On the Categorical Möbius Calculus

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Abstract

In [4] Rota introduced the incidence algebra of a locally finite partially ordered set together with its Möbius function as a unifying platform behind several seemingly different counting principles in diverse areas of combinatorics. Haigh [2] introduced the notion of a category algebra of a small category, thus generalizing the notion of an incidence algebra. Those finite categories in whose category algebras the Möbius function may be defined as an inverse of the zeta function are termed Möbius categories. The purpose of this paper is twofold. First, the notion of a Möbius category is generalized to include not only finite but also locally finite posets. Second, some of the combinatorial results of [4], including the Möbius inversion formula, are explored in the case of Möbius categories.

Introduction

Let $\mathbf{R}$ be a commutative ring with identity and $\mathbf{P} = \langle P, \leq \rangle$ be a locally finite poset. The **incidence algebra** $\mathbf{I}(\mathbf{P}, \mathbf{R})$ of $\mathbf{P}$ over $\mathbf{R}$ [4] has as its carrier $I(\mathbf{P}, \mathbf{R})$ the set of all $\mathbf{R}$-valued functions of two variables $f(x, y)$, defined for all $x, y \in P$, and such that $f(x, y) = 0$ if $x \not\leq y$, i.e.,

$$I(\mathbf{P}, \mathbf{R}) = \{f : P \times P \to R : x \not\leq y \text{ implies } f(x, y) = 0, \text{ for all } x, y \in P\}.$$

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Addition and scalar multiplication are defined pointwise and the product fg of two functions $f, g \in I(\mathbf{P}, \mathbf{R})$ is defined by

$$fg(x,y) = \sum_{x \le z \le y} f(x,z)g(z,y).$$

The multiplicative identity of the algebra $\mathbf{I}(\mathbf{P}, \mathbf{R})$ is given by the **Kronecker delta function**

$$\delta(x,y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{otherwise} \end{cases}$$

The **zeta function** $\zeta(x,y) \in I(\mathbf{P}, \mathbf{R})$ is defined by

$$\zeta(x,y) = \begin{cases} 1, & \text{if } x \le y \\ 0, & \text{otherwise} \end{cases}$$

It is proved in [4] that ζ is invertible in $I(\mathbf{P}, \mathbf{R})$ and its inverse μ is called the **Möbius function** of $\mathbf{P}$ over $\mathbf{R}$. For some of the history of the incidence algebra and the Möbius function the reader is referred to [4]. Many more results and references can be found in [5]. In [4], the Möbius inversion formula is then proved along with a result relating the Möbius function of the product of two posets with the Möbius functions of the factors. This allows the formulation of the Inclusion-Exclusion Principle as an easy application of these results on the Boolean algebra of all subsets of a finite set. Several more propositions and theorems are then obtained that relate the Möbius functions of two posets that are connected via a monotonic function or a Galois connection. Finally, applications are presented of these results in a wide variety of contexts such as finite lattices, geometric lattices, lattice representations, graph colorings and network flows.

In [2], the incidence algebra $\mathbf{I}(\mathbf{P}, \mathbf{R})$ of a poset $\mathbf{P}$ over the ring $\mathbf{R}$ is generalized to the category algebra of a small category over a ring.

Let $\mathbf{C}$ be a small category and $\mathbf{R}$ be a commutative ring with identity as before. For basic notation pertaining to category-theoretic notions the reader is referred to [3] or [1]. The **category algebra RC** of $\mathbf{C}$ over $\mathbf{R}$ [2] is the $\mathbf{R}$-algebra whose carrier RC consists of all functions $f : \mathrm{Mor}(\mathbf{C}) \to R$, such that f has finite **support**, i.e., $\mathrm{supp}(f) = \{\alpha \in \mathrm{Mor}(\mathbf{C}) : f(\alpha) \neq 0\}$ is finite. Addition and scalar multiplication are defined pointwise and the product of two functions $f, g \in RC$ is given by

$$fg(\alpha) = \sum_{\{\beta,\gamma \in \mathrm{Mor}(\mathbf{C}) : \alpha = \gamma\beta\}} f(\beta)g(\gamma).$$

The restriction that the functions in RC have finite support ensures that the sum defining the product of two functions at a morphism has finitely many non-zero summands and, hence, is well defined in $\mathbf{R}$.

With this definition of RC the multiplicative identity element, i.e., the Kronecker delta function, $\delta : \mathrm{Mor}(\mathbf{C}) \to R$ exists if and only if $|\mathbf{C}|$, the collection of objects of $\mathbf{C}$, is finite. This, however, makes this notion deficient, since algebras of infinite locally finite posets (viewed as categories) do not possess multiplicative identities. In this sense, therefore, Haigh's category algebras do not generalize Rota's incidence algebras in an entirely satisfactory way. This is one of the motivations for slightly modifying the definition of a category algebra.

Definitions and Results

Let $\mathbf{C}$ be a small category. Given two objects C_1 and C_2 in $\mathbf{C}$, denote by $[C_1, C_2]$ the full subcategory of $\mathbf{C}$ generated by the collection of all objects C of $\mathbf{C}$, such that there exist morphisms $C_1 \xrightarrow{f} C \xrightarrow{g} C_2$ in $\mathbf{C}$. By analogy with posets, $[C_1, C_2]$ will be said to be the **segment** in $\mathbf{C}$ with **endpoints** C_1 and C_2.

Definition 1 *A small category* $\mathbf{C}$ *is said to be* **segment finite** *if, for every* $C_1, C_2 \in |\mathbf{C}|$*, the segment* $[C_1, C_2]$ *is finite.*

Segment finiteness is a stronger notion from the categorical notion of local finiteness since it requires not only that the number of arrows between any two objects C_1, C_2 be finite but, in addition, that there exist finitely many objects and arrows in the full subcategory $[C_1, C_2]$. The two notions do coincide, however, when restricted to poset categories. Note for future reference that, trivially, every finite category is segment finite and that, according to the previous remark, every locally finite poset is segment finite when viewed as a category in the usual way.

Given two categories $\mathbf{C}, \mathbf{D}$, the **product category** $\mathbf{C} \times \mathbf{D}$ is defined as usual. If both $\mathbf{C}$ and $\mathbf{D}$ are segment finite, then $\mathbf{C} \times \mathbf{D}$ is also a segment finite category. By $\mathbf{C}^*$ is denoted the **dual category** of $\mathbf{C}$, which is also segment finite in case $\mathbf{C}$ is a segment finite category.

Let $\mathbf{C}$ be a segment finite category and $\mathbf{R}$ be a commutative ring with identity. Define the **category algebra** of $\mathbf{C}$ over $\mathbf{R}$ as the $\mathbf{R}$-algebra $\mathbf{RC}$ having as universe RC the collection of all functions $f : \mathrm{Mor}(\mathbf{C}) \to R$. Addition and scalar multiplication are performed point-wise and the product of two functions $f, g \in RC$ is given by

$$fg(\alpha) = \sum_{\{\beta,\gamma \in \mathrm{Mor}(\mathbf{C}) : \alpha = \gamma\beta\}} f(\beta) g(\gamma). \tag{1}$$

Multiplication is well-defined since the segment finiteness of $\mathbf{C}$ ensures that the sum appearing on the right-hand side of (1) is finite and thus well-defined in $\mathbf{R}$. Also, it

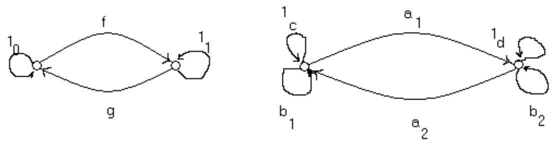

Figure 1: Two finite categories. The first is not Möbius. The second is Möbius over **Q** but is not Möbius over **Z**.

is now obvious that the function $\delta : \mathrm{Mor}(\mathbf{C}) \to R$, with

$$\delta(\alpha) = \begin{cases} 1, & \text{if } \alpha = i_C, \text{ for some } C \in |\mathbf{C}| \\ 0, & \text{otherwise} \end{cases}$$

is a multiplicative identity in **RC**. δ is called the **Kronecker delta function**.

This notion of a category algebra generalizes properly both the notion of a category algebra of [2] and of an incidence algebra of [4].

The **zeta function** $\zeta : \mathrm{Mor}(\mathbf{C}) \to R$ is now the constant function

$$\zeta(\alpha) = 1, \text{ for all } \alpha \in \mathrm{Mor}(\mathbf{C}).$$

If a unique 2-sided inverse of ζ exists in **RC**, it is called the **Möbius function** of **C over R** and denoted by μ. If μ exists in **RC** then **C** is called a **Möbius category over R**.

The two examples of Figure 1 illustrate the two possibilities that may arise in terms of the existence of μ. The finite category on the left is not a Möbius category, since, by considering first the identity arrow 1_0 and then the arrow f, it is easily seen that the conditions $\mu(1_0) + \mu(f) = 1$ and $\mu(1_0) + \mu(f) = 0$ must hold simultaneously. The category pictured on the right is borrowed from [2]. Its composition (apart from the action of the identities) is given by the table

	α_1	α_2	β_1	β_2
α_1		β_2	α_1	
α_2	β_1			α_2
β_1		α_2	β_1	
β_2	α_1			β_2

and, as Haigh points out, it is Möbius over $\mathbf{Q}$ with

$$\mu(1_c) = \mu(1_d) = 1, \quad \mu(\alpha_1) = \mu(\alpha_2) = \mu(\beta_1) = \mu(\beta_2) = -\frac{1}{3}.$$

but it is not Möbius over $\mathbf{Z}$.

If $\mathbf{C}$ is a locally finite poset (viewed as a category), then μ is its Möbius function in the sense of [4] and, if $\mathbf{C}$ is a finite Möbius category, then μ is the Möbius function of $\mathbf{C}$ over $\mathbf{R}$ introduced in [2]. To see that the definition here is strictly more encompassing than the ones presented by Rota and Haigh, consider the category $\mathbf{C}$ with collection of objects $\mathbb{N} = \{0, 1, 2, \ldots\}$ and whose arrows are freely generated by the collection of arrows

$$f_{ij} : i \to i+1, \quad i = 0, 1, \ldots, j = 0, 1 \ldots, 2^i.$$

This means that $\mathbf{C}(n, m) = \emptyset$ if $n > m$, and that the arrows in $\mathbf{C}(n, m)$ with $n \leq m$ are in 1-1 correspondence with paths of the form $f_{ni_n} f_{n+1, i_{n+1}} \cdots f_{m-1, i_{m-1}}$, $0 \leq i_j \leq 2^j$, for all $n \leq j < m$. This is neither a finite category nor a locally finite poset but it is a segment finite category, which is Möbius, with Möbius function given by

$$\mu(\alpha) = \begin{cases} 1, & \text{if } \alpha = i_n, \text{ for some } n \in \mathbb{N} \\ -1, & \text{if } \alpha : n \to n+1, \text{ for some } n \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}.$$

With this definition the following proposition holds. It generalizes Proposition 3.2 of [2] to the case of segment finite categories and it can be proved in exactly the same way.

Proposition 2 *A segment finite category* $\mathbf{C}$ *is a Möbius category over* $\mathbf{R}$ *if and only if the following set of equations has a unique solution in* $\mathbf{R}$.

1. *For all* $C \in |\mathbf{C}|$,

$$\sum_{\{\beta, \gamma \in \mathrm{Mor}(\mathbf{C}) : \gamma\beta = 1_C\}} \mu(\beta) = 1, \qquad \sum_{\{\beta, \gamma \in \mathrm{Mor}(\mathbf{C}) : \gamma\beta = 1_C\}} \mu(\gamma) = 1.$$

2. *For each non-identity* $\alpha \in \mathrm{Mor}(\mathbf{C})$,

$$\sum_{\{\beta, \gamma \in \mathrm{Mor}(\mathbf{C}) : \gamma\beta = \alpha\}} \mu(\beta) = 0, \qquad \sum_{\{\beta, \gamma \in \mathrm{Mor}(\mathbf{C}) : \gamma\beta = \alpha\}} \mu(\gamma) = 0.$$

The following proposition is the analog of Proposition 3.4 from [2]. Its proof from [2], however, is not valid in the case of a segment finite category, since conditions 1 and 2 in this case do not imply that the length of a maximal chain of composable non-identity morphisms in $\mathbf{C}$ is finite. Therefore a different approach will be adopted for its proof. The proof presented here does not only generalize the proposition itself but also provides a proof for Proposition 1 in [4].

Proposition 3 *Let* $\mathbf{C}$ *be a segment finite category such that*

1. *the composite of every pair of composable non-identity morphisms in* $\mathbf{C}$ *is a non-identity morphism and*

2. *every endomorphism in* $\mathbf{C}$ *is an identity.*

Then $\mathbf{C}$ *is a Möbius category over* $\mathbf{Z}$*, the ring of integers.*

Proof:

$\mu : \mathrm{Mor}(\mathbf{C}) \to R$ has to be defined, such that $\mu\zeta = \zeta\mu = \delta$. First, consider the case of an identity morphism 1_C, for some $C \in |\mathbf{C}|$. By conditions 1 and 2, $\sum_{\{\beta,\gamma:\gamma\beta=1_C\}} \mu(\beta) = \mu(1_C)$, whence, by Proposition 2,1., $\mu(1_C) = 1$. Now, consider an arbitrary $\alpha \in \mathbf{C}(C, D)$. By Proposition 2, 2., $\sum_{\{\beta,\gamma:\gamma\beta=\alpha\}} \mu(\beta) = 0$, whence $\mu(1_C) + \mu(\alpha) + \sum_{\{\beta,\gamma\neq\alpha:\gamma\beta=\alpha\}} \mu(\beta) = 0$, i.e., $\mu(\alpha) = -1 - \sum_{\{\beta,\gamma\neq\alpha:\gamma\beta=\alpha\}} \mu(\beta)$. Note that each of the sums required to compute the Möbius functions on the right are strictly smaller than the sum for $\mu(\alpha)$ because of conditions 1 and 2. $\mu(\alpha)$ may thus be computed by iterating the last formula until the problem is reduced down to the case of identity morphisms. ■

Proposition 4 (Generalized Möbius Inversion Formula) *Let* $\mathbf{C}$ *be a Möbius category,* $C_0 \in |\mathbf{C}|$ *and* $f : |\mathbf{C}| \to R$*, such that, if* $\mathbf{C}(C_0, C) = \emptyset$*, then* $f(C) = 0$*. Suppose that*

$$g(C) = \sum_{D \xrightarrow{\alpha} C} f(D).$$

Then

$$f(C) = \sum_{D \xrightarrow{\alpha} C} g(D)\mu(\alpha).$$

Proof:

First, notice that because of the condition that if $\mathbf{C}(C_0, C) = \emptyset$ then $f(C) = 0$ and segment finiteness, the sum defining g is finite and, therefore, g is well-defined.

$$\begin{tikzcd}
 & C \\
D \arrow[ur, "\alpha"] & \\
 & E \arrow[ul, "\beta"] \arrow[uu, "\gamma"']
\end{tikzcd}$$

We have

$$\begin{array}{rcl} \sum_{D \xrightarrow{\alpha} C} g(D)\mu(\alpha) & = & \sum_{D \xrightarrow{\alpha} C} \sum_{E \xrightarrow{\beta} D} f(E)\mu(\alpha) \\ & = & \sum_{D \xrightarrow{\alpha} C} \sum_{E \xrightarrow{\beta} D} f(E)\zeta(\beta)\mu(\alpha) \\ & = & \sum_{E} f(E) \sum_{E \xrightarrow{\beta} D \xrightarrow{\alpha} C} \mu(\alpha)\zeta(\beta) \\ & = & \sum_{E} f(E) \sum_{E \xrightarrow{\gamma} C} \delta(\gamma) \\ & = & f(C). \end{array}$$

■

Similarly, it can be shown following the dual reasoning that the following holds

Proposition 5 (Dual of the Generalized Möbius Inversion Formula) *Let* $\mathbf{C}$ *be a Möbius category,* $C_0 \in |\mathbf{C}|$ *and* $f : |\mathbf{C}| \to R$*, such that, if* $\mathbf{C}(C, C_0) = \emptyset$*, then* $f(C) = 0$*. Suppose that*

$$g(C) = \sum_{C \xrightarrow{\alpha} D} f(D).$$

Then

$$f(C) = \sum_{C \xrightarrow{\alpha} D} g(D)\mu(\alpha).$$

Proposition 6 (Duality) *Let* $\mathbf{C}$ *be a Möbius category and* μ *its Möbius function. Then, the dual* $\mathbf{C}^*$ *is also a Möbius category and its Möbius function* μ^* *is given by* $\mu^*(\alpha^*) = \mu(\alpha)$*, for all* $\alpha \in \mathrm{Mor}(\mathbf{C})$.

Proof:

As noted before, $\mathbf{C}^*$ is also segment finite. To show that it is Möbius it suffices to show that $\mu^* : \mathrm{Mor}(\mathbf{C}^*) \to R$, defined by $\mu^*(\alpha^*) = \mu(\alpha)$, for all $\alpha^* \in \mathrm{Mor}(\mathbf{C}^*)$, is a two sided inverse of ζ^*. We have

$$\begin{array}{rcl} \mu^*\zeta^*(\alpha^*) & = & \sum_{\{\beta^*, \gamma^* : \gamma^*\beta^* = \alpha^*\}} \mu^*(\beta^*)\zeta^*(\gamma^*) \\ & = & \sum_{\{\beta, \gamma : \beta\gamma = \alpha\}} \mu(\beta)\zeta(\gamma) \\ & = & \zeta\mu(\alpha) \\ & = & \delta(\alpha) \\ & = & \delta^*(\alpha^*), \end{array}$$

and, similarly, for $\zeta^*\mu^*$. ■

Proposition 7 *Let* $\mathbf{C}$ *be a Möbius category,* μ *its Möbius function and* $\mathbf{S} = [C_0, C_1]$ *a segment in* $\mathbf{C}$*. Then* $\mathbf{S}$ *is a Möbius category and its Möbius function* $\mu_{\mathbf{S}}$ *is the restriction of* μ *on* $\mathrm{Mor}(\mathbf{S})$.

Proof:
Since every segment of $\mathbf{S} = [C_0, C_1]$ is a segment of $\mathbf{C}$, $\mathbf{S}$ is also segment finite if $\mathbf{C}$ is. To show that it is Möbius, it suffices to show that, for all $\alpha \in \mathrm{Mor}(\mathbf{S})$, $\zeta\mu(\alpha) = \delta(\alpha)$. For this, observe that the sum $\sum_{\{\beta,\gamma\in \mathrm{Mor}(\mathbf{S}):\gamma\beta=\alpha\}} \zeta(\beta)\mu(\gamma)$ is exactly the same with the sum $\sum_{\{\beta,\gamma\in \mathrm{Mor}(\mathbf{C}):\gamma\beta=\alpha\}} \zeta(\beta)\mu(\gamma)$. ■

Proposition 8 (Product Category) *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be two Möbius categories with Möbius functions* $\mu_{\mathbf{C}}$ *and* $\mu_{\mathbf{D}}$*, respectively. Then* $\mathbf{C}\times\mathbf{D}$ *is also a Möbius category with Möbius function* $\mu_{\mathbf{C}\times\mathbf{D}} = \mu_{\mathbf{C}} \times \mu_{\mathbf{D}}$.

Proof:
Once the observation that, $\mathbf{C}\times\mathbf{D}$ is segment finite if both $\mathbf{C}$ and $\mathbf{D}$ are, is made, the proof follows mutatis mutandis from the corresponding proof for locally finite posets. ■

Now, the main results of [4], tying the Möbius functions of two finite posets that are related via a Galois connection or a monotone mapping, are explored in the case of Möbius categories that are related via an adjunction or a functor, respectively.

First, Theorem 1 of [4] is generalized to

Theorem 9 *Let* $\mathbf{C}, \mathbf{D}$ *be two Möbius categories, with Möbius functions* $\mu_{\mathbf{C}}$ *and* $\mu_{\mathbf{D}}$*, respectively, and* $\langle F, G, \eta, \epsilon\rangle : \mathbf{C} \to \mathbf{D}^{\mathrm{op}}$ *an adjunction. Suppose that* $D_1, D_2 \in |\mathbf{D}|, D_1 \neq D_2$*, such that* $G(D_2) = G(D)$ *implies* $D_2 = D$*, for all* $D \in |\mathbf{D}|$*. Then*

$$\sum_{D_1 \xrightarrow{\alpha} D_2} \mu_{\mathbf{D}}(\alpha) = \sum_{\{G(D_2)\xrightarrow{\beta} C : F(C) = D_1\}} \mu_{\mathbf{C}}(\beta).$$

Proof:
Note, first that, for $C_0 \in |\mathbf{C}|, D_0 \in |\mathbf{D}|$,

$$\sum_{C_0 \xrightarrow{\alpha} C} \sum_{G(D_0)\xrightarrow{\beta} C} \delta_{\mathbf{C}}(\beta) = \sum_{D_0 \xrightarrow{\gamma} F(C_0)} \zeta_{\mathbf{D}}(\gamma). \tag{2}$$

In fact, we have

$$\sum_{C_0 \xrightarrow{\alpha} C} \sum_{G(D_0)\xrightarrow{\beta} C} \delta_{\mathbf{C}}(\beta) = \sum_{C_0 \xrightarrow{\beta} G(D_0)} 1 = \sum_{D_0 \xrightarrow{\gamma} F(C_0)} 1 = \sum_{D_0 \xrightarrow{\gamma} F(C_0)} \zeta_{\mathbf{D}}(\gamma),$$

since $\langle F, G, \eta, \epsilon\rangle : \mathbf{C} \to \mathbf{D}$ is an adjunction.

Now, set, for all $C \in |\mathbf{C}|$,

$$f(C) = \sum_{G(D_0)\xrightarrow{\beta} C} \delta_{\mathbf{C}}(\beta), \quad g(C) = \sum_{D_0 \xrightarrow{\gamma} F(C)} \zeta_{\mathbf{D}}(\gamma).$$

Then, by (2), we get $\sum_{C_0 \xrightarrow{\alpha} C} f(C) = g(C_0)$, whence, by Proposition 5, we obtain $f(C_0) = \sum_{C_0 \xrightarrow{\alpha} C} g(C)\mu_{\mathbf{C}}(\alpha)$. This means that

$$\sum_{G(D_0) \xrightarrow{\beta} C_0} \delta_{\mathbf{C}}(\beta) = \sum_{C_0 \xrightarrow{\alpha} C} \sum_{D_0 \xrightarrow{\gamma} F(C)} \zeta_{\mathbf{D}}(\gamma)\mu_{\mathbf{C}}(\alpha). \tag{3}$$

Now, since C_0 is arbitrary, we may set $C_0 = G(D_2)$, whence

$$\sum_{G(D_0) \xrightarrow{\beta} G(D_2)} \delta_{\mathbf{C}}(\beta) = \sum_{G(D_2) \xrightarrow{\alpha} C} \sum_{D_0 \xrightarrow{\gamma} F(C)} \zeta_{\mathbf{D}}(\gamma)\mu_{\mathbf{C}}(\alpha).$$

Note that, by the hypothesis, the only possible choice for D_0 that makes the left-hand side non-zero is $D_0 = D_2$, and in this case the sum is equal to 1. Thus, the sum may be replaced by $\sum_{D_2 \xrightarrow{\beta} D_0} \delta_{\mathbf{D}}(\beta)$. Then, we obtain

$$\sum_{D_0 \xrightarrow{\beta} D_2} \delta_{\mathbf{D}}(\beta) = \sum_{G(D_2) \xrightarrow{\alpha} C} \sum_{D_0 \xrightarrow{\gamma} F(C)} \zeta_{\mathbf{D}}(\gamma)\mu_{\mathbf{C}}(\alpha).$$

Therefore, we have

$$\begin{aligned}
\textstyle\sum_{D_1 \xrightarrow{\alpha} D_2} \mu_{\mathbf{D}}(\alpha) &= \textstyle\sum_{D_1 \xrightarrow{\alpha} D_2} \mu_{\mathbf{D}}\delta_{\mathbf{D}}(\alpha) \\
&= \textstyle\sum_{D_1 \xrightarrow{\alpha} D_2} \sum_{\{\beta,\gamma:\gamma\beta=\alpha\}} \mu_{\mathbf{D}}(\beta)\delta_{\mathbf{D}}(\gamma) \\
&= \textstyle\sum_{D_1 \xrightarrow{\beta} D_0} \sum_{D_0 \xrightarrow{\gamma} D_2} \mu_{\mathbf{D}}(\beta)\delta_{\mathbf{D}}(\gamma) \\
&= \textstyle\sum_{D_1 \xrightarrow{\beta} D_0} \mu_{\mathbf{D}}(\beta) \sum_{G(D_2) \xrightarrow{\alpha} C} \sum_{D_0 \xrightarrow{\gamma} F(C)} \mu_{\mathbf{C}}(\alpha)\zeta_{\mathbf{D}}(\gamma) \\
&= \textstyle\sum_{G(D_2) \xrightarrow{\alpha} C} \sum_{D_1 \xrightarrow{\beta} F(C)} \mu_{\mathbf{C}}(\alpha)\delta_{\mathbf{D}}(\beta) \\
&= \textstyle\sum_{G(D_2) \xrightarrow{\alpha} C : F(C) = D_1} \mu_{\mathbf{C}}(\alpha).
\end{aligned}$$

■

Based on Theorem 2 of [4] the following conjecture may be formulated:

Let $\mathbf{C}, \mathbf{D}$ *be two finite Möbius categories,* $C_0 \in |\mathbf{C}|, D_0 \in |\mathbf{D}|$, *and* $F : \mathbf{D} \to \mathbf{C}$ *a functor. Suppose that the inverse image of every segment* $[C_0, C_1]$ *in* $\mathbf{C}$ *is a segment* $[D_0, D_1]$ *in* $\mathbf{D}$ *and that the inverse image of* C_0 *contains at least two objects in* $\mathbf{D}$. *Then, for all* $C \in |\mathbf{C}|$, *with* $\mathbf{C}(C_0, C) \neq \emptyset$,

$$\sum_{D_0 \xrightarrow{\alpha} D : F(D) = C} \mu_{\mathbf{D}}(\alpha) = 0.$$

This conjecture, however, turns out not to hold in general in the case of finite Möbius categories. A counterexample is provided by the finite Möbius category over $\mathbf{Q}$ of Haigh, mapped to the trivial category by means of the constant functor.

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