. Suppose a* Marriage *ontology* P_3 *reuses the* People *ontology:*

$$(=1\ (1:\mathsf{marriedTo}).(1:\mathsf{Woman}))\sqsubseteq 3:\mathsf{Monogamist} \tag{5}$$

$$3:\mathsf{MarriedPerson}\sqsubseteq\forall(1:\mathsf{marriedTo}).(3:\mathsf{MarriedPerson}) \tag{6}$$

$$3:\mathsf{NuclearFamily}\sqsubseteq\exists(3:\mathsf{hasMember}).(1:\mathsf{Child}) \tag{7}$$

A complex concept in P_3 *may be constructed using an imported role* (6)*, an imported concept* (7)*, or both an imported role and an imported concept* (5)*.*

[2]This follows from the reduction from $\mathcal{SHOIQP}$ to $\mathcal{SHOIQ}$ given in the section titled "Reduction to Ordinary DL".

[3]For some subsets of $\mathcal{SHOIQP}$, this restriction may be relaxed. For example, $\mathcal{ALCHIOP}$ with unrestricted use of imported roles can be reduced to the DL $\mathcal{ALBO}$ (Schmidt & Tishkovsky, 2007) (extending $\mathcal{ALCO}$ with boolean role operators, role inclusion, inverse of roles and domain and range restriction operators), which is known to be decidable (Bao et al., 2008a).

Example 3 The use of nominals. *Suppose the* Work *ontology P_2, defined above, is augmented with additional knowledge from a* Calendar *ontology P_4, to obtain an augmented* Work *ontology. Suppose P_4 contains the following axiom:*

$$4{:}\mathsf{WeekDay} \quad = \quad \{4{:}Mon, 4{:}Tue, 4{:}Wed, 4{:}Thu, 4{:}Fri\},$$

where the nominals are shown in italic font. Suppose the new version of P_2 contains the following additional axioms:

$$\begin{aligned} 4:Fri &\sqsubseteq \exists(2:\mathsf{hasDressingCode}).(2:\mathsf{CasualDress}) \\ \top_2 &\sqsubseteq \exists(2:\mathsf{hasDressingCode}^-).(4:\mathsf{WeekDay}) \end{aligned}$$

Semantics

A $\mathcal{SHOIQP}$ ontology has *localized semantics* in the sense that each package has its own local interpretation domain. Formally, for a $\mathcal{SHOIQP}$ ontology $\Sigma = \{P_i\}$, a *distributed interpretation* is a tuple $\mathcal{I} = \langle \{\mathcal{I}_i\}, \{r_{ij}\}_{P_i \in P_j^+} \rangle$, where $\mathcal{I}_i$ is a *local interpretation* of package P_i, with (a not necessarily non-empty) domain $\Delta^{\mathcal{I}_i}$, $r_{ij} \subseteq \Delta^{\mathcal{I}_i} \times \Delta^{\mathcal{I}_j}$ is the *(image) domain relation* for the interpretation of the direct or indirect importing relation from P_i to P_j. For convenience, we use $r_{ii} = \mathrm{id}_{\Delta^{\mathcal{I}_i}} := \{(x,x) | x \in \Delta^{\mathcal{I}_i}\}$ to denote the identity mapping in the local domain $\Delta^{\mathcal{I}_i}$. Taking this convention into account, the distributed interpretation $\mathcal{I} = \langle \{\mathcal{I}_i\}, \{r_{ij}\}_{P_i \in P_j^+} \rangle$ may also be denoted by $\mathcal{I} = \langle \{\mathcal{I}_i\}, \{r_{ij}\}_{P_i \in P_j^*} \rangle$.

To facilitate our further discussion of interpretations, the following notational conventions will be used throughout. Given i, j, such that $P_i \in P_j^*$, for every $x \in \Delta^{\mathcal{I}_i}$, $A \subseteq \Delta^{\mathcal{I}_i}$ and $S \subseteq \Delta^{\mathcal{I}_i} \times \Delta^{\mathcal{I}_i}$, define[4] (please see Figure 2 and 3 for illustration):

$$\begin{aligned} r_{ij}(A) &= \{y \in \Delta^{\mathcal{I}_j} | \exists x \in A, (x,y) \in r_{ij}\}, && \text{(concept image)} \\ r_{ij}(S) &= r_{ij} \circ S \circ r_{ij}^- && \text{(role image)} \\ &= \{(z,w) \in \Delta^{\mathcal{I}_j} \times \Delta^{\mathcal{I}_j} | \exists (x,y) \in S, (x,z) \in r_{ij} \wedge (y,w) \in r_{ij}\}, \\ S(x) &= \{y \in \Delta^{\mathcal{I}_i} | (x,y) \in S\} && \text{(successor set)} \end{aligned}$$

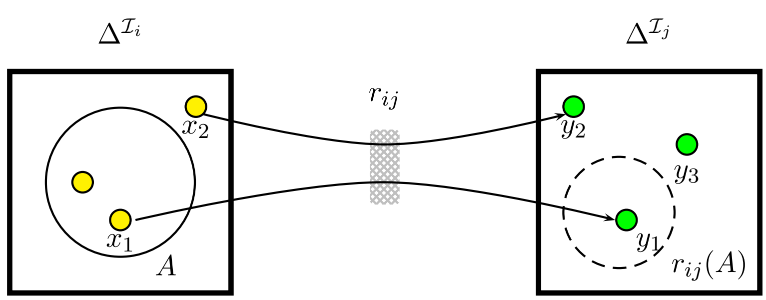


Figure 2. Concept Image

[4] In this chapter, $f_1 \circ ... \circ f_n$ denotes the composition of n relations $f_1, ..., f_n$, i.e., $(f_1 \circ ... \circ f_n)(x) = f_1(...f_n(x))$.

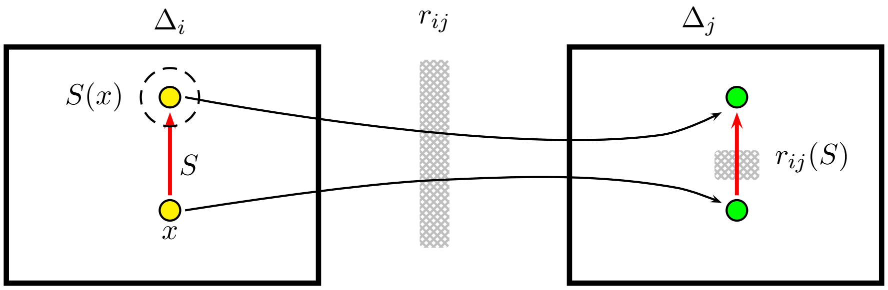


Figure 3. Successor Set and Role Image

Moreover, let ρ be the equivalence relation on $\bigcup_i \Delta^{\mathcal{I}_i}$ generated by the collection of ordered pairs $\bigcup_{P_i \in P_j^*} r_{ij}$. This is the symmetric and transitive closure of the set $\bigcup_{P_i \in P_j^*} r_{ij}$. Define, for every i, j, $\rho_{ij} = \rho \cap (\Delta^{\mathcal{I}_i} \times \Delta^{\mathcal{I}_j})$.

Each of the local interpretations $\mathcal{I}_i = \langle \Delta^{\mathcal{I}_i}, \cdot^{\mathcal{I}_i} \rangle$ consists of a domain $\Delta^{\mathcal{I}_i}$ and an interpretation function $\cdot^{\mathcal{I}_i}$, which maps every concept name to a subset of $\Delta^{\mathcal{I}_i}$, every role name to a subset of $\Delta^{\mathcal{I}_i} \times \Delta^{\mathcal{I}_i}$ and every nominal name to an element in $\Delta^{\mathcal{I}_i}$. We require that the interpretation function $\cdot^{\mathcal{I}}$ satisfies the following equations, where R is a j-role, S is a locally simple j-role, C, D are concepts:

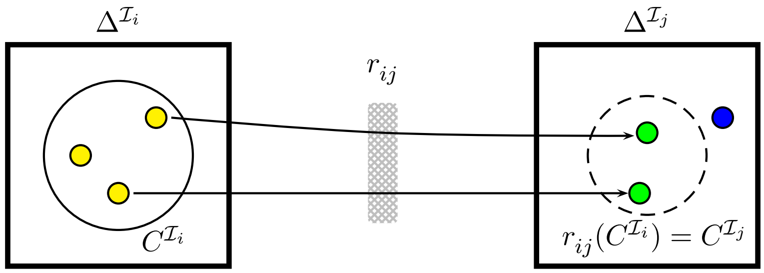
$$
\begin{aligned}
R^{\mathcal{I}_i} &= (R^{\mathcal{I}_i})^+, \text{ if } \mathsf{Trans}_i(R) \in \mathcal{R}_i \\
(R^-)^{\mathcal{I}_i} &= \{(x,y) | (y,x) \in R^{\mathcal{I}_i}\} \\
(C \sqcap D)^{\mathcal{I}_i} &= C^{\mathcal{I}_i} \cap D^{\mathcal{I}_i} \\
(C \sqcup D)^{\mathcal{I}_i} &= C^{\mathcal{I}_i} \cup D^{\mathcal{I}_i} \\
(\neg_j C)^{\mathcal{I}_i} &= r_{ji}(\Delta^{\mathcal{I}_j}) \backslash C^{\mathcal{I}_i} \\
(\exists R.C)^{\mathcal{I}_i} &= \{x \in r_{ji}(\Delta^{\mathcal{I}_j}) | \exists y \in \Delta^{\mathcal{I}_i}, (x,y) \in R^{\mathcal{I}_i} \wedge y \in C^{\mathcal{I}_i}\} \\
(\forall R.C)^{\mathcal{I}_i} &= \{x \in r_{ji}(\Delta^{\mathcal{I}_j}) | \forall y \in \Delta^{\mathcal{I}_i}, (x,y) \in R^{\mathcal{I}_i} \rightarrow y \in C^{\mathcal{I}_i}\} \\
(\geqslant nS.C)^{\mathcal{I}_i} &= \{x \in r_{ji}(\Delta^{\mathcal{I}_j}) |\ |\{y \in \Delta^{\mathcal{I}_i} | (x,y) \in S^{\mathcal{I}_i} \wedge y \in C^{\mathcal{I}_i}\}| \geqslant n\} \\
(\leqslant nS.C)^{\mathcal{I}_i} &= \{x \in r_{ji}(\Delta^{\mathcal{I}_j}) |\ |\{y \in \Delta^{\mathcal{I}_i} | (x,y) \in S^{\mathcal{I}_i} \wedge y \in C^{\mathcal{I}_i}\}| \leqslant n\}
\end{aligned}
$$

Note that, when $i = j$, since $r_{ii} = \mathrm{id}_{\Delta^{\mathcal{I}_i}}$, $(\neg_j C)^{\mathcal{I}_i}$ reduces to the usual negation $(\neg_i C)^{\mathcal{I}_i} = \Delta^{\mathcal{I}_i} \backslash C^{\mathcal{I}_i}$. Similarly, the other semantic definitions also reduce to the usual DL semantic definitions.

For an example of contextualized negation, suppose $A = C^{\mathcal{I}_i}$ in Figure 2. Then $(\neg_i C)^{\mathcal{I}_j}$ contains only y_2 but not y_3. On the other hand, $(\neg_j C)^{\mathcal{I}_j}$ contains both y_2 and y_3.

A local interpretation $\mathcal{I}_i$ *satisfies* a role inclusion axiom $R_1 \sqsubseteq R_2$ iff $R_1^{\mathcal{I}_i} \subseteq R_2^{\mathcal{I}_i}$ and a GCI $C \sqsubseteq D$ iff $C^{\mathcal{I}_i} \subseteq D^{\mathcal{I}_i}$. $\mathcal{I}_i$ is a *model* of P_i, denoted by $\mathcal{I}_i \vDash P_i$, if it satisfies all axioms in P_i.

The proposed semantics of $\mathcal{SHOIQP}$ is motivated by the need to overcome some of the limitations of existing approaches that can be traced back to the arbitrary construction of domain relations and the lack of support for contextualized interpretation. Specifically, we seek a semantics that satisfies the following desiderata:



An image domain relation in P-DL is one-to-one, i.e., it is a partial injective function. It is not necessarily total, i.e., some individuals of $C^{\mathcal{I}_i}$ may not be mapped to $\Delta^{\mathcal{I}_j}$.

Figure 4. One-to-One Domain Relation

- **Preservation of concept unsatisfiability**. The intuition is that an unsatisfiable concept expression should never be reused so as to be interpreted as a satisfiable concept. Formally, we say that a domain relation r_{ij} *preserves the unsatisfiability* of a concept C, that appears in both P_i and P_j, if whenever $C^{\mathcal{I}_i} = \emptyset$, it is necessarily the case that $C^{\mathcal{I}_j} = \emptyset$.
- **Transitive reusability of knowledge**. The intention is that the consequences of some of the axioms in one module can be propagated in a transitive fashion to other ontology modules. For example, if a package P_i asserts that $C \sqsubseteq D$, and P_j directly or indirectly imports that axiom from P_i, then it should be the case that $C \sqsubseteq D$ is also valid from the *point of view* of P_j.
- **Contextualized interpretation of knowledge**. The idea is that the interpretation of assertions in each ontology module is constrained by their context. When knowledge, e.g., axioms, in that module is reused by other modules, the interpretation of the reused knowledge should be constrained by the context in which the knowledge is being reused.
- **Improved expressivity**. Ideally, the language should support

1. both inter-module concept inclusion and concept construction using foreign concepts, roles and nominals;
2. more general reuse of roles and of nominals than allowed by existing approaches.

A major goal of this chapter is to explore the constraints that need to be imposed on local interpretations so that the resulting semantics for $\mathcal{SHOIQP}$ satisfies the desiderata enumerated above. These constraints are presented in the following:

Definition 1 *An interpretation* $\mathcal{I} = \langle \{\mathcal{I}_i\}, \{r_{ij}\}_{P_i \in P_j^*} \rangle$ *is a* model *of a* $\mathcal{SHOIQP}$ *KB* $\Sigma = \{P_i\}$*, denoted as* $\mathcal{I} \vDash \Sigma$*, if* $\bigcup_i \Delta^{\mathcal{I}_i} \neq \emptyset$*, i.e., at least one of the local interpretation domains is non-empty*[5]*, and the following conditions are satisfied:*

1. For all i, j*,* r_{ij} *is one-to-one, i.e., it is an injective partial function.*

[5] This agrees with conventional model-theoretic semantics, where an ordinary model (of a single package) is assumed to have a non-empty domain.

2. Compositional Consistency*: For all i, j, k s.t. $P_i \in P_k^*$ and $P_k \in P_j^*$, we have $\rho_{ij} = r_{ij} = r_{kj} \circ r_{ik}$.*

3. For every i-concept name C that appears in P_j, we have $r_{ij}(C^{\mathcal{I}_i}) = C^{\mathcal{I}_j}$.

4. For every i-role R that appears in P_j, we have $R^{\mathcal{I}_j} = r_{ij}(R^{\mathcal{I}_i})$.

5. Cardinality Preservation for Roles*: For every i-role R that appears in P_j and every $(x, x') \in r_{ij}$, $y \in R^{\mathcal{I}_i}(x)$ iff $r_{ij}(y) \in R^{\mathcal{I}_j}(x')$.*

6. For every i-nominal o that appears in P_j, $(o^{\mathcal{I}_i}, o^{\mathcal{I}_j}) \in r_{ij}$.

7. $\mathcal{I}_i \vDash P_i$, for every i.

The proposed semantics for $\mathcal{SHOIQP}$ is an extension of the semantics for $\mathcal{ALCP}_\mathcal{C}$ (Bao et al., 2006d), which uses Conditions 1,2,3 and 7 above, and borrows Condition 5 from the semantics of Semantic Importing (Pan et al., 2006).

Intuitively, one-to-oneness (Condition 1, see Figure 4) and compositional consistency (Condition 2, Figure 5) ensure that the parts of local domains connected by domain relations match perfectly. Conditions 3 and 4 ensure consistency between the interpretations of concepts and of roles in their home package and the interpretations in the packages that import them. Condition 5 (Figure 6) ensures that r_{ij} is a total bijection from $R^{\mathcal{I}_i}(x)$ to $R^{\mathcal{I}_j}(r_{ij}(x))$. In particular, the sizes $|R^{\mathcal{I}_i}(x)|$ and $|R^{\mathcal{I}_j}(r_{ij}(x))|$ are always equal in different local domains. Condition 6 ensures the uniqueness of nominals. In Section 4, we will show that Conditions 1-7 are minimally sufficient to guarantee that the desiderata for the semantics of $\mathcal{SHOIQP}$ as outlined above are indeed satisfied.

Note that Condition 2 implies that if P_i and P_j mutually (possibly indirectly) import one another, then $r_{ij} = \rho_{ij} = \rho_{ji}^- = r_{ji}^-$ and r_{ij} is a total function from $\Delta^{\mathcal{I}_i}$ to $\Delta^{\mathcal{I}_j}$. However, if $P_j \notin P_i^*$, then r_{ji} does not exist (even if r_{ij} exists). In that case, r_{ij} is not necessarily a total function.

Definition 2 *An ontology Σ is* consistent as witnessed by a package P_w *of Σ if P_w^* has a model $\mathcal{I} = \langle\{\mathcal{I}_i\}, \{r_{ij}\}_{P_i \in P_j^+}\rangle$, such that $\Delta^{\mathcal{I}_w} \neq \emptyset$. A concept C is* satisfiable as witnessed by P_w *if there is a model $\mathcal{I}$ of P_w^*, such that $C^{\mathcal{I}_w} \neq \emptyset$. A concept subsumption $C \sqsubseteq D$* is valid as witnessed by P_w*, denoted by $C \sqsubseteq_w D$, if, for every model $\mathcal{I}$ of P_w^*, $C^{\mathcal{I}_w} \subseteq D^{\mathcal{I}_w}$.*

Hence, in $\mathcal{SHOIQP}$, the questions of consistency, satisfiability and subsumption are always answered from the local point of view of a *witness package* and it is possible that different packages draw different conclusions from their own points of view.

The following examples show some inference problems that a P-DL ontology can tackle. Precise proofs for general cases will be given in the section titled "Properties of Semantic Importing".

Example 4 *Transitive subsumption propagation. Given three packages: $P_1 : \{1 : A \sqsubseteq 1 : B\}$, $P_2 : \{1 : B \sqsubseteq 2 : C\}$, $P_3 : \{2 : C \sqsubseteq 3 : D\}$, the subsumption query $1 : A \sqsubseteq 3 : D$ is answered in the affirmative as witnessed by P_3.*

Example 5 *Detection of inter-module unsatisfiability. Given two packages $P_1 : \{1 : B \sqsubseteq 1 : F\}$, $P_2 : \{2 : P \sqsubseteq 1 : B, 2 : P \sqsubseteq \neg 1 : F\}$, $2 : P$ is unsatisfiable as witnessed by P_2.*

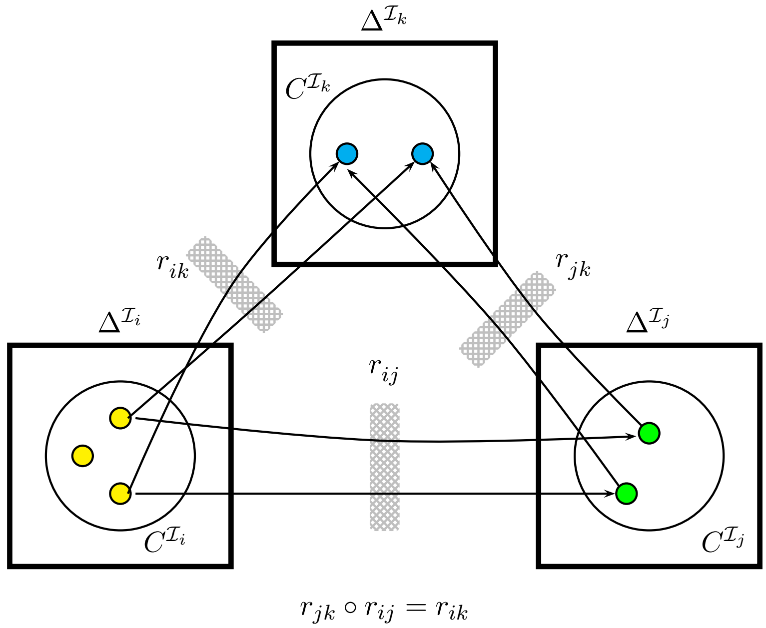


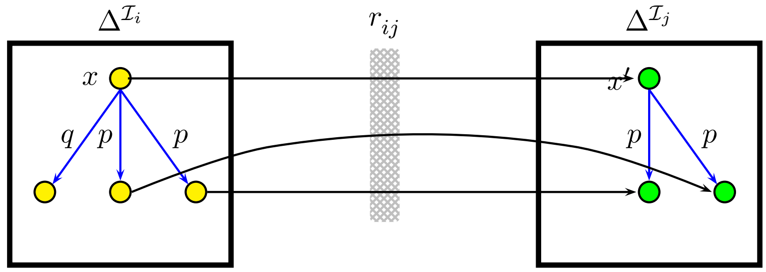
Figure 5. Compositionally Consistent Domain Relation

Example 6 *Reasoning from a local point of view. Given two packages* $P_1 : \{1 : A \sqsubseteq 1 : C\}$, $P_2 : \{1 : A \sqsubseteq \exists\, 2 : R.(2 : B), 2 : B \sqsubseteq 1 : A \sqcap (\neg\, 1 : C)\}$*, consider the satisfiability of* $1 : A$ *as witnessed by* P_1 *and* P_2*, respectively. It is easy to see* A *is satisfiable when witnessed by* P_1*, but unsatisfiable when witnessed by* P_2*. Thus, inferences in P-DL are always drawn from the point of view of a witness package. Different witnesses can draw different conclusions, since they operate on different domains and have access to different pieces of knowledge.*

Discussion: Relation Between the Semantics of P-DL and Partially-Overlapping Local Domain Semantics. In (Catarci & Lenzerini, 1993) a semantics based on partially overlapping domains was proposed for terminology mappings between ontology modules. In that framework, a global interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ is given together with local domains $\Delta^{\mathcal{I}_i}$, that are subsets of $\Delta^{\mathcal{I}}$. Any two local domains may be partially overlapping. Moreover, inclusions between concepts are of the following two forms:

- $i : C \sqsubseteq_{\text{ext}} j : D$ (extensional inclusion), with semantics $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$, and
- $i : C \sqsubseteq_{\text{int}} j : D$ (intentional inclusion), with semantics $C^{\mathcal{I}} \cap \Delta^{\mathcal{I}_i} \cap \Delta^{\mathcal{I}_j} \subseteq D^{\mathcal{I}} \cap \Delta^{\mathcal{I}_i} \cap \Delta^{\mathcal{I}_j}$.

Since P-DL semantics does not envision a global point of view, extensional inclusion has no corresponding notion in P-DL semantics. In addition, P-DL semantics differs significantly from this approach in that, while both intentional and extensional inclusions are not directional, the semantic importing in P-DL is. To make this distinction clearer, consider two packages P_i and P_j, such that $P_i \mapsto P_j$. Let C, D be two i-concept names that are imported by P_j and consider the interpretation where $\Delta^{\mathcal{I}_i} = \{x, y, z\}, \Delta^{\mathcal{I}_j} = \{y, z\}$, $C^{\mathcal{I}_i} = \{x, y\}, D^{\mathcal{I}_i} = \{y, z\}$ and $r_{ij} = \{\langle y, y\rangle, \langle z, z\rangle\}$. Then, in P-DL, from the point of view of package P_i, we have $C^{\mathcal{I}_i} = \{x, y\} \not\subseteq \{y, z\} = D^{\mathcal{I}_i}$. Therefore, $\mathcal{I} \not\models_i C \sqsubseteq D$. Simi-



If an i-role p is imported by P_j, then every pair of p instances must have a "preimage" pair in Δ_i. The cardinality preservation condition for roles, illustrated in this figure, requires that, if an individual x in $\Delta^{\mathcal{I}_i}$ has an image individual x' in $\Delta^{\mathcal{I}_j}$, then each of its p-neighbors must have an image in $\Delta^{\mathcal{I}_j}$ which is a p-neighbor of x'.

Figure 6. Cardinality Preservation for Roles

larly, from the point of view of package P_j, we have $C^{\mathcal{I}_j} = r_{ij}(C^{\mathcal{I}_i}) = r_{ij}(\{x, y\}) = \{y\} \subseteq \{y, z\} = r_{ij}(\{y, z\}) = r_{ij}(D^{\mathcal{I}_i}) = D^{\mathcal{I}_j}$. Therefore, $\mathcal{I} \models_j C \sqsubseteq D$. However, in the partially overlapping domain semantics of (Catarci & Lenzerini, 1993), $C =_{\text{int}} D$ holds from both P_i's and P_j's point of view.

Thus, in spite of the fact that the intersection of two sets is "seen equally" from both sets' points of view, the example that was presented above illustrates that the way concept names are interpreted in these models still preserves some form of directionality in the subsumption reasoning.

Despite this subtle semantic difference between the partially overlapping domain semantics of (Catarci & Lenzerini, 1993) and the semantics of P-DL presented here, it is still possible to provide P-DL with a different kind of overlapping-domain-style semantics. More precisely, in the proof of Lemma 4, it is shown how one may combine the various local domains of a P-DL interpretation into one global domain. The P-DL model satisfies a given subsumption $C \sqsubseteq D$ from a witness P_i's point of view if and only if the global model satisfies an appropriately constructed *subjective* translation $\#_i(C) \sqsubseteq \#_i(D)$ of the given subsumption (see Section 3). Moreover, in the proof of Lemma 3, it is shown how, conversely, starting from a global domain, one may construct a P-DL model with various local domains; if the aforementioned subjective translation of a subsumption is satisfied in the global domain, then the original subsumption is satisfied from P_i's point of view. If the two constructions are composed, starting from the original P-DL model one obtains another equivalent model that is based on a partially-overlapping-style domain semantics. However, due to the interpretations of the translations of the concept names in this model, directionality is still preserved, unlike the situation in the ordinary partially overlapping domain semantics of (Catarci & Lenzerini, 1993).

Since any ordinary P-DL model gives rise to an equivalent model with partially-overlapping-style semantics, it is natural to ask as to why we do not choose the latter as the basis for the semantics of P-DL. The main reason has to do with the fact that, in many applications, local models are populated independently of one another before semantic relations between their individuals are physically established. Moreover, the main motivation for introducing modular description logics is to allow autonomous groups to independently develop knowledge bases (ontologies). Additionally, the semantics of P-DL is derived from the Local Model Semantics (Ghidini & Giunchiglia, 2001). A main feature of the proposed P-DL semantics is the directionality (and subjectivity) of domain relations, which is not preserved by the partially-overlapping-domain semantics. The preservation of the directionality of domain relations, keeps open the possibility of extensions of P-DL to settings where the use of partially-overlapping-domain semantics is infeasible, e.g., when transitive knowledge propagation needs to be limited to only trusted entities.

□ (**End of Discussion**)

As immediate consequences of the proposed semantics for the P-DL $\mathcal{SHOIQP}$, extensions of various versions of the De Morgan's Law may be proven. Those deal with both the ordinary propositional logical connectives, including local negations, and with the quantifiers, as shown in the following lemma.

Lemma 1 *Let $P_i \mapsto P_j$, C, D be concepts, R a k-role, such that* $\mathsf{Sig}(C) \cup \mathsf{Sig}(D) \cup \{R\} \subseteq \mathsf{Sig}(P_i) \cap \mathsf{Sig}(P_j)$*. Then, the following equalities hold from the point of view of P_j:*

1. $\neg_i C = \top_i \sqcap \neg_j C$;
2. $\neg_i (C \sqcap D) = \neg_i C \sqcup \neg_i D$;
3. $\neg_i (C \sqcup D) = \neg_i C \sqcap \neg_i D$;
4. $\neg_i (\exists R.C) = \neg_i \top_k \sqcup \forall R.\neg_j C$;
5. $\neg_i (\forall R.C) = \neg_i \top_k \sqcup \exists R.\neg_j C$;
6. $\neg_i (\leq nR.C) = \neg_i \top_k \sqcup \geq (n+1)R.C$;
7. $\neg_i (\geq (n+1)R.C) = \neg_i \top_k \sqcup \leq nR.C$.

Proof:

- For Equation 1, we have

$$\begin{aligned}(\top_i \sqcap \neg_j C)^{\mathcal{I}_j} &= \top_i^{\mathcal{I}_j} \cap (\neg_j C)^{\mathcal{I}_j} && \text{(by the definition of } \cdot^{\mathcal{I}_j}\text{)}\\ &= r_{ij}(\Delta^{\mathcal{I}_i}) \cap (\Delta^{\mathcal{I}_j} \backslash C^{\mathcal{I}_j}) && \text{(by the definition of } \cdot^{\mathcal{I}_j}\text{)}\\ &= r_{ij}(\Delta^{\mathcal{I}_i}) \backslash C^{\mathcal{I}_j} && \text{(since } r_{ij}(\Delta^{\mathcal{I}_i}) \subseteq \Delta^{\mathcal{I}_j}\text{)}\\ &= (\neg_i C)^{\mathcal{I}_j}. && \text{(by the definition of } (\neg_i C)^{\mathcal{I}_j}\text{)}\end{aligned}$$

- For Equation 2,

$$\begin{aligned}&(\neg_i (C \sqcap D))^{\mathcal{I}_j}\\ =\ & r_{ij}(\Delta^{\mathcal{I}_i}) \backslash (C \sqcap D)^{\mathcal{I}_j} && \text{(by the definition of } \cdot^{\mathcal{I}_j}\text{)}\\ =\ & r_{ij}(\Delta^{\mathcal{I}_i}) \backslash (C^{\mathcal{I}_j} \cap D^{\mathcal{I}_j}) && \text{(by the definition of } \cdot^{\mathcal{I}_j}\text{)}\\ =\ & (r_{ij}(\Delta^{\mathcal{I}_i}) \backslash C^{\mathcal{I}_j}) \cup (r_{ij}(\Delta^{\mathcal{I}_i}) \backslash D^{\mathcal{I}_j}) && \text{(set-theoretically)}\\ =\ & (\neg_i C)^{\mathcal{I}_j} \cup (\neg_i D)^{\mathcal{I}_j} && \text{(by the definition of } \cdot^{\mathcal{I}_j}\text{)}\\ =\ & (\neg_i C \sqcup \neg_i D)^{\mathcal{I}_j}. && \text{(by the definition of } \cdot^{\mathcal{I}_j}\text{)}\end{aligned}$$

- The proof of Equation 3 is dual to that of Equation 2.
- For Equation 4, we first note that $P_k \xrightarrow{R} P_i$ and

$$\begin{aligned}\forall R.\neg_j C &\sqsubseteq_j \top_k && \text{(since } (\forall R.\neg_j C)^{\mathcal{I}_j} \subseteq (\top_k)^{\mathcal{I}_j} \text{ by the definitions)}\\ &\sqsubseteq_j \top_i, && \text{(since } (\top_k)^{\mathcal{I}_j} = r_{kj}(\Delta^{\mathcal{I}_k}) = r_{ij} \circ r_{ki}(\Delta^{\mathcal{I}_k}) \subseteq r_{ij}(\Delta^{\mathcal{I}_i})\text{)}\end{aligned}$$

$$
\begin{aligned}
(\neg_i(\exists R.C))^{\mathcal{I}_j} &= r_{ij}(\Delta^{\mathcal{I}_i})\backslash(\exists R.C)^{\mathcal{I}_j} && \text{(by the definition of } \cdot^{\mathcal{I}_j}) \\
&= r_{ij}(\Delta^{\mathcal{I}_i})\backslash\{x \in r_{kj}(\Delta^{\mathcal{I}_k}) | \exists y \in \Delta^{\mathcal{I}_j}, (x,y) \in R^{\mathcal{I}_j} \wedge y \in C^{\mathcal{I}_j}\} \\
& && \text{(by the definition of } (\exists R.C)^{\mathcal{I}_j}) \\
&= (r_{ij}(\Delta^{\mathcal{I}_i})\backslash r_{kj}(\Delta^{\mathcal{I}_k})) \cup (r_{ij}(\Delta^{\mathcal{I}_i}) \cap \{x \in r_{kj}(\Delta^{\mathcal{I}_k})| \\
&\qquad \forall y \in \Delta^{\mathcal{I}_j}, (x,y) \in R^{\mathcal{I}_j} \rightarrow y \notin C^{\mathcal{I}_j}\}) \\
& && \text{(set-theoretically)} \\
&= (\neg_i \top_k)^{\mathcal{I}_j} \cup (\top_i^{\mathcal{I}_j} \cap (\forall R.\neg_j C)^{\mathcal{I}_j}) \\
& && \text{(by the definition of } \cdot^{\mathcal{I}_j}) \\
&= (\neg_i \top_k \sqcup (\top_i \sqcap \forall R.\neg_j C))^{\mathcal{I}_j} && \text{(by the definition of } \cdot^{\mathcal{I}_j}) \\
&= (\neg_i \top_k \sqcup (\forall R.\neg_j C))^{\mathcal{I}_j}. && \text{(since } \forall R.\neg_j C \sqsubseteq_j \top_i)
\end{aligned}
$$

- The proof of Equation 5 is dual to that of Equation 4.
- For Equation 6, as for Equation 4, we have $\geq(n+1)R.C \sqsubseteq_j \top_i$.

$$
\begin{aligned}
(\neg_i(\leq nR.C))^{\mathcal{I}_j} &= r_{ij}(\Delta^{\mathcal{I}_i})\backslash(\leq nR.C)^{\mathcal{I}_j} && \text{(by the definition of } \cdot^{\mathcal{I}_j}) \\
&= r_{ij}(\Delta^{\mathcal{I}_i})\backslash\{x \in r_{kj}(\Delta^{\mathcal{I}_k})|\ |\{y \in \Delta^{\mathcal{I}_j} | (x,y) \in R^{\mathcal{I}_j} \\
&\qquad \wedge y \in C^{\mathcal{I}_j}\}| \leq n\} \\
& && \text{(by the definition of } (\leq nR.C)^{\mathcal{I}_j}) \\
&= r_{ij}(\Delta^{\mathcal{I}_i})\backslash r_{kj}(\Delta^{\mathcal{I}_k}) \cup (r_{ij}(\Delta^{\mathcal{I}_i}) \cap \{x \in r_{kj}(\Delta^{\mathcal{I}_k})| \\
&\qquad |\{y \in \Delta^{\mathcal{I}_j} | (x,y) \in R^{\mathcal{I}_j} \wedge y \in C^{\mathcal{I}_j}\}| \geq n+1\}) \\
& && \text{(set-theoretically)} \\
&= (\neg_i \top_k)^{\mathcal{I}_j} \cup (\top_i^{\mathcal{I}_j} \cap (\geq(n+1)R.C)^{\mathcal{I}_j}) \\
& && \text{(by the definition of } \cdot^{\mathcal{I}_j}) \\
&= (\neg_i \top_k \sqcup (\top_i \sqcap \geq(n+1)R.C))^{\mathcal{I}_j} \\
& && \text{(by the definition of } \cdot^{\mathcal{I}_j}) \\
&= (\neg_i \top_k \sqcup \geq(n+1)R.C)^{\mathcal{I}_j}. \\
& && \text{(since } \geq(n+1)R.C \sqsubseteq_j \top_i)
\end{aligned}
$$

- The proof of Equation 7 follows the dual steps to those in the proof of Equation 6.

Q.E.D.

Note that when $i = j = k$, the equations in Lemma 1 reduce to the ordinary versions of De Morgan's Law in DL. These equations are helpful in simplifying proofs of other properties of $\mathcal{SHOIQP}$. Also note that, under the same hypotheses as those in Lemma 1,

$$
\begin{aligned}
(\exists R.C)^{\mathcal{I}_j} &= \{x \in r_{kj}(\Delta^{\mathcal{I}_k}) | \exists y \in \Delta^{\mathcal{I}_j}, (x,y) \in R^{\mathcal{I}_j} \wedge y \in C^{\mathcal{I}_j}\} \\
&= \{x \in r_{kj}(\Delta^{\mathcal{I}_k})|\ |\{y \in \Delta^{\mathcal{I}_j} | (x,y) \in R^{\mathcal{I}_j} \wedge y \in C^{\mathcal{I}_j}\}| \geq 1\} \\
&= (\geq 1R.C)^{\mathcal{I}_j} \\
(\forall R.C)^{\mathcal{I}_j} &= \{x \in r_{kj}(\Delta^{\mathcal{I}_k}) | \forall y \in \Delta^{\mathcal{I}_j}, (x,y) \in R^{\mathcal{I}_j} \rightarrow y \in C^{\mathcal{I}_j}\} \\
&= \{x \in r_{kj}(\Delta^{\mathcal{I}_k})|\ |\{y \in \Delta^{\mathcal{I}_j} | (x,y) \in R^{\mathcal{I}_j} \wedge y \notin C^{\mathcal{I}_j}\}| \leq 0\} \\
&= (\leq 0R.\neg_j C)^{\mathcal{I}_j}
\end{aligned}
$$

Hence, proofs involving existential restriction and value restriction may be reduced to those involving the corresponding number restrictions[6]. In what follows, we will only

[6]Note that R may not be a locally simple role in which case it cannot be used in number restrictions. However, the formulas above still allow us in practice to rephrase arguments involving existential restriction

consider negation, conjunction and at-most number restriction as concept constructors since, as we have just pointed out, arguments for other constructors can be reduced to them.

In the next lemma, it is asserted that Condition 3 of Definition 1 holds not only for concept names, but, in fact, for arbitrary concepts. Beyond its own intrinsic interest, it becomes handy in Section 4 in showing that the package description logic $\mathcal{SHOIQP}$ supports monotonicity of reasoning and transitive reusability of modules.

Lemma 2 *Let Σ be a $\mathcal{SHOIQP}$ ontology, P_i, P_j two packages in Σ such that $P_i \in P_j^+$, C a concept such that $\mathsf{Sig}(C) \subseteq \mathsf{Sig}(P_i) \cap \mathsf{Sig}(P_j)$, and R a role name such that $R \in \mathsf{Sig}(P_i) \cap \mathsf{Sig}(P_j)$. If $\mathcal{I} = \langle \{\mathcal{I}_u\}, \{r_{uv}\}_{P_u \in P_v^+} \rangle$ is a model of Σ, then $r_{ij}(C^{\mathcal{I}_i}) = C^{\mathcal{I}_j}$ and $r_{ij}(R^{\mathcal{I}_i}) = R^{\mathcal{I}_j}$.*

Proof: For a k-role name R, such that $R \in \mathsf{Sig}(P_i) \cap \mathsf{Sig}(P_j)$, we have $r_{ij}(R^{\mathcal{I}_i}) = r_{ij} \circ r_{ki}(R^{\mathcal{I}_k}) = r_{kj}(R^{\mathcal{I}_k}) = R^{\mathcal{I}_j}$.

To prove the claim for concepts, structural induction on the concept formula C will be used.

If C is a k-concept name or a k-nominal name, we have

$$\begin{array}{rcll} r_{ij}(C^{\mathcal{I}_i}) & = & r_{ij}(r_{ki}(C^{\mathcal{I}_k})) & \text{(by the definition of } C^{\mathcal{I}_i}) \\ & = & r_{kj}(C^{\mathcal{I}_k}) & \text{(by compositional consistency)} \\ & = & C^{\mathcal{I}_j}. & \text{(by the definition of } C^{\mathcal{I}_j}) \end{array}$$

For $C = \neg_k D$ and $r_{ij}(D^{\mathcal{I}_i}) = D^{\mathcal{I}_j}$, we have

$$\begin{array}{rcll} r_{ij}(C^{\mathcal{I}_i}) & = & r_{ij}((\neg_k D)^{\mathcal{I}_i}) & \text{(since } C = \neg_k D) \\ & = & r_{ij}(r_{ki}(\Delta^{\mathcal{I}_k}) \backslash D^{\mathcal{I}_i}) & \text{(by the definition of } (\neg_k D)^{\mathcal{I}_i}) \\ & = & r_{ij}(r_{ki}(\Delta^{\mathcal{I}_k})) \backslash r_{ij}(D^{\mathcal{I}_i}) & \text{(since } r_{ij} \text{ is one-to-one)} \\ & = & r_{kj}(\Delta^{\mathcal{I}_k}) \backslash D^{\mathcal{I}_j} & \text{(by compositional consistency and} \\ & & & \text{the induction hypothesis)} \\ & = & (\neg_k D)^{\mathcal{I}_j} & \text{(by the definition of } (\neg_k D)^{\mathcal{I}_j}) \\ & = & C^{\mathcal{I}_j}. & \text{(since } C = \neg_k D) \end{array}$$

For $C = D \sqcap E$, assuming inductively that $r_{ij}(D^{\mathcal{I}_i}) = D^{\mathcal{I}_j}$ and $r_{ij}(E^{\mathcal{I}_i}) = E^{\mathcal{I}_j}$, we have

$$\begin{array}{rcll} r_{ij}(C^{\mathcal{I}_i}) & = & r_{ij}((D \sqcap E)^{\mathcal{I}_i}) & \text{(since } C = D \sqcap E) \\ & = & r_{ij}(D^{\mathcal{I}_i} \cap E^{\mathcal{I}_i}) & \text{(by the definition of } \cdot^{\mathcal{I}_i}) \\ & = & r_{ij}(D^{\mathcal{I}_i}) \cap r_{ij}(E^{\mathcal{I}_i}) & \text{(since } r_{ij} \text{ is one-to-one)} \\ & = & D^{\mathcal{I}_j} \cap E^{\mathcal{I}_j} & \text{(by the induction hypothesis)} \\ & = & (D \sqcap E)^{\mathcal{I}_j} & \text{(by the definition of } \cdot^{\mathcal{I}_j}) \\ & = & C^{\mathcal{I}_j}. & \text{(since } C = D \sqcap E) \end{array}$$

Let $C = {\leq}nR.D$, with R a k-role, and assume inductively that $r_{ij}(D^{\mathcal{I}_i}) = D^{\mathcal{I}_j}$. We first prove two auxiliary claims.

---

or universal restriction into corresponding arguments on number restrictions (for $n = 1$ or $n = 0$) regardless of the simplicity of R.

Claim 1: Let $x' = r_{ij}(x)$. Then $r_{ij} : R^{\mathcal{I}_i}(x) \rightarrow R^{\mathcal{I}_j}(x')$ is a total bijection.
Proof: r_{ij} is a one-to-one function by definition. It is onto because

$$\begin{aligned} R^{\mathcal{I}_j}(x') &= r_{ij}(R^{\mathcal{I}_i})(x') \\ &= (r_{ij} \circ R^{\mathcal{I}_i} \circ r_{ij}^{-})(r_{ij}(x)) \\ &= (r_{ij} \circ R^{\mathcal{I}_i})(x) \\ &= r_{ij}(R^{\mathcal{I}_i}(x)) \end{aligned}$$

By cardinality preservation (item 5 in Definition 1), r_{ij} is a total function from $R^{\mathcal{I}_i}(x)$ to $R^{\mathcal{I}_j}(x')$, whence r_{ij} is a total bijection from $R^{\mathcal{I}_i}(x)$ to $R^{\mathcal{I}_j}(x')$. Q.E.D.

Claim 2: Let $x' = r_{ij}(x)$. Then $r_{ij} : R^{\mathcal{I}_i}(x) \cap D^{\mathcal{I}_i} \rightarrow R^{\mathcal{I}_j}(x') \cap D^{\mathcal{I}_j}$ is also a total bijection.
Proof: r_{ij} is one-to-one and total on $R^{\mathcal{I}_i}(x)$ by Claim 1. Hence,

$$r_{ij}(R^{\mathcal{I}_i}(x) \cap D^{\mathcal{I}_i}) = r_{ij}(R^{\mathcal{I}_i}(x)) \cap r_{ij}(D^{\mathcal{I}_i}) = R^{\mathcal{I}_j}(x') \cap D^{\mathcal{I}_j}.$$

Thus, r_{ij} is onto $R^{\mathcal{I}_j}(x') \cap D^{\mathcal{I}_j}$, whence the claim holds. Q.E.D.

Using the two claims, we now obtain

$$\begin{aligned} x' \in r_{ij}((\leq nR.D)^{\mathcal{I}_i}) &\Leftrightarrow \exists x \in (\leq nR.D)^{\mathcal{I}_i} \text{ such that } x' = r_{ij}(x) \\ &\qquad \text{(by the definition of } r_{ij}) \\ &\Leftrightarrow \exists x \in r_{ki}(\Delta^{\mathcal{I}_k}), |R^{\mathcal{I}_i}(x) \cap D^{\mathcal{I}_i}| \leq n \wedge x' = r_{ij}(x) \\ &\qquad \text{(by the definition of } (\leq nR.D)^{\mathcal{I}_i}) \\ &\Leftrightarrow x' \in r_{kj}(\Delta^{\mathcal{I}_k}), |R^{\mathcal{I}_j}(x') \cap D^{\mathcal{I}_j}| \leq n \\ &\qquad (\Rightarrow\text{: by compositional consistency and Claim 2}) \\ &\qquad (\Leftarrow\text{: by compositional consistency and Claim 2}) \\ &\Leftrightarrow x' \in (\leq nR.D)^{\mathcal{I}_j} \quad \text{(by the definition of } (\leq nR.D)^{\mathcal{I}_j}) \end{aligned}$$

Q.E.D.

Reduction to Ordinary DL

In this section, we present a translation from concept formulas that appear in a given package of a $\mathcal{SHOIQP}$ KB Σ to concept formulas of a $\mathcal{SHOIQ}$ KB $\Sigma^\star$. The $\mathcal{SHOIQ}$ KB $\Sigma^\star$ is constructed in such a way that the top concept $\top_w$, associated with a specific package P_w of Σ, is satisfiable by $\Sigma^\star$ in the ordinary DL sense if and only if Σ itself is consistent *from the point of view of* P_w (see Theorem 1). (Note that the $\mathcal{SHOIQ}$ KB $\Sigma^\star$ is dependent on the importing relations present in the $\mathcal{SHOIQP}$ KB Σ). This shows that the consistency problem in $\mathcal{SHOIQP}$ is reducible to the satisfiability problem in $\mathcal{SHOIQ}$, which is known to be NExpTime-complete (Tobies, 2000, 2001). This has the consequence that the problems of concept satisfiability, concept subsumption and consistency in $\mathcal{SHOIQP}$ are also NExpTime-complete (see Theorem 2). Moreover, as will be seen in Section 4, this result plays a central role in showing that some of the desiderata presented in Section 2.2

are satisfied by $\mathcal{SHOIQP}$. For instance, Reasoning Exactness, Monotonicity of Reasoning, Transitive Reusability of Knowledge and Preservation of Unsatisfiability are all features of $\mathcal{SHOIQP}$, which are shown to hold by employing the translation from $\mathcal{SHOIQP}$ to $\mathcal{SHOIQ}$.

The reduction $\Re$ from a $\mathcal{SHOIQP}$ KB $\Sigma = \{P_i\}$ to a $\mathcal{SHOIQ}$ KB $\Sigma^\star$ can be obtained as follows: the signature of $\Sigma^\star$ is the union of the local signatures of the component packages together with a global top $\top$, a global bottom $\bot$ and local top concepts $\top_i$, for all i, i.e., $\mathsf{Sig}(\Sigma^\star) = \bigcup_i(\mathsf{Loc}(P_i) \cup \{\top_i\}) \cup \{\top, \bot\}$, and

a) For all i, j, k such that $P_i \in P_k^*$, $P_k \in P_j^*$, $\top_i \sqcap \top_j \sqsubseteq \top_k$ is added to $\Sigma^\star$.

b) For each GCI $X \sqsubseteq Y$ in P_j, $\#_j(X) \sqsubseteq \#_j(Y)$ is added to Σ. The mapping $\#_j()$ is defined below.

c) For each role inclusion $X \sqsubseteq Y$ in P_j, $X \sqsubseteq Y$ is added to $\Sigma^\star$.

d) For each i-concept name or i-nominal name C in P_i, $i : C \sqsubseteq \top_i$ is added to $\Sigma^\star$.

e) For each i-role name R in P_i, $\top_i$ is stipulated to be its domain and range, i.e., $\top \sqsubseteq \forall R^-.\top_i$ and $\top \sqsubseteq \forall R.\top_i$ are added to $\Sigma^\star$.

f) For each i-role name R in P_j, the following axioms are added to $\Sigma^\star$:

– $\exists R.\top_j \sqsubseteq \top_j$ (local domain);

– $\exists R^-.\top_j \sqsubseteq \top_j$ (local range).

g) For each i-role name, add $\mathsf{Trans}(R)$ to $\Sigma^\star$ if $\mathsf{Trans}_i(R)$.

The mapping $\#_j()$ is adapted from a similar one for DDL (Borgida & Serafini, 2003) with modifications to facilitate context preservation whenever name importing occurs. For a formula X used in P_j, $\#_j(X)$ is:

- X, for a j-concept name or a j-nominal name.
- $X \sqcap \top_j$, for an i-concept name or an i-nominal name X.
- $\neg\#_j(Y) \sqcap \top_i \sqcap \top_j$, for $X = \neg_i Y$, where Y is a concept.
- $(\#_j(X_1) \oplus \#_j(X_2)) \sqcap \top_j$, for a concept $X = X_1 \oplus X_2$, where $\oplus = \sqcap$ or $\oplus = \sqcup$.
- $(\otimes R.\#_j(X')) \sqcap \top_i \sqcap \top_j$, for a concept $X = (\otimes R.X')$, where $\otimes \in \{\exists, \forall, \leq n, \geq n\}$ and R is an i-role.

For example, if C, D are concept names and R a role name,

$$\begin{aligned}
\#_j(\neg_i\ i : C) &= \neg(C \sqcap \top_j) \sqcap \top_i \sqcap \top_j \\
\#_j(j : D \sqcup i : C) &= (D \sqcup (C \sqcap \top_j)) \sqcap \top_j \\
\#_j(\forall(j : R).(i : C)) &= \forall R.(C \sqcap \top_j) \sqcap \top_j \\
\#_j(\exists(i : R).(i : C)) &= \exists R.(C \sqcap \top_j) \sqcap \top_i \sqcap \top_j
\end{aligned}$$

It should be noted that $\#_j()$ is *contextualized* so as to allow a given formula to have different interpretations when it appears in different packages. See also the Discussion subsection in Section 2.2.

Properties of Semantic Importing

In this section, we further justify the proposed semantics for $\mathcal{SHOIQP}$. More specifically, we present the main results showing that $\mathcal{SHOIQP}$ satisfies the desiderata listed in Section 2.

The first main theorem shows that the consistency problem of a $\mathcal{SHOIQP}$ ontology w.r.t. a witness package P_w can be reduced to a satisfiability problem of a $\mathcal{SHOIQ}$ concept w.r.t. an integrated ontology *from the point of view of that witness package*, namely, $\Re(P_w^*)$. Note that there is no single universal integrated ontology for all packages. Each package, *sees* an integrated ontology (depending on the witness package and all the packages that are directly or indirectly imported by the witness package), and hence different packages can witness different consequences.

Theorem 1 *A $\mathcal{SHOIQP}$ KB Σ is consistent as witnessed by a package P_w if and only if $\top_w$ is satisfiable with respect to $\Re(P_w^*)$.*

Proof: Sufficiency is proven in Lemma 3 and necessity in Lemma 4.

Lemma 3 *Let Σ be a $\mathcal{SHOIQP}$ KB and P_w a package of Σ. If $\top_w$ is satisfiable with respect to $\Re(P_w^*)$, then Σ is consistent as witnessed by P_w.*

Proof: If $\top_w$ is satisfiable with respect to $\Re(P_w^*)$, then $\Re(P_w^*)$ has at least one model $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$, such that $\top_w^{\mathcal{I}} \neq \emptyset$. Our goal is to construct a model of P_w^* from $\mathcal{I}$, such that $\Delta^{\mathcal{I}_w} \neq \emptyset$. For each package P_i, a local interpretation $\mathcal{I}_i$ is constructed as a projection of $\mathcal{I}$ in the following way:

- $\Delta^{\mathcal{I}_i} = \top_i^{\mathcal{I}}$.
- For every concept name C that appears in P_i, $C^{\mathcal{I}_i} = C^{\mathcal{I}} \cap \top_i^{\mathcal{I}}$.
- For every role name R that appears in P_i, $R^{\mathcal{I}_i} = R^{\mathcal{I}} \cap (\top_i^{\mathcal{I}} \times \top_i^{\mathcal{I}})$.
- For every nominal name o that appears in P_i, $o^{\mathcal{I}_i} = o^{\mathcal{I}}$.

For every pair i, j, such that $P_i \in P_j^*$, we define

$$r_{ij} = \{(x, x) | x \in \Delta^{\mathcal{I}_i} \cap \Delta^{\mathcal{I}_j}\}.$$

Clearly, we have $\Delta^{\mathcal{I}_w} = \top_w^{\mathcal{I}} \neq \emptyset$, by the hypothesis. So it suffices, now, to show that $\langle \{\mathcal{I}_i\}, \{r_{ij}\}_{P_i \in P_j^*} \rangle$ is a model of the modular ontology P_w^*, i.e., that it satisfies the seven conditions postulated in Definition 1.

First, it is clear from the definition that each r_{ij} is in fact a one-to-one relation.

Second, we must show that Compositional Consistency holds.

• Suppose that $P_i \in P_j^*$, $x \in \Delta^{\mathcal{I}_i}$, $y \in \Delta^{\mathcal{I}_j}$, and $(x, y) \in \rho_{ij}$. Therefore, x and y must be connected by some chain of domain relations and/or inverse domain relations according to the definition of ρ_{ij}. Because all domain relations are identities, this implies that $x = y \in \Delta^{\mathcal{I}_i} \cap \Delta^{\mathcal{I}_j}$, whence, once more by the definition of r_{ij}, we obtain that $(x, y) \in r_{ij}$. This proves that $\rho_{ij} \subseteq r_{ij}$.

• Assume that i, j, k such that $P_i \in P_k^*$, $P_k \in P_j^*$ and $(x, y) \in r_{ij}$. Then $x = y \in \Delta^{\mathcal{I}_i} \cap \Delta^{\mathcal{I}_j}$. Since, in that case, $\top_i \sqcap \top_j \sqsubseteq \top_k$, this implies that $x \in \Delta^{\mathcal{I}_k}$, whence $x \in \Delta^{\mathcal{I}_i} \cap \Delta^{\mathcal{I}_j} \cap \Delta^{\mathcal{I}_k}$, showing that $(x, x) \in r_{ik}$ and $(x, x) \in r_{kj}$. Therefore $r_{ij} \subseteq r_{kj} \circ r_{ik}$.

• From the definition of ρ_{ij}, we have $r_{kj} \circ r_{ik} \subseteq \rho_{ij}$.

Hence, $\rho_{ij} = r_{ij} = r_{kj} \circ r_{ik}$, for $P_i \in P_k^*$ and $P_k \in P_j^*$.

Next, it is shown that Conditions 3,4 and 6 of Definition 1 hold for the distributed interpretation. Let X be an i-concept name or an i-nominal name. Then, we have that

$$\begin{aligned} r_{ij}(X^{\mathcal{I}_i}) &= X^{\mathcal{I}_i} \cap \Delta^{\mathcal{I}_j} && \text{(by the definition of } r_{ij}\text{)} \\ &= X^{\mathcal{I}} \cap \Delta^{\mathcal{I}_i} \cap \Delta^{\mathcal{I}_j} && \text{(by the definition of } X^{\mathcal{I}_i}\text{)} \\ &= X^{\mathcal{I}} \cap \Delta^{\mathcal{I}_j} && \text{(since } i : X \sqsubseteq \top_i\text{)} \\ &= X^{\mathcal{I}_j}. && \text{(by the definition of } X^{\mathcal{I}_j}\text{)} \end{aligned}$$

For X an i-role name, the same equalities hold with all local interpretation domains replaced by their cartesian squares.

To show Cardinality Preservation for Roles, suppose that R is an i-role in P_j and that $(x, x') \in r_{ij}$, i.e., $x = x' \in \Delta^{\mathcal{I}_i} \cap \Delta^{\mathcal{I}_j}$. Then, we have

$$\begin{array}{rlll} y \in R^{\mathcal{I}_i}(x) & \text{iff} & (x, y) \in R^{\mathcal{I}_i} & \text{(by the definition of } R^{\mathcal{I}_i}(x)\text{)} \\ & \text{iff} & (x, y) \in R^{\mathcal{I}} & \text{(since } R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}_i} \times \Delta^{\mathcal{I}_i}\text{)} \\ & \text{iff} & (x', y) \in R^{\mathcal{I}} \cap (\Delta^{\mathcal{I}_j} \times \Delta^{\mathcal{I}_j}) & \text{(by the local domain and local} \\ & & & \text{range axioms, and } x = x' \in \Delta^{\mathcal{I}_j}\text{)} \\ & \text{iff} & (x', y) \in R^{\mathcal{I}_j} & \text{(by the definition of } R^{\mathcal{I}_j}\text{)} \\ & \text{iff} & y = r_{ij}(y) \in R^{\mathcal{I}_j}(x'). & \text{(by the definition of } R^{\mathcal{I}_j}(x')\text{)} \end{array}$$

Thus, cardinality preservation for roles holds.

Finally, it remains to show that Condition 7 of Definition 1 holds, i.e., that $\mathcal{I}_j$ is a model of P_j, for every j.

For every role inclusion of the form $R \sqsubseteq S$ in P_j, R and S must be j-roles (by our restriction on the use of imported roles), whence we have that

$$\begin{array}{rcll} R^{\mathcal{I}_j} & = & R^{\mathcal{I}} \cap (\Delta^{\mathcal{I}_j} \times \Delta^{\mathcal{I}_j}) & \text{(by the definition of } R^{\mathcal{I}_j}\text{)} \\ & \subseteq & S^{\mathcal{I}} \cap (\Delta^{\mathcal{I}_j} \times \Delta^{\mathcal{I}_j}) & \text{(since } R \sqsubseteq S \text{ holds in the integrated ontology)} \\ & = & S^{\mathcal{I}_j}. & \text{(by the definition of } S^{\mathcal{I}_j}\text{)} \end{array}$$

For a role R that appears in P_j, we have that $\mathsf{Trans}_j(R)$ if and only if $\mathsf{Trans}(R)$, whence

$$\begin{array}{rcll} (R^{\mathcal{I}_j})^+ & = & (R^{\mathcal{I}} \cap (\Delta^{\mathcal{I}_j} \times \Delta^{\mathcal{I}_j}))^+ & \text{(by the definition of } R^{\mathcal{I}_j}\text{)} \\ & = & (R^{\mathcal{I}})^+ \cap (\Delta^{\mathcal{I}_j} \times \Delta^{\mathcal{I}_j}) & \text{(set-theoretically)} \\ & = & R^{\mathcal{I}} \cap (\Delta^{\mathcal{I}_j} \times \Delta^{\mathcal{I}_j}) & \text{(since } \mathsf{Trans}(R) \text{ holds)} \\ & = & R^{\mathcal{I}_j}. & \text{(by the definition of } R^{\mathcal{I}_j}\text{)} \end{array}$$

Finally, suppose that $C \sqsubseteq D$ is a concept inclusion in P_j. Then we must have $\#_j(C)^{\mathcal{I}} \subseteq \#_j(D)^{\mathcal{I}}$, whence, to prove that $C^{\mathcal{I}_j} \subseteq D^{\mathcal{I}_j}$, it suffices to show that, for every concept formula X that appears in P_j, we have $\#_j(X)^{\mathcal{I}} = X^{\mathcal{I}_j}$. We do this by structural induction on X. We will consider in detail only concepts constructed using negation, conjunction and number restriction. All other constructors may be handled similarly.

For the basis of the induction, if X is a j-concept name or a j-nominal name, then we have $\#_j(X) = X$, whence $\#_j(X)^{\mathcal{I}} = X^{\mathcal{I}} = X^{\mathcal{I}} \cap \Delta^{\mathcal{I}_j} = X^{\mathcal{I}_j}$, whereas, if X is an i-concept name or an i-nominal name, with $i \neq j$, we have $\#_j(X)^{\mathcal{I}} = (X \sqcap \top_j)^{\mathcal{I}} = X^{\mathcal{I}} \cap \Delta^{\mathcal{I}_j} = X^{\mathcal{I}_j}$.

Suppose, next, as the induction hypothesis, that for concepts C and D appearing in P_j, $\#_j(C)^{\mathcal{I}} = C^{\mathcal{I}_j}$ and $\#_j(D)^{\mathcal{I}} = D^{\mathcal{I}_j}$, and also note that $\#_j(R) = R$ for every i-role R appearing in P_j. Thus, we have

$$\begin{array}{rcll} \#_j(\neg_i C)^{\mathcal{I}} & = & (\neg \#_j(C) \sqcap \top_i \sqcap \top_j)^{\mathcal{I}} & \text{(by the definition of } \#_j(\neg_i C)\text{)} \\ & = & (\neg \#_j(C))^{\mathcal{I}} \cap \top_i^{\mathcal{I}} \cap \top_j^{\mathcal{I}} & \text{(by the definition of } \cdot^{\mathcal{I}}\text{)} \\ & = & (\Delta^{\mathcal{I}} \backslash \#_j(C)^{\mathcal{I}}) \cap \Delta^{\mathcal{I}_i} \cap \Delta^{\mathcal{I}_j} & \text{(by the definition of } \cdot^{\mathcal{I}}\text{)} \\ & = & (\Delta^{\mathcal{I}_i} \cap \Delta^{\mathcal{I}_j}) \backslash \#_j(C)^{\mathcal{I}} & \text{(since } \Delta^{\mathcal{I}_i} \cap \Delta^{\mathcal{I}_j} \subseteq \Delta^{\mathcal{I}}\text{)} \\ & = & r_{ij}(\Delta^{\mathcal{I}_i}) \backslash C^{\mathcal{I}_j} & \text{(by the definition of } r_{ij} \text{ and} \\ & & & \text{the induction hypothesis)} \\ & = & (\neg_i C)^{\mathcal{I}_j}. & \text{(by the definition of } (\neg_i C)^{\mathcal{I}_j}\text{)} \end{array}$$

$$
\begin{array}{rcll}
\#_j(C \sqcap D)^{\mathcal{I}} & = & (\#_j(C) \sqcap \#_j(D) \sqcap \top_j)^{\mathcal{I}} & \text{(by the definition of } \#_j(C \sqcap D)\text{)} \\
& = & \#_j(C)^{\mathcal{I}} \cap \#_j(D)^{\mathcal{I}} \cap \Delta^{\mathcal{I}_j} & \text{(by the definition of } \cdot^{\mathcal{I}}\text{)} \\
& = & C^{\mathcal{I}_j} \cap D^{\mathcal{I}_j} \cap \Delta^{\mathcal{I}_j} & \text{(by the induction hypothesis)} \\
& = & C^{\mathcal{I}_j} \cap D^{\mathcal{I}_j} & \text{(since } C^{\mathcal{I}_j}, D^{\mathcal{I}_j} \subseteq \Delta^{\mathcal{I}_j}\text{)} \\
& = & (C \sqcap D)^{\mathcal{I}_j}. & \text{(by the definition of } \cdot^{\mathcal{I}_j}\text{)}
\end{array}
$$

$$
\begin{array}{rcll}
\#_j(\leq nR.C)^{\mathcal{I}} & = & ((\leq nR.\#_j(C)) \sqcap \top_i \sqcap \top_j)^{\mathcal{I}} & \\
& = & (\leq nR.\#_j(C))^{\mathcal{I}} \cap \Delta^{\mathcal{I}_i} \cap \Delta^{\mathcal{I}_j} & \\
& = & \{x \in \Delta^{\mathcal{I}} |\ |R^{\mathcal{I}}(x) \cap (\#_j(C))^{\mathcal{I}}| \leq n\} \cap \Delta^{\mathcal{I}_i} \cap \Delta^{\mathcal{I}_j} & \\
& = & \{x \in \Delta^{\mathcal{I}_i} \cap \Delta^{\mathcal{I}_j} |\ |R^{\mathcal{I}}(x) \cap C^{\mathcal{I}_j}| \leq n\} & \\
& = & \{x \in \Delta^{\mathcal{I}_i} \cap \Delta^{\mathcal{I}_j} |\ |R^{\mathcal{I}_j}(x) \cap C^{\mathcal{I}_j}| \leq n\} & (*) \\
& = & \{x \in r_{ij}(\Delta^{\mathcal{I}_i}) |\ |R^{\mathcal{I}_j}(x) \cap C^{\mathcal{I}_j}| \leq n\} & \\
& = & (\leq nR.C)^{\mathcal{I}_j} &
\end{array}
$$

(*) holds because $R^{\mathcal{I}_j} \subseteq R^{\mathcal{I}}$ and for any $y \in R^{\mathcal{I}}(x) \cap C^{\mathcal{I}_j}$, $y \in \Delta^{\mathcal{I}_j}$, hence $y \in R^{\mathcal{I}_j}(x) \cap C^{\mathcal{I}_j}$. Q.E.D.

Next, we proceed to show the reverse implication.

Lemma 4 *Let Σ be a $\mathcal{SHOIQP}$ KB. If Σ is consistent as witnessed by a package P_w, then $\top_w$ is satisfiable with respect to $\Re(P_w^*)$.*

Proof: Suppose that Σ is consistent as witnessed by P_w. Thus, it has a distributed model $\langle \{\mathcal{I}_i\}, \{r_{ij}\}_{P_i \in P_j^*} \rangle$, such that $\Delta^{\mathcal{I}_w} \neq \emptyset$. We proceed to construct a model $\mathcal{I}$ of $\Re(P_w^*)$ by merging individuals that are related via chains of image domain relations or their inverses. More precisely, for every element x in the distributed model, we define its equivalence class $\overline{x} = \{y | (x, y) \in \rho\}$ where ρ is the symmetric and transitive closure of the set $\bigcup_{P_i \in P_j^*} r_{ij}$. Moreover, for a set S, we define $\overline{S} = \{\bar{x} | x \in S\}$ and for a binary relation R, we define $\overline{R} = \{(\overline{x}, \overline{y}) | (x, y) \in R\}$.

<u>Claim 3</u>: (a) For all i and for all $\overline{x}$, $|\overline{x} \cap \Delta^{\mathcal{I}_i}| \leq 1$.
(b) For all i and any set $S \subseteq \Delta^{\mathcal{I}_i}$, $|S| = |\overline{S}|$.
(c) For all i and all sets $A_1, A_2 \subseteq \Delta^{\mathcal{I}_i}$, $\overline{A_1 \backslash A_2} = \overline{A_1} \backslash \overline{A_2}$.
(d) For all i and for all $S \subseteq \Delta^{\mathcal{I}_i} \times \Delta^{\mathcal{I}_i}$, $(\overline{S})^+ = (\overline{S^+})$.

Proof: (a) Suppose $u, v \in \overline{x} \cap \Delta^{\mathcal{I}_i}, u \neq v$. Then, since $r_{ii} = \mathrm{id}_{\Delta^{\mathcal{I}_i}}$ and ρ is the equivalence relation generated by the union of the r_{ij}'s, there must exist a $y \in (\overline{x} \backslash \Delta^{\mathcal{I}_i}) \cap \Delta^{\mathcal{I}_j}$ for some j, which implies that $\{(v, y), (u, y)\} \subseteq \rho_{ij} = r_{ij}$ or $\{(y, u), (y, v)\} \subseteq \rho_{ji} = r_{ji}$, contradicting the assumption that domain relations are one-to-one. Hence $|\overline{x} \cap \Delta^{\mathcal{I}_i}| \leq 1$.

In what follows, we denote by $\overline{x}|_i$ the element (if it exists) in $\Delta^{\mathcal{I}_i}$ that belongs to $\overline{x}$, i.e., $\overline{x}|_i \in \Delta^{\mathcal{I}_i} \cap \overline{x}$.

(b) We prove this statement by showing that $f : x \rightarrow \overline{x}$ is a total bijection from S to $\overline{S}$. f is a total and onto function by the definition of $\overline{S}$. f is injective because for $\overline{x} \in \overline{S}$, if there are two distinct x_1, x_2 in S, such that $\overline{x_1} = \overline{x_2} = \overline{x}$, then $\{x_1, x_2\} \subseteq \overline{x} \cap \Delta^{\mathcal{I}_i}$, contradicting (a).

(c) This statement holds because

$$\begin{array}{rcll}\overline{x} \in \overline{A_1}\backslash\overline{A_2} & \leftrightarrow & (\overline{x} \in \overline{A_1} \text{ and } \overline{x} \notin \overline{A_2}) & \\ & \leftrightarrow & \exists x', \{x'\} = \overline{x} \cap \Delta^{\mathcal{I}_i}, x' \in A_1\backslash A_2 & \text{(by Part (a))} \\ & \leftrightarrow & \overline{x} \in \overline{A_1\backslash A_2} & \end{array}$$

d) First we prove $(\overline{S})^+ \subseteq \overline{S^+}$. This holds because $S \subseteq S^+$, hence $(\overline{S})^+ \subseteq (\overline{S^+})^+$; on the other hand, if $(x,y) \in (\overline{S^+})^+$, there exist $x_1, ..., x_n$ such that $(x, x_1), (x_n, y)$ and (x_{k-1}, x_k) (for $k = 2, ..., n$) $\in \overline{S^+}$, hence $(x|_i, x_1|_i), (x_n|_i, y|_i)$ and $(x_{k-1}|_i, x_k|_i) \in S^+$ (for $k = 2, ..., n$), hence $(x|_i, y|_i) \in S^+$, thus $(x, y) \in \overline{S^+}$.

In the other direction, if $(x,y) \in \overline{S^+}$, then $(x|_i, y|_i) \in S^+$, hence there exist $x_1, ..., x_n$ such that $(x|_i, x_1), (x_n, y|_i)$ and $(x_{k-1}, x_k) \in S$ (for $k = 2, ..., n$), therefore $(x, \overline{x_1}), (\overline{x_n}, y)$ and $(\overline{x_{k-1}}, \overline{x_k}) \in \overline{S}$ (for $k = 2, ..., n$), thus $(x, y) \in (\overline{S})^+$. Claim 3 Q.E.D.

We now proceed to define a model of Σ. Let $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ be defined as follows:

- $\top^{\mathcal{I}} = \Delta^{\mathcal{I}} = \overline{\bigcup_i \Delta^{\mathcal{I}_i}}$, and $\bot^{\mathcal{I}} = \emptyset$.
- For every i-name X, $X^{\mathcal{I}} := \overline{X^{\mathcal{I}_i}}$.
- For every i, $\top_i^{\mathcal{I}} = \overline{\Delta^{\mathcal{I}_i}}$.

We must show that $\mathcal{I}$ is a model of $\Re(P_w^*)$, such that $\top_w^{\mathcal{I}} \neq \emptyset$.

We have $\top_w^{\mathcal{I}} = \overline{\Delta^{\mathcal{I}_w}} \neq \emptyset$, by the hypothesis.

a) Suppose, next that i, j, k are such that $P_i \in P_k^*$ and $P_k \in P_j^*$. To see that $\top_i \sqcap \top_j \sqsubseteq \top_k$ holds in $\mathcal{I}$, suppose that $\overline{x} \in (\top_i \sqcap \top_j)^{\mathcal{I}} = \top_i^{\mathcal{I}} \cap \top_j^{\mathcal{I}} = \overline{\Delta^{\mathcal{I}_i}} \cap \overline{\Delta^{\mathcal{I}_j}}$. Then $\overline{x} \in \overline{\Delta^{\mathcal{I}_i}}$ and $\overline{x} \in \overline{\Delta^{\mathcal{I}_j}}$, therefore $(\overline{x}|_i, \overline{x}|_j) \in \rho_{ij} = r_{kj} \circ r_{ik}$. Hence, there exists $x' \in \Delta^{\mathcal{I}_k}$, such that $(\overline{x}|_i, x') \in r_{ik} \subseteq \rho$ and $(x', \overline{x}|_j) \in r_{kj} \subseteq \rho$, implying $\overline{x} = \overline{x'} \in \overline{\Delta^{\mathcal{I}_k}} = \top_k^{\mathcal{I}}$.

b) is discussed at the end of the proof.

c) For every role inclusion $R \sqsubseteq S$ in P_j, since both R and S must be j-roles, we obtain

$$\begin{array}{rcll} R^{\mathcal{I}} & = & \overline{R^{\mathcal{I}_j}} & \text{(by the definition of } R^{\mathcal{I}}) \\ & \subseteq & \overline{S^{\mathcal{I}_j}} & \text{(since } R \sqsubseteq S \text{ is in } P_j) \\ & = & S^{\mathcal{I}} & \text{(by the definition of } S^{\mathcal{I}}) \end{array}$$

d) If C is an i-concept name or an i-nominal name, then we do have $C \sqsubseteq \top_i$, since $C^{\mathcal{I}} = \overline{C^{\mathcal{I}_i}} \subseteq \overline{\Delta^{\mathcal{I}_i}} = \top_i^{\mathcal{I}}$.

e) If R is an i-role, then $R^{\mathcal{I}} = \overline{R^{\mathcal{I}_i}} \subseteq \overline{\Delta^{\mathcal{I}_i}} \times \overline{\Delta^{\mathcal{I}_i}} = \top_i^{\mathcal{I}} \times \top_i^{\mathcal{I}}$, whence the domain and range of $R^{\mathcal{I}}$ are both restricted to $\top_i^{\mathcal{I}}$.

f) Next, let R be an i-role name in P_j. It must be shown that $\exists R.\top_j \sqsubseteq \top_j$ and $\exists R^-.\top_j \sqsubseteq \top_j$ are both valid in $\mathcal{I}$. Only the first subsumption will be shown. The second follows using a similar argument. For any $\overline{x}$,

$$
\begin{aligned}
\overline{x} \in (\exists R.\top_j)^{\mathcal{I}} \quad &\Rightarrow \quad \exists \overline{y}, (\overline{x}, \overline{y}) \in R^{\mathcal{I}} \text{ and } \overline{y} \in \top_j^{\mathcal{I}} \\
&\Rightarrow \quad \exists \overline{y}, (\overline{x}, \overline{y}) \in \overline{R^{\mathcal{I}_i}} \text{ and } \overline{y} \in \overline{\Delta^{\mathcal{I}_j}} \\
&\Rightarrow \quad \exists x' = \overline{x}|_i, y' = \overline{y}|_i, (x', y') \in R^{\mathcal{I}_i} \text{ and } y'' = \overline{y}|_j, (y', y'') \in \rho_{ij} = r_{ij} \\
&\Rightarrow \quad \exists x'' \in \Delta^{\mathcal{I}_j} \text{ and } (x', x'') \in r_{ij} = \rho_{ij} \\
&\qquad\qquad \text{(because } r_{ij} \text{ is a total bijection from } (R^-)^{\mathcal{I}_i}(y') \text{ to} \\
&\qquad\qquad (R^-)^{\mathcal{I}_j}(y'') \text{ by Claim 1 in the proof of Lemma 2.)} \\
&\Rightarrow \quad \overline{x} = \overline{x'} = \overline{x''} \in \overline{\Delta^{\mathcal{I}_j}} = \top_j^{\mathcal{I}}
\end{aligned}
$$

g) For a transitive i-role R, we have $R^{\mathcal{I}+} = (\overline{R^{\mathcal{I}_i}})^+ = \overline{R^{\mathcal{I}_i+}} = \overline{R^{\mathcal{I}_i}} = R^{\mathcal{I}}$ (the second equality is by Claim 3 part (d)).

b): For concept inclusions, we first prove, by induction on the structure of concepts, that for any concept E appearing in P_j,

$$\#_j(E)^{\mathcal{I}} = \overline{E^{\mathcal{I}_j}}. \tag{8}$$

For the basis of the induction, let E be a concept such that $\mathsf{Sig}(E) \subseteq \mathsf{Sig}(P_i) \cap \mathsf{Sig}(P_j)$:

Claim 4: $\overline{E^{\mathcal{I}_i}} \cap \overline{\Delta^{\mathcal{I}_j}} = \overline{r_{ij}(E^{\mathcal{I}_i})} = \overline{E^{\mathcal{I}_j}}$
Proof:

$$
\begin{aligned}
\overline{E^{\mathcal{I}_i}} \cap \overline{\Delta^{\mathcal{I}_j}} &= \{\overline{x} | x \in E^{\mathcal{I}_i}\} \cap \{\overline{x'} | x' \in \Delta^{\mathcal{I}_j}\} && \text{(by definition)} \\
&= \{\overline{x'} | \exists x \in E^{\mathcal{I}_i} \wedge x' \in \Delta^{\mathcal{I}_j} \wedge \overline{x} = \overline{x'}\} \\
&= \{\overline{x'} | \exists x \in E^{\mathcal{I}_i} \wedge x' \in \Delta^{\mathcal{I}_j} \wedge (x, x') \in \rho_{ij} = r_{ij}\} \\
& && \text{(by compositional consistency)} \\
&= \{\overline{x'} | x' \in r_{ij}(E^{\mathcal{I}_i})\} && \text{(by the definition of } r_{ij}(\cdot)) \\
&= \overline{r_{ij}(E^{\mathcal{I}_i})} && \text{(by definition)} \\
&= \overline{E^{\mathcal{I}_j}} && \text{(since } r_{ij}(E^{\mathcal{I}_i}) = E^{\mathcal{I}_j}) \text{ Q.E.D.}
\end{aligned}
$$

The proof of the basis case of the induction is concluded as follows: if E is an i-concept name or an i-nominal name, then

$$
\begin{aligned}
\#_j(E)^{\mathcal{I}} &= (E \sqcap \top_j)^{\mathcal{I}} \\
&= E^{\mathcal{I}} \cap \top_j^{\mathcal{I}} \\
&= \overline{E^{\mathcal{I}_i}} \cap \overline{\Delta^{\mathcal{I}_j}} \\
&= \overline{E^{\mathcal{I}_j}}. && \text{(by Claim 4)}
\end{aligned}
$$

For the induction step, assume that for concepts C and D appearing in P_j, we have that $\#_j(C)^{\mathcal{I}} = \overline{C^{\mathcal{I}_j}}$ and $\#_j(D)^{\mathcal{I}} = \overline{D^{\mathcal{I}_j}}$.

If $E = \neg_i C$, then

$$
\begin{array}{rcll}
\#_j(E)^{\mathcal{I}} & = & \#_j(\neg_i C)^{\mathcal{I}} & \text{(since } E = \neg_i C\text{)} \\
& = & (\neg \#_j(C) \sqcap \top_i \sqcap \top_j)^{\mathcal{I}} & \text{(by the definition of } \#_j(\neg_i C)\text{)} \\
& = & (\Delta^{\mathcal{I}} \backslash \#_j(C)^{\mathcal{I}}) \cap \top_i^{\mathcal{I}} \cap \top_j^{\mathcal{I}} & \text{(by the definition of } \cdot^{\mathcal{I}}\text{)} \\
& = & (\Delta^{\mathcal{I}} \backslash (\overline{C^{\mathcal{I}_j}})) \cap \overline{\Delta^{\mathcal{I}_i}} \cap \overline{\Delta^{\mathcal{I}_j}} & \text{(by the induction hypothesis)} \\
& = & (\overline{\Delta^{\mathcal{I}_i}} \cap \overline{\Delta^{\mathcal{I}_j}}) \backslash (\overline{C^{\mathcal{I}_j}}) & \text{(since } \overline{\Delta^{\mathcal{I}_i}} \cap \overline{\Delta^{\mathcal{I}_j}} \subseteq \Delta^{\mathcal{I}}\text{)} \\
& = & \overline{(r_{ij}(\Delta^{\mathcal{I}_i}))} \backslash (\overline{C^{\mathcal{I}_j}}) & \text{(by Claim 4)} \\
& = & \overline{(r_{ij}(\Delta^{\mathcal{I}_i}) \backslash C^{\mathcal{I}_j})} & \text{(by Claim 3c)} \\
& = & \overline{(\neg_i C)^{\mathcal{I}_j}} & \text{(by the definition of } (\neg_i C)^{\mathcal{I}_j}\text{)} \\
& = & \overline{E^{\mathcal{I}_j}}. &
\end{array}
$$

If $E = C \sqcap D$, then

$$
\begin{array}{rcll}
\#_j(E)^{\mathcal{I}} & = & \#_j(C \sqcap D)^{\mathcal{I}} & \text{(since } E = C \sqcap D\text{)} \\
& = & (\#_j(C) \sqcap \#_j(D) \sqcap \top_j)^{\mathcal{I}} & \text{(by the definition of } \#_j(C \sqcap D)\text{)} \\
& = & \#_j(C)^{\mathcal{I}} \cap \#_j(D)^{\mathcal{I}} \cap \top_j^{\mathcal{I}} & \text{(by the definition of } \cdot^{\mathcal{I}}\text{)} \\
& = & \overline{C^{\mathcal{I}_j}} \cap \overline{D^{\mathcal{I}_j}} \cap \overline{\Delta^{\mathcal{I}_j}} & \text{(by the induction hypothesis)} \\
& = & \overline{C^{\mathcal{I}_j}} \cap \overline{D^{\mathcal{I}_j}} & \text{(since } \overline{C^{\mathcal{I}_j}} \cap \overline{D^{\mathcal{I}_j}} \subseteq \overline{\Delta^{\mathcal{I}_j}}\text{)} \\
& = & \{\overline{x} | x \in C^{\mathcal{I}_j}\} \cap \{\overline{x} | x \in D^{\mathcal{I}_j}\} & \text{(by the definition of } \overline{(\cdot)}\text{)} \\
& = & \{\overline{x} | x \in C^{\mathcal{I}_j} \cap D^{\mathcal{I}_j}\} & \text{(follows from Claim 3a)} \\
& = & \overline{(C \sqcap D)^{\mathcal{I}_j}} & \text{(by the definition of } \cdot^{\mathcal{I}_j}\text{)} \\
& = & \overline{E^{\mathcal{I}_j}}. &
\end{array}
$$

For $E = \leq nR.C$, where R is an i-role, we first need to show:

Claim 5: If $\overline{x} \in \overline{\Delta^{\mathcal{I}_i}} \cap \overline{\Delta^{\mathcal{I}_j}}$, then $(\overline{x}, \overline{y}) \in \overline{R^{\mathcal{I}_i}}$ iff $(\overline{x}|_j, \overline{y}|_j) \in R^{\mathcal{I}_j}$, for any y.
Proof:

$$
\begin{array}{rl}
& \overline{x} \in \overline{\Delta^{\mathcal{I}_i}} \cap \overline{\Delta^{\mathcal{I}_j}} \text{ and } (\overline{x}, \overline{y}) \in \overline{R^{\mathcal{I}_i}} \\
\Rightarrow & (\overline{x}|_i, \overline{y}|_i) \in R^{\mathcal{I}_i} \text{ and } (\overline{x}|_i, \overline{x}|_j) \in \rho_{ij} = r_{ij} \\
\Rightarrow & \exists y' \in \Delta^{\mathcal{I}_j}, (\overline{x}|_j, y') \in R^{\mathcal{I}_j}, \text{ and } (\overline{y}|_i, y') \in r_{ij} = \rho_{ij} \\
& \qquad \text{(because } r_{ij} \text{ is a total bijection from } R^{\mathcal{I}_i}(\overline{x}|_i) \text{ to} \\
& \qquad R^{\mathcal{I}_j}(\overline{x}|_j) \text{ by Claim 1 in the proof of Lemma 2.)} \\
\Rightarrow & (\overline{x}|_j, \overline{y}|_j) = (\overline{x}|_j, y') \in R^{\mathcal{I}_j}
\end{array}
$$

and conversely

$$
\begin{array}{rll}
& \overline{x} \in \overline{\Delta^{\mathcal{I}_i}} \cap \overline{\Delta^{\mathcal{I}_j}} \text{ and } (\overline{x}|_j, \overline{y}|_j) \in R^{\mathcal{I}_j} & \\
\Rightarrow & (\overline{x}|_j, \overline{y}|_j) \in r_{ij}(R^{\mathcal{I}_i}) & \text{(since } r_{ij}(R^{\mathcal{I}_i}) = R^{\mathcal{I}_j}\text{)} \\
\Rightarrow & \exists x', y' \in \Delta^{\mathcal{I}_i}, (x', \overline{x}|_j) \in r_{ij}, (y', \overline{y}|_j) \in r_{ij}, (x', y') \in R^{\mathcal{I}_i} & \\
\Rightarrow & (\overline{x}, \overline{y}) \in \overline{R^{\mathcal{I}_i}}. & \text{Q.E.D.}
\end{array}
$$

Based on Claims 3,4 and 5, we have:

$$
\begin{array}{rcl}
\#_j(E)^{\mathcal{I}} & = & \#_j(\leq nR.C)^{\mathcal{I}} \qquad\qquad\qquad\qquad (\text{since } E = \leq nR.C) \\
& = & (\leq nR.\#_j(C) \sqcap \top_i \sqcap \top_j)^{\mathcal{I}} \quad (\text{by the definition of } \#_j(\leq nR.C)) \\
& = & \{\overline{x}|\ |\{\overline{y}|(\overline{x},\overline{y}) \in R^{\mathcal{I}} \wedge \overline{y} \in \#_j(C)^{\mathcal{I}}\}| \leq n\} \cap \top_i^{\mathcal{I}} \cap \top_j^{\mathcal{I}} \\
& & \qquad\qquad\qquad\qquad\qquad\qquad (\text{by the definition of } \cdot^{\mathcal{I}}) \\
& = & \{\overline{x}|\ |\{\overline{y}|(\overline{x},\overline{y}) \in \overline{R^{\mathcal{I}_i}} \wedge \overline{y} \in \overline{C^{\mathcal{I}_j}}\}| \leq n\} \cap \overline{\Delta^{\mathcal{I}_i}} \cap \overline{\Delta^{\mathcal{I}_j}} \\
& & (\text{by the definitions of } R^{\mathcal{I}}, \top_i^{\mathcal{I}}, \top_j^{\mathcal{I}} \text{ and the induction hypothesis}) \\
& = & \{\overline{x} \in \overline{\Delta^{\mathcal{I}_i}} \cap \overline{\Delta^{\mathcal{I}_j}}|\ |\{\overline{y}|_j \in \Delta^{\mathcal{I}_j}|(\overline{x},\overline{y}) \in \overline{R^{\mathcal{I}_i}} \wedge \overline{y} \in \overline{C^{\mathcal{I}_j}}\}| \leq n\} \\
& & \qquad\qquad\qquad\qquad\qquad\qquad (\text{by Claim 3b}) \\
& = & \{\overline{x} \in \overline{\Delta^{\mathcal{I}_i}} \cap \overline{\Delta^{\mathcal{I}_j}}|\ |\{\overline{y}|_j \in \Delta^{\mathcal{I}_j}|(\overline{x}|_j,\overline{y}|_j) \in R^{\mathcal{I}_j} \wedge \overline{y}|_j \in C^{\mathcal{I}_j}\}| \leq n\} \\
& & \qquad\qquad\qquad\qquad\qquad\qquad (\text{by Claim 5}) \\
& = & \{\overline{x} \in \overline{r_{ij}(\Delta^{\mathcal{I}_i})}|\ |\{\overline{y}|_j \in \Delta^{\mathcal{I}_j}|(\overline{x}|_j,\overline{y}|_j) \in R^{\mathcal{I}_j} \wedge \overline{y}|_j \in C^{\mathcal{I}_j}\}| \leq n\} \\
& & \qquad\qquad\qquad\qquad\qquad\qquad (\text{by Claim 4}) \\
& = & \{\overline{x}|x \in r_{ij}(\Delta^{\mathcal{I}_i}),\ |\{z \in \Delta^{\mathcal{I}_j}|(x,z) \in R^{\mathcal{I}_j} \wedge z \in C^{\mathcal{I}_j}\}| \leq n\} \\
& & \qquad\qquad\qquad\qquad\qquad\qquad (\text{by Claim 3a}) \\
& = & \overline{(\leq nR.C)^{\mathcal{I}_j}} \qquad\qquad (\text{by the definition of } (\leq nR.C)^{\mathcal{I}_j}) \\
& = & \overline{E^{\mathcal{I}_j}}.
\end{array}
$$

Finally, using Equation (8), we have that

$$
\begin{array}{rcll}
\#_j(C)^{\mathcal{I}} & = & \overline{C^{\mathcal{I}_j}} & (\text{by Equation (8)}) \\
& \subseteq & \overline{D^{\mathcal{I}_j}} & (\text{since } C \sqsubseteq D \text{ is in } P_j) \\
& = & \#_j(D)^{\mathcal{I}}. & (\text{by Equation (8)})
\end{array}
$$

Lemma 4 Q.E.D.

Using Theorem 1 and the fact that concept satisfiability in $\mathcal{SHOIQ}$ is NExpTime-complete (Tobies, 2000, 2001), we obtain

Theorem 2 *The concept satisfiability, concept subsumption and consistency problems in $\mathcal{SHOIQP}$ are* NExpTime*-complete.*

The next theorem shows that concept subsumption problems in $\mathcal{SHOIQP}$ can be reduced to concept subsumption problems in $\mathcal{SHOIQ}$.

Theorem 3 (Reasoning Exactness) *For a $\mathcal{SHOIQP}$ KB $\Sigma = \{P_i\}$, $C \sqsubseteq_j D$ iff $\Re(P_j^*) \models \#_j(C) \sqsubseteq \#_j(D)$.*

Proof: As usual, we reduce subsumption to (un)satisfiability. It follows directly from Theorem 1 that P_j^* and $C \sqcap \neg_j D$ have a common model if and only if $\Re(P_j^*)$ and $\#_j(C) \sqcap \neg\#_j(D) \sqcap \top_j$ have a common model. Since $\#_j(C) \sqsubseteq \top_j$, this holds if and only if $\Re(P_j^*)$ and $\#_j(C) \sqcap \neg\#_j(D)$ have a common model. Thus, $\Re(P_j^*) \models \#_j(C) \sqsubseteq \#_j(D)$.

Q.E.D.

Discussion of Desiderata. To show that the package description logic $\mathcal{SHOIQP}$ supports transitive reusability and preservation of unsatisfiability, we prove the monotonicity of reasoning in $\mathcal{SHOIQP}$.

Theorem 4 (Monotonicity and Transitive Reusability) *Suppose* $\Sigma = \{P_i\}$ *is a* $\mathcal{SHOIQP}$ *KB,* $P_i \in P_j^+$ *and* C, D *are concepts, such that* $\mathsf{Sig}(C) \cup \mathsf{Sig}(D) \subseteq \mathsf{Sig}(P_i) \cap \mathsf{Sig}(P_j)$. *If* $C \sqsubseteq_i D$, *then* $C \sqsubseteq_j D$.

Proof: Suppose that $C \sqsubseteq_i D$. Thus, for every model $\mathcal{I}$ of P_i^*, $C^{\mathcal{I}_i} \subseteq D^{\mathcal{I}_i}$. Now consider a model $\mathcal{J}$ of P_j^*. Since $P_i \in P_j^*$, $\mathcal{J}$ is also an interpretation of P_i^*. If $\bigcup_{P_k \in P_i^*} \Delta^{\mathcal{J}_k} = \emptyset$, then the conclusion holds trivially. Otherwise, $\mathcal{J}$ is a model of P_i^* and, therefore, $C^{\mathcal{J}_i} \subseteq D^{\mathcal{J}_i}$. Hence, $r_{ij}(C^{\mathcal{J}_i}) \subseteq r_{ij}(D^{\mathcal{J}_i})$, whence, by Lemma 2, $C^{\mathcal{J}_j} \subseteq D^{\mathcal{J}_j}$. This proves that $C \sqsubseteq_j D$. Q.E.D.

Theorem 4 ensures that when some part of an ontology module is reused, the restrictions asserted by it, e.g., domain restrictions on roles, will not be relaxed in a way that prohibits the reuse of imported knowledge. Theorem 4 also ensures that consequences of imported knowledge can be transitively propagated across importing chains.

In the special case where $D = \bot$, we obtain the following corollary:

Corollary 1 (Preservation of Unsatisfiability) *For a* $\mathcal{SHOIQP}$ *knowledge base* $\Sigma = \{P_i\}$ *and* $P_i \in P_j^+$, *if* $C \sqsubseteq_i \bot$ *then* $C \sqsubseteq_j \bot$.

Finally, the semantics of $\mathcal{SHOIQP}$ ensures that the interpretation of an axiom in an ontology module is constrained by its *context*, as seen from the reduction to a corresponding integrated ontology: $C \sqsubseteq D$ in P_j is mapped to $\#_j(C) \sqsubseteq \#_j(D)$, where $\#_j(C)$ and $\#_j(D)$ are now relativized to the corresponding local domain of P_j.

When a package P_i is directly or indirectly reused by another package P_j, some axioms in P_i may be effectively "propagated" to module P_j (i.e., may influence inference from the point of view of P_j). P-DL semantics ensures that such axiom propagation will affect only the "overlapping" domain $r_{ij}(\Delta^{\mathcal{I}_i}) \cap \Delta^{\mathcal{I}_j}$ and not the entire domain $\Delta^{\mathcal{I}_j}$.

Example 7 *For instance, in Figure 1, package* P_1 *contains an axiom* $\neg_1\mathsf{Child} \sqsubseteq \mathsf{Adult}$ *and package* P_2 *imports* P_1. *The assertion* $\neg_1\mathsf{Child} \sqsubseteq \mathsf{Adult}$ *is made within the implicit context of people, i.e. every individual that is not a child is an adult. Thus, every individual within the domain of people is either a* Child *or an* Adult ($\top_1 \sqsubseteq \mathsf{Child} \sqcup \mathsf{Adult}$). *However, it is not necessarily the case in* P_2 *that* $\top_2 \sqsubseteq \mathsf{Child} \sqcup \mathsf{Adult}$. *For example, an* Employer *in the domain of Work may be an organization which is not a member of the domain of People. In fact, since* $r_{12}(\Delta^{\mathcal{I}_1}) \subseteq \Delta^{\mathcal{I}_2}$, $\Delta^{\mathcal{I}_1} \backslash \mathsf{Child}^{\mathcal{I}_1} \subseteq \mathsf{Adult}^{\mathcal{I}_1}$, *i.e.,* $\Delta^{\mathcal{I}_1} = \mathsf{Child}^{\mathcal{I}_1} \cup \mathsf{Adult}^{\mathcal{I}_1}$, *does not necessarily imply* $\Delta^{\mathcal{I}_2} = \mathsf{Child}^{\mathcal{I}_2} \cup \mathsf{Adult}^{\mathcal{I}_2}$.

Hence, the effect of an axiom is always limited to its originally designated context. Consequently, it is not necessary to explicitly restrict the use of the ontology language to ensure locality of axioms, as is required, for instance, by conservative extensions (Grau et al., 2007). Instead, the locality of axioms follows directly from the semantics of $\mathcal{SHOIQP}$.

P-DL with Unrestricted Role Inclusion

The P-DL Family $\mathcal{ALCHIO}(\neg)\mathcal{P}$

We now proceed to show that the two P-DLs $\mathcal{ALCHIO}(\neg)_{\mathcal{CR}}\mathcal{P}$ and $\mathcal{ALCHIO}(\neg)_{\mathcal{R}}\mathcal{P}$, that together constitute the family $\mathcal{ALCHIO}(\neg)\mathcal{P}$, obtained by extending the P-DL $\mathcal{ALCP}$ to allow role importing, general role inclusions (and hence role mappings between ontologies), inverse roles, nominals, nominal importing, and negation on roles, are decidable. The syntax of both P-DLs in $\mathcal{ALCHIO}(\neg)\mathcal{P}$ can be obtained from $\mathcal{ALCHIOP}$ with (contextualized) negations on roles. Thus, roles of a package P_j in both P-DLs in $\mathcal{ALCHIO}(\neg)\mathcal{P}$ are defined inductively by the following grammar:

$$R := P|R^-|\neg_k R$$

where P is a local or imported role name, and P_j imports P_k. A role of the form $\neg_k R$ is called a *k-negated role*. The semantics of role negation is given by $(\neg_k R)^{\mathcal{I}_j} = (r_{kj}(\Delta^{\mathcal{I}_k}) \times r_{kj}(\Delta^{\mathcal{I}_k}))\backslash R^{\mathcal{I}_j}$.

Depending on whether negated roles can be used or not in concept inclusions, the two members of the family $\mathcal{ALCHIO}(\neg)\mathcal{P}$ are given by:

- $\mathcal{ALCHIO}(\neg)_{\mathcal{CR}}\mathcal{P}$: negated roles can be used in both concept and role inclusions. If an i-role name P is imported by P_j, we require that the cardinality preservation condition holds for both P and $\neg_i P$.
- $\mathcal{ALCHIO}(\neg)_{\mathcal{R}}\mathcal{P}$: negated roles can only be used in role inclusions. In this variant, we only require cardinality preservation for imported role names but not their negations.

Consideration of these two P-DLs and the respective conditions imposed in each case are motivated by the desire to achieve transitive reusability of knowledge using a *minimal* set of restrictions on domain relations between local models.

The decidability proofs of the P-DLs in $\mathcal{ALCHIO}(\neg)\mathcal{P}$ use a reduction to the decidable DL $\mathcal{ALBO}$ (Schmidt & Tishkovsky, 2007). The logic $\mathcal{ALBO}$ extends $\mathcal{ALC}$ with boolean role operators, role inclusions, inverses of roles, domain and range restriction operators and nominals.

In $\mathcal{ALBO}$, roles are defined inductively by the following grammar:

$$R := P|R \sqcap R|\neg R|R^-|(R \downharpoonright C)|(R \upharpoonright C)$$

where P is a role name and C is a concept. The semantics of $\mathcal{ALBO}$ is defined as an extension of that of $\mathcal{ALCHIO}$ with the following additional constraints on interpretations (where $\Delta^{\mathcal{I}}$ is the interpretation domain):

$$\begin{aligned}
(\neg R)^{\mathcal{I}} &= (\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}})\backslash R^{\mathcal{I}} \\
(R \sqcap S)^{\mathcal{I}} &= R^{\mathcal{I}} \cap S^{\mathcal{I}} \\
(R \downharpoonright C)^{\mathcal{I}} &= R^{\mathcal{I}} \cap (\Delta^{\mathcal{I}} \times C^{\mathcal{I}}) \\
(R \upharpoonright C)^{\mathcal{I}} &= R^{\mathcal{I}} \cap (C^{\mathcal{I}} \times \Delta^{\mathcal{I}})
\end{aligned}$$

We use the abbreviation $R \updownarrow C = (R \downharpoonright C) \upharpoonright C$.

Decidability of P-DL $\mathcal{ALCHIO}(\neg)_{\mathcal{CR}}\mathcal{P}$

A reduction $\Re$ from an $\mathcal{ALCHIO}(\neg)_{\mathcal{CR}}\mathcal{P}$ KB $\Sigma_d = \{P_i\}$ to an $\mathcal{ALBO}$ KB Σ can be established based on the reduction of P-DL $\mathcal{SHOIQP}$ to $\mathcal{SHOIQ}$, as presented in the section titled "Reduction to Ordinary DL", with a couple of modifications to handle role inclusions: $\#_j()$ is also applied to roles and that a negated local domain and a negated local range axiom for roles are added to the $\mathcal{ALBO}$ KB Σ, as detailed below.

- The signature of Σ is the union of the local signatures of the component packages together with a global top $\top$, a global bottom $\bot$ and local top concepts $\top_i$, for all i, i.e., $\mathsf{Sig}(\Sigma) = \bigcup_i \mathsf{Loc}(P_i) \cup \{\top_i\} \cup \{\top, \bot\}$.
- For all i, j, k such that $P_i \in P_k^*$, $P_k \in P_j^*$, $\top_i \sqcap \top_j \sqsubseteq \top_k$ is added to Σ.
- For each GCI or role inclusion $X \sqsubseteq Y$ in P_j, $\#_j(X) \sqsubseteq \#_j(Y)$ is added to Σ. The mapping $\#_j()$ is defined below.
- For each i-concept name or i-nominal name C in P_i, $i : C \sqsubseteq \top_i$ is added to Σ.
- For each i-role name R in P_i, its domain and range is $\top_i$, i.e., $\top \sqsubseteq \forall R^-.\top_i$ and $\top \sqsubseteq \forall R.\top_i$ are added to Σ.
- For each i-role name R in P_j, the following axioms are added to Σ:

– $\exists R.\top_j \sqsubseteq \top_j$; (local domain)
– $\exists R^-.\top_j \sqsubseteq \top_j$; (local range)
– $\exists((\neg R) \updownarrow \top_i).\top_j \sqsubseteq \top_j$; (negated local domain)
– $\exists((\neg R) \updownarrow \top_i)^-.\top_j \sqsubseteq \top_j$; (negated local range)

For a formula X used in P_j, $\#_j(X)$ is:

- X, for a j-(concept, role or nominal) name.
- $X \sqcap \top_j$, for an i-concept name or an i-nominal name X.
- $X \updownarrow \top_j$, for an i-role name.
- $\#_j(Y)^-$, for a role $X = Y^-$.
- $\neg\#_j(X) \sqcap \top_i \sqcap \top_j$, for $\neg_i X$, where X is a concept.
- $\neg\#_j(Y) \updownarrow (\top_i \sqcap \top_j)$, for a role $X = \neg_i Y$.
- $(\#_j(X_1) \oplus \#_j(X_2)) \sqcap \top_j$, for a concept $X = X_1 \oplus X_2$, where $\oplus = \sqcap$ or $\oplus = \sqcup$.
- $(\oplus\#_j(R).\#_j(X')) \sqcap \top_i \sqcap \top_j$, for a concept $X = (\oplus R.X')$, where $\oplus \in \{\exists, \forall\}$ and R is an i-role or an i-negated role.

The following lemma shows that the consistency problem in $\mathcal{ALCHIO}(\neg)\mathcal{P}$ can be reduced to the concept satisfiability problem in $\mathcal{ALBO}$:

Lemma 5 *An $\mathcal{ALCHIO}(\neg)_{\mathcal{CR}}\mathcal{P}$ KB Σ_d is consistent as witnessed by a package P_w if and only if $\top_w$ is satisfiable with respect to $\Re(P_w^*)$.*

Proof sketch: The proof is similar to the proof of Theorem 1. The main modification concerns the reduction of role inclusion axioms. The basic idea is that, given a distributed model of Σ_d, we can construct an ordinary model of $\Re(P_w^*)$ by "merging" individuals connected by domain relations. Given a model of $\Re(P_w^*)$, we can construct a distributed model of Σ_d by "copying shared individuals" into local interpretation domains.

For the "if" direction, if $\top_w$ is satisfiable with respect to $\Re(P_w^*)$, then $\Re(P_w^*)$ has at least one model $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$, such that $\top_w^{\mathcal{I}} \neq \emptyset$. Our goal is to construct a model of P_w^* from $\mathcal{I}$, such that $\Delta^{\mathcal{I}_w} \neq \emptyset$. For each package P_i, a local interpretation $\mathcal{I}_i$ is constructed in the following way:

- $\Delta^{\mathcal{I}_i} = \top_i^{\mathcal{I}}$.
- For every concept name C in P_i, $C^{\mathcal{I}_i} = C^{\mathcal{I}} \cap \top_i^{\mathcal{I}}$.
- For every role name R in P_i, $R^{\mathcal{I}_i} = R^{\mathcal{I}} \cap (\top_i^{\mathcal{I}} \times \top_i^{\mathcal{I}})$.
- For every nominal name o that appears in P_i, $o^{\mathcal{I}_i} = o^{\mathcal{I}}$.

For every pair i, j, such that $P_i \in P_j^*$, we define

$$r_{ij} = \{(x, x) | x \in \Delta^{\mathcal{I}_i} \cap \Delta^{\mathcal{I}_j}\}.$$

Clearly, we have $\Delta^{\mathcal{I}_w} = \top_w^{\mathcal{I}} \neq \emptyset$. So it suffices to show that $\langle\{\mathcal{I}_i\}, \{r_{ij}\}_{P_i \in P_j^*}\rangle$ is a model of P_w^*. The proof is similar to that of Lemma 3. We will only show that if $\#_j(X) \sqsubseteq \#_j(Y)$ is satisfied by $\mathcal{I}$, then $X \sqsubseteq Y$ is satisfied by $\mathcal{I}_j$. To accomplish this, it suffices to show that for any role X in the signature of P_j, $\#_j(X)^{\mathcal{I}} = X^{\mathcal{I}_j}$:

- If X is a j-role name, $\#_j(X)^{\mathcal{I}} = X^{\mathcal{I}_j}$ by definition.
- If X is a i-role name, $i \neq j$, $\#_j(X)^{\mathcal{I}} = (X \updownarrow \top_j)^{\mathcal{I}} = X^{\mathcal{I}} \cap (\Delta^{\mathcal{I}_j} \times \Delta^{\mathcal{I}_j}) = X^{\mathcal{I}_j}$.
- If $X = Y^-$ and $\#_j(Y)^{\mathcal{I}} = Y^{\mathcal{I}_j}$, then $\#_j(X)^{\mathcal{I}} = (\#_j(Y)^-)^{\mathcal{I}} = (\#_j(Y)^{\mathcal{I}})^- = (Y^{\mathcal{I}_j})^- = (Y^-)^{\mathcal{I}_j} = X^{\mathcal{I}_j}$.
- If $X = \neg_i Y$ and $\#_j(Y)^{\mathcal{I}} = Y^{\mathcal{I}_j}$, then $\#_j(X)^{\mathcal{I}} = (\neg\#_j(Y) \updownarrow (\top_i \sqcap \top_j))^{\mathcal{I}} = ((\Delta^{\mathcal{I}_i} \cap \Delta^{\mathcal{I}_j}) \times (\Delta^{\mathcal{I}_i} \cap \Delta^{\mathcal{I}_j}))\backslash Y^{\mathcal{I}_j} = (\neg_i Y)^{\mathcal{I}_j} = X^{\mathcal{I}_j}$.

For the "only if" direction, suppose that Σ_d is consistent as witnessed by P_w. Thus, Σ_d has a distributed model $\langle\{\mathcal{I}_i\}, \{r_{ij}\}_{P_i \in P_j^*}\rangle$, such that $\Delta^{\mathcal{I}_w} \neq \emptyset$. We construct a model $\mathcal{I}$ of $\Re(P_w^*)$ by merging individuals that are related via chains of image domain relations or their inverses. More precisely, for every element x in the distributed model, we define, as before, its equivalence class $\overline{x} = \{y | (x, y) \in \rho\}$, where ρ is the symmetric and transitive closure of the set $\bigcup_{P_i \in P_j^*} r_{ij}$. For a set S, we define $\overline{S} = \{\bar{x} | x \in S\}$ and, for a binary relation R, we define $\overline{R} = \{(\overline{x}, \overline{y}) | (x, y) \in R\}$.

Now, let $\mathcal{I} = \langle\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}\rangle$ be defined as follows:

- $\Delta^{\mathcal{I}} = \overline{\bigcup_i \Delta^{\mathcal{I}_i}}$.
- For every i-name X, $X^{\mathcal{I}} := \overline{X^{\mathcal{I}_i}}$.
- For every i, $\top_i^{\mathcal{I}} = \overline{\Delta^{\mathcal{I}_i}}$.

We denote by $\overline{x}|_i$ the element (if it exists) in $\Delta^{\mathcal{I}_i}$ that belongs to $\overline{x}$, i.e., $\overline{x}|_i \in \Delta^{\mathcal{I}_i} \cap \overline{x}$.

The proof that $\mathcal{I}$ is a model of $\Re(P_w^*)$, with $\top_w^{\mathcal{I}} \neq \emptyset$, is similar to that of Lemma 4. We only show that for every role inclusion $X \sqsubseteq Y \in P_j$, we have that $\mathcal{I}$ satisfies $\#_j(X) \sqsubseteq \#_j(Y)$. We prove this by showing that, for any role R the appears in P_j, $\overline{R^{\mathcal{I}_j}} = \#_j(R)^{\mathcal{I}}$, again using induction on the structure of R. We only show the case for negated roles. The other cases (local roles, imported roles and inverse roles) can be handled similarly. When $R = \neg_i S$ and $\overline{S^{\mathcal{I}_j}} = \#_j(S)^{\mathcal{I}}$, we have that $\overline{R^{\mathcal{I}_j}} = \overline{(\neg_i S)^{\mathcal{I}_j}} = \overline{(r_{ij}(\Delta^{\mathcal{I}_i}) \times r_{ij}(\Delta^{\mathcal{I}_i}))\backslash S^{\mathcal{I}_j}} = ((\top_i \sqcap \top_j)^{\mathcal{I}} \times (\top_i \sqcap \top_j)^{\mathcal{I}})\backslash\overline{S^{\mathcal{I}_j}} = (\neg\#_j(S) \updownarrow (\top_i \sqcap \top_j))^{\mathcal{I}} = \#_j(R)^{\mathcal{I}}$. Q.E.D.

Decidability of P-DL $\mathcal{ALCHIO}(\neg)_{\mathcal{R}}\mathcal{P}$

The decidability proof of $\mathcal{ALCHIO}(\neg)_{\mathcal{R}}\mathcal{P}$ also uses a reduction to $\mathcal{ALBO}$ and is very similar to that of $\mathcal{ALCHIO}(\neg)_{\mathcal{CR}}\mathcal{P}$. Since negated roles appear only in role inclusions and cardinality preservation is not required for negated roles, in the reduction from an $\mathcal{ALCHIO}(\neg)_{\mathcal{R}}\mathcal{P}$ ontology to an $\mathcal{ALBO}$ ontology, the negated local domain and the negated local range axioms are not needed. Note that, in $\mathcal{ALCHIO}(\neg)_{\mathcal{CR}}\mathcal{P}$, if P_j imports a role from P_i, then, due to cardinality preservation on both role names and negated roles, r_{ij} has to be either empty or a total function. In $\mathcal{ALCHIO}(\neg)_{\mathcal{R}}\mathcal{P}$, on the other hand, there is no such requirement. This allows some increased flexibility in role mappings while, at the same time, maintaining the autonomy of ontology modules.

From the above reductions from $\mathcal{ALCHIO}(\neg)_{\mathcal{CR}}\mathcal{P}$ and $\mathcal{ALCHIO}(\neg)_{\mathcal{R}}\mathcal{P}$ to $\mathcal{ALBO}$ and the fact that the complexity of $\mathcal{ALBO}$ is NExpTime-complete (Schmidt & Tishkovsky, 2007) we obtain the following complexity result.

Theorem 5 *The consistency problem and concept satisfiability problem in $\mathcal{ALCHIO}(\neg)_{\mathcal{CR}}\mathcal{P}$ and $\mathcal{ALCHIO}(\neg)_{\mathcal{R}}\mathcal{P}$ are in* NEXPTIME.

Discussion of the P-DL Semantics

Necessity of P-DL Constraints on Domain Relations

The constraints on domain relations in the semantics of $\mathcal{SHOIQP}$, as given in Definition 1, are minimal in the sense that if we drop any of them, we can no longer satisfy the desiderata summarized in the section titled "Semantics".

Dropping Condition 1 of Definition 1 (one-to-one domain relations) leads to difficulties in preservation of concept unsatisfiability. For example, if the domain relations are not injective, then $C_1 \sqsubseteq_i \neg_i C_2$, i.e., $C_1 \sqcap C_2 \sqsubseteq_i \bot$, does not ensure $C_1 \sqcap C_2 \sqsubseteq_j \bot$ when P_j imports P_i. If the domain relations are not partial functions, multiple individuals in $\Delta^{\mathcal{I}_j}$ may be images of the same individual in $\Delta^{\mathcal{I}_i}$ via r_{ij}, whence unsatisfiability of a complex concept can no longer be preserved when both number restriction and role importing are allowed. Thus, if R is an i-role name and C is an i-concept name, $\geq 2R.C \sqsubseteq_i \bot$ does not imply $\geq 2R.C \sqsubseteq_j \bot$.

Dropping Condition 2 of Definition 1 (compositional consistency of domain relations) would result in violation of the transitive reusability requirement, in particular, and of the monotonicity of inference based on imported knowledge, in general. In the absence of compositional consistency of domain relations, the importing relations would be like bridge rules in DDL, in that they are localized w.r.t. the connected pairs of modules without supporting compositionality (Zimmermann & Euzenat, 2006).

In the absence of Conditions 3 and 4 of Definition 1, the reuse of concept and role names would be purely syntactical, i.e., the local interpretations of imported concepts and role names would be unconstrained by their interpretations in their home package.

Condition 5 (cardinality preservation of role instances) is needed to ensure the consistency of local interpretations of complex concepts that use number restrictions.

Condition 6 is needed to ensure that concepts that are nominals can only have one instance. Multiple "copies" of such an instance are effectively identified with a single instance via domain relations.

Finally, Condition 7, i.e., that $\mathcal{I}_i \vDash P_i$, for every i, is self-explanatory.

Contextualized Negation

Contextualized negation has been studied in logic programming (Polleres, 2006; Polleres et al., 2006). Existing modular ontology languages DDL and $\mathcal{E}$-Connections do not explicitly support contextualized negation in their respective syntax. However, in those formalisms, a negation is always interpreted with respect to the local domain of the module in which the negation occurs, not the union of all local domains. Thus, in fact, both DDL and $\mathcal{E}$-Connections implicitly support contextualized negation.

The P-DL syntax and semantics, proposed in this work, support a more general use of contextualized negation so that a package can use, besides its own negation, the negations of its imported packages.

Directionality of Importing

There appears to be some apparent confusion in the literature regarding whether the constraints imposed by P-DL allow the importing relations in P-DL to be indeed directional (Grau & Kutz, 2007). As noted in (Grau & Kutz, 2007), if it is indeed the case that a P-DL model $\mathcal{I}$ satisfies $r_{ij}(s^{\mathcal{I}_i}) = s^{\mathcal{I}_j}$ if only if it satisfies $r_{ji}(s^{\mathcal{I}_j}) = s^{\mathcal{I}_i}$, for any symbol s such that $P_i \xrightarrow{s} P_j$ (Definition 18 and Proposition 19 in (Grau & Kutz, 2007)) it must follow that a P-DL ontology can be reduced to an equivalent imports-free ontology. Then, a shared symbol s of P_i and P_j must have the same interpretation from the point of view of both P_i and P_j, i.e., $s^{\mathcal{I}_i} = s^{\mathcal{I}_j}$. However, according to our definition of model (Definition 1), it is *not* the case that a P-DL model $\mathcal{I}$ satisfies $r_{ij}(s^{\mathcal{I}_i}) = s^{\mathcal{I}_j}$ if and only if it satisfies $r_{ji}(s^{\mathcal{I}_j}) = s^{\mathcal{I}_i}$, for any symbol s such that $P_i \xrightarrow{s} P_j$. As noted by Bao et al. (Bao et al., 2006b,c):

- P-DL semantics does not require the existence of both r_{ij} and r_{ji}. Their joint existence is only required when P_i and P_j mutually import one another. Hence, even if $r_{ij}(s^{\mathcal{I}_i}) = s^{\mathcal{I}_j}$, it is possible that the corresponding r_{ji} may not exist in which case $r_{ji}(s^{\mathcal{I}_j})$ is undefined.
- Domain relations are *not necessarily* total functions. Hence, it need not be the case that *every* individual of $\Delta^{\mathcal{I}_i}$ is mapped (by the one-to-one domain relation r_{ij}) to an individual of $\Delta^{\mathcal{I}_j}$.
- Satisfiability and consistency have only contextualized meaning in P-DL. If P_j is not in P_i^*, then models of P_i^* need not be models of P_j^*. This is made clear in Definition 2, where satisfiability and consistency are always considered from the point of view of a witness package.

In the following subsection, we will present an additional example (Example 8) that illustrates the directionality of importing in P-DL.

P-DL Consistency and TBox Consistency

In the section titled "Reduction to Ordinary DL", we have shown how to reduce a $\mathcal{SHOIQP}$ P-DL ontology to a corresponding DL ($\mathcal{SHOIQ}$) ontology. We have further shown (Theorem 1) that determining the consistency of a $\mathcal{SHOIQP}$ ontology from the point of view of a package P_w can be reduced to the satisfiability of a $\mathcal{SHOIQ}$ concept with respect to a $\mathcal{SHOIQ}$ ontology obtained by integrating the packages imported by P_w. However, it is important to note that this reduction of $\mathcal{SHOIQP}$ is *different* from a reduction based on S-compatibility as defined in (Grau & Kutz, 2007).

Definition 3 (Expansion) *Let A-interpretation denote an interpretation over a signature A. An S-interpretation* $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$ *is an expansion of an* S'*-interpretation* $\mathcal{J}' = (\Delta^{\mathcal{J}'}, \cdot^{\mathcal{J}'})$ *if*

(1) $S' \subseteq S$,

(2) $\Delta^{\mathcal{J}'} \subseteq \Delta^{\mathcal{J}}$, *and*
(3) $s^{\mathcal{J}} = s^{\mathcal{J}'}$, *for every* $s \in S'$.

Definition 4 (S-compatibility) *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be TBoxes expressed in a description logic $\mathcal{L}$, and let S be the shared part of their signatures. We say that $\mathcal{T}_1$ and $\mathcal{T}_2$ are S-compatible if there exists an S-interpretation $\mathcal{J}$, that can be expanded to a model $\mathcal{J}_1$ of $\mathcal{T}_1$ and to a model $\mathcal{J}_2$ of $\mathcal{T}_2$.*

As the following example illustrates, a P-DL ontology is not always reducible to the imports-free ontology that is obtained by simply taking the union of the modules (packages).

Example 8 *Let $\mathcal{T}_1 = \{D \sqcup \neg D \sqsubseteq C\}$, $\mathcal{T}_2 = \{C \sqsubseteq \bot\}$. The shared signature $S = \{C\}$ and $\mathcal{T}_1$ and $\mathcal{T}_2$ are not S-compatible. However, suppose we have a* P-DL *ontology such that $\mathcal{T}_1 \xrightarrow{C} \mathcal{T}_2$ and negation in $\mathcal{T}_1$ becomes contextualized negation $\neg_1$. Then we have a model:*

$$\begin{aligned} \Delta_1 &= C^{\mathcal{I}_1} = D^{\mathcal{I}_1} = \{x\} \\ \Delta_2 &= \{y\}, C^{\mathcal{I}_2} = \emptyset \\ r_{12} &= r_{21} = \emptyset \end{aligned}$$

On the other hand, all models of a P-DL *ontology where $\mathcal{T}_2 \xrightarrow{C} \mathcal{T}_1$ have empty Δ_1. Thus, the whole ontology is consistent as witnessed by $\mathcal{T}_2$ but inconsistent as witnessed by $\mathcal{T}_1$. This example demonstrates that* P-DL *importing is directional.*

The next example shows that, in the presence of nominals, the P-DL consistency problem is not reducible to the consistency of an imports-free ontology obtained by simply combining the P-DL modules.

Example 9 (Use of Nominals) *Consider the following TBoxes:*

$$\begin{aligned} \mathcal{T}_1 &= \{\top \sqsubseteq i \sqcup j, \quad i \sqcap j \sqsubseteq \bot\} \\ \mathcal{T}_2 &= \{\top \sqsubseteq i\}, \end{aligned}$$

with the shared signature $S = \{i\}$, where i, j are nominals. $\mathcal{T}_1$ and $\mathcal{T}_2$ are S-compatible but $\mathcal{T}_1 \cup \mathcal{T}_2$ is not consistent. Suppose we have a P-DL *ontology with $\mathcal{T}_1 \xrightarrow{i} \mathcal{T}_2$. Since "$\top$" only has contextualized meaning in* P-DL*, these TBoxes in fact should be represented as*

$$\begin{aligned} \mathcal{T}_1 &= \{\top_1 \sqsubseteq i \sqcup j, \quad i \sqcap j \sqsubseteq \bot\} \\ \mathcal{T}_2 &= \{\top_2 \sqsubseteq i\} \end{aligned}$$

Now, there exists a model for this P-DL *ontology:*

$$\begin{aligned} \Delta_1 &= \{x, y\}, i^{\mathcal{I}_1} = \{x\}, j^{\mathcal{I}_1} = \{y\} \\ \Delta_2 &= \{x'\}, i^{\mathcal{I}_2} = \{x'\} \\ r_{12} &= \{(x, x')\} \end{aligned}$$

In general, the reduction from P-DL modules to imports-free TBoxes with shared signatures based on S-compatibility, as suggested by (Grau & Kutz, 2007), does not preserve the semantics of P-DL. Thus, there is a fundamental difference between the two settings: P-DL has no universal top concept and, as a result, P-DL axioms have only localized effect. In the case of imports-free TBoxes, in the absence of contextualized semantics, it is not possible to ensure that the effects of axioms are localized. Consequently, it is not possible to reduce reasoning with a P-DL ontology with modules $\{\mathcal{T}_i\}$ to standard DL reasoning over the union of all ontology modules $\mathcal{T} = \mathcal{T}_1 \cup ... \cup \mathcal{T}_n$.

In contrast, in the previous section we have shown that such a reduction from reasoning in P-DL from the point of view of a witness package to reasoning with a suitably constructed DL (as shown in the section "Reduction to Ordinary DL") is possible. Nevertheless, relying on such a reduction is not attractive in practice, because it requires the integration of the ontology modules, which may be prohibitively expensive. More importantly, in many scenarios encountered in practice, e.g., in peer-to-peer applications, centralized reasoning with an integrated ontology is simply infeasible. Hence, work in progress is aimed at developing federated reasoners for P-DL that do not require the integration of different ontology modules (see, e.g., (Bao et al., 2006d)).

Summary

In this paper, we have introduced a modular ontology language, package-based description logic $\mathcal{SHOIQP}$, that allows reuse of knowledge from multiple ontologies. A $\mathcal{SHOIQP}$ ontology consists of multiple ontology modules each of which can be viewed as a $\mathcal{SHOIQ}$ ontology. Concept, role and nominal names can be shared by "importing" relations among modules.

The proposed language supports contextualized interpretation, i.e., interpretation from the point of view of a specific package. We have established a minimal set of constraints on domain relations, i.e., the relations between individuals in different local domains, that allow the preservation of the satisfiability of concept expressions, the monotonicity of inference, and the transitive reuse of knowledge.

Ongoing work is aimed at developing a distributed reasoning algorithm for $\mathcal{SHOIQP}$ by extending the results of (Bao et al., 2006d) and (Pan et al., 2006), as well as an OWL extension capturing the syntax of $\mathcal{SHOIQP}$. We are also exploring several variants of P-DL, based on a more in-depth analysis of the properties of the domain relations and the preservation of satisfiability of concept subsumptions across modules.

References

Bao, J., Caragea, D., & Honavar, V. (2006a). A distributed tableau algorithm for package-based description logics. In *the 2nd International Workshop on Context Representation and Reasoning (CRR 2006), co-located with ECAI 2006.*

Bao, J., Caragea, D., & Honavar, V. (2006b). Modular ontologies - a formal investigation of semantics and expressivity. In *Asian Semantic Web Conference (ASWC)* (p. 616-631).

Bao, J., Caragea, D., & Honavar, V. (2006c). On the semantics of linking and importing in modular ontologies. In *International Semantic Web Conference (ISWC)* (p. 72-86).

Bao, J., Caragea, D., & Honavar, V. (2006d). A tableau-based federated reasoning algorithm for modular ontologies. In *IEEE/WIC/ACM International Conference on Web Intelligence* (p. 404-410). IEEE Press.

Bao, J., Slutzki, G., & Honavar, V. (2007). A semantic importing approach to knowledge reuse from multiple ontologies. In *AAAI* (p. 1304-1309).

Bao, J., Voutsadakis, G., Slutzki, G., & Honavar, V. (2008a). On the decidability of role mappings between modular ontologies. In *AAAI* (p. 400-405).

Bao, J., Voutsadakis, G., Slutzki, G., & Honavar, V. (2008b). Ontology modularization. In C. Parent, S. Spaccapietra, & H. Stuckenschmidt (Eds.), (chap. Package-based Description Logics). Berlin: Springer (In Press).

Berners-Lee, T., Hendler, J., & Lassila, O. (2001, May). The semantic web. *Scientific American*, *284*(5), 34–43. Available from `http://www.scientificamerican.com/2001/0501issue/0501berners-lee.html`

Borgida, A., & Serafini, L. (2003). Distributed description logics: Assimilating information from peer sources. *Journal of Data Semantics*, *1*, 153-184.

Bouquet, P., Giunchiglia, F., Harmelen, F. van, Serafini, L., & Stuckenschmidt, H. (2003). C-OWL: Contextualizing ontologies. In D. Fensel, K. P. Sycara, & J. Mylopoulos (Eds.), *International Semantic Web Conference* (Vol. 2870, p. 164-179). Springer.

Catarci, T., & Lenzerini, M. (1993). Representing and using interschema knowledge in cooperative information systems. In *CoopIS* (p. 55-62).

Ghidini, C., & Giunchiglia, F. (2001). Local models semantics, or contextual reasoning=locality+compatibility. *Artificial Intelligence*, *127*(2), 221-259.

Ghilardi, S., Lutz, C., & Wolter, F. (2006). Did i damage my ontology? a case for conservative extensions in description logics. In *KR* (p. 187-197).

Grau, B. C., Horrocks, I., Kazakov, Y., & Sattler, U. (2007). A logical framework for modularity of ontologies. In *IJCAI* (p. 298-303).

Grau, B. C., Horrocks, I., Kutz, O., & Sattler, U. (2006). Will my ontologies fit together? In *Proc. of the 2006 Description Logic Workshop (DL 2006)* (Vol. 189). CEUR (`http://ceur-ws.org/`). Available from `download/2006/CHKS.pdf`

Grau, B. C., & Kutz, O. (2007). Modular ontology languages revisited. In *Workshop on Semantic Web for Collaborative Knowledge Acquisition (SWECKA), co-located with IJCAI.*

Grau, B. C., Parsia, B., & Sirin, E. (2004). Working with multiple ontologies on the semantic web. In S. A. McIlraith, D. Plexousakis, & F. van Harmelen (Eds.), *International Semantic Web Conference* (Vol. 3298, p. 620-634). Springer.

Horrocks, I., & Sattler, U. (2005). A Tableaux Decision Procedure for $\mathcal{SHOIQ}$. In *IJCAI* (p. 448-453).

Horrocks, I., Sattler, U., & Tobies, S. (1999). Practical reasoning for expressive description logics. In *LPAR* (p. 161-180).

Lutz, C., Walther, D., & Wolter, F. (2007). Conservative extensions in expressive description logics. In *IJCAI* (p. 453-458).

Pan, J., Serafini, L., & Zhao, Y. (2006). Semantic import: An approach for partial ontology reuse. In *1st International Workshop on Modular Ontologies (WoMo 2006), co-located with ISWC.*

Polleres, A. (2006). Logic programs with contextually scoped negation. In *WLP* (p. 129-136).

Polleres, A., Feier, C., & Harth, A. (2006). Rules with contextually scoped negation. In *ESWC* (p. 332-347).

Schmidt, R., & Tishkovsky, D. (2007). Deciding ALBO with tableau. In *Proceedings of the 20th International Workshop on Description Logics (dl-2007)* (pp. 135–146). Bozen-Bolzano University Press. Available from `\url{http://www.cs.man.ac.uk/~schmidt/publications/SchmidtTishkovskyDL07.html}`

Tobies, S. (2000). The complexity of reasoning with cardinality restrictions and nominals in expressive description logics. *J. Artif. Intell. Res. (JAIR)*, *12*, 199-217.

Tobies, S. (2001). *Complexity results and practical algorithms for logics in knowledge representation.* Phd thesis, LuFG Theoretical Computer Science, RWTH-Aachen, Germany.

Zimmermann, A., & Euzenat, J. (2006). Three semantics for distributed systems and their relations with alignment composition. In *International Semantic Web Conference* (p. 16-29).