Categorical Abstract Algebraic Logic:
Structure Systems and Łoś' Theorem

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Abstract

Following work on abstracting the concept of an algebra to that of an algebraic system and of an ordered algebra to that of an ordered algebraic system, the notion of a first-order structure is abstracted to obtain structure systems. The algebraic part of a structure system is an algebraic system rather than an algebra as is the case in the ordinary first-order structures. This abstraction is accompanied by the introduction of a suitably modified notion of a countable first-order language with the aim of developing a first-order model theory of structure systems and, therefore, axiomatizing classes of structure systems. After introducing some basic constructions on structure systems, including the ultraproduct construction, an analog of Łoś' Ultraproduct Theorem is provided for structure systems.

1 Introduction

A very important part of the theory of *abstract algebraic logic* deals with a characterization of certain classes of logical matrices and of reduced logical matrices that form the matrix semantics of sentential logics. Results of this kind serve in characterizing classes of logics based on closure properties of their matrix semantics. Theorem 3.15 of [19] summarizes the main characterization results of this type and more details, including proofs and commentary concerning original sources, are provided in Czelakowski's comprehensive treatise [7].

A critical part in relating classes of logical matrices with classes of reduced logical matrices is played by the Leibniz operator and the Leibniz congruences, first introduced by Blok and Pigozzi [2], with the goal of providing an intrinsic characterization of algebraizable logics. Reduction by the Leibniz congruence of a logical matrix is the operation that leads from a class of logical matrices to the corresponding reduced class.

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Bloom's work [3], relating sentential logic with universal Horn logic without equality, shows that the theory of logical matrices also forms part of the theory of first-order languages with a single unary relation, in which the unary relation is modeled via the filter of the logical matrix. This relation was the basis that led a decade ago Raimon Elgueta [13, 14, 15, 16] (and, in part, in joint work with Czelakowski [8] and with Jansana [17]) and Pillar Dellunde [9, 10] (and, in part, jointly with Casanovas and Jansana [5] and with Jansana [11]) to consider, in the context of abstract algebraic logic, first-order logic without equality and its model theory.

In [13], Elgueta begins the study of several aspects of the model-theory of equality-free first-order structures. In the first section, he introduces basic notation and constructions for equality-free first order logic that carry over, almost without change, from the case of first-order logic with equality. In Section 2, the notion of Leibniz equality is introduced for arbitrary first-order structures without equality. It constitutes a weak form of equality that replaces genuine equality in this equality-free context. The inspiration for its consideration comes from its role in the theory of logical matrices, as established by Blok and Pigozzi [2]. Based on Leibniz equality, Leibniz quotients of structures are introduced in Section 3. Finally, in Sections 4 and 5 Elgueta proves the main lemmas and the main theorems, respectively, including characterization theorems for several classes (elementary, universal, universal Horn, universal atomic) of structures defined in equality-free first-order logic.

In recent work by the author on the algebraization of π-institutions [25, 26, 27, 28], it has become clear that the role played by algebras in the theory of sentential logics and of their matrix and algebraic models is now played by algebraic systems. These are set-valued functors, whose algebraic nature is given in the form of a category of natural transformations on the functor. Further, if endowed with a partial ordering system, algebraic systems give rise to ordered algebraic systems, properties of whose classes were recently explored in [29, 30, 31, 32], inspired by analogous work of Pałasińska and Pigozzi [24] on the theory of partially-ordered algebras. Partially-ordered algebras form a generalization of universal algebras and they, in turn, are special cases of first-order structures. Thus, it is a natural endeavor to seek to extend the theory of first-order structures to structures that would generalize in the same direction partially ordered algebraic systems and to explore properties of their equality-free first-order model theory, following the lead of the works of Elgueta and of Dellunde.

In fact, inspired by the works of Elgueta and of Dellunde, the concept of a structure system is introduced in this paper. Structure systems are generalizations of both first-order structures and partially ordered algebraic systems. A first-order language is also introduced, a slight variant of the ordinary notion, that allows us to syntactically study structure systems. Several elementary results are presented on structure systems but the most important is an analog of Łoś' Ultraproduct Theorem for structure systems. This line of research is to be continued in forthcoming work by the author in which analogs of many other properties, analogous to those studied by Elgueta and Dellunde, as pertaining to classes of structure systems are studied.

For general concepts and notation from category theory the reader is referred to any of [1, 4, 23]. For an overview of the current state of affairs in abstract algebraic logic the review article [19], the monograph [18] and the book [7] are all excellent references. To follow recent developments on the categorical side of the subject the reader may refer to the series of papers [25]-[28] (see also additional references therein). Finally, the original reference for Łos' Ultraproduct Theorem is the paper [21], whereas standard references on model theory, all of which contain treatments of Łos' Theorem and related results, are the books by Chang and Keisler [6], Hodges [20], Marker [22] and Doets [12].

2 Basic Definitions

A **clone category** is a category $\mathbf{F}$ with objects all finite natural numbers that is isomorphic to the category of natural transformations $\mathbf{N}$ on a given functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ (see [28] for the definition of a category of natural transformations on a set-valued functor) via an isomorphism that preserves projections, and, as a consequence, also preserves objects. Note that in previous papers on the subject, the symbol N was used to denote a category of natural transformations on a functor. In the present work, boldfaced symbols will be used to denote categories. So $\mathbf{N}$ in place of N will be preferred.

A **(structure system) language** is a triple $\mathcal{L} = \langle \mathbf{F}, R, \rho \rangle$, where $\mathbf{F}$ is a clone category, R is a nonempty set of relation symbols and $\rho : R \to \omega$ is an arity function.

An **$\mathcal{L}$-term** is an arrow $t \in \mathbf{F}(n, 1)$, for some $n \in \omega$. The collection of all $\mathcal{L}$-terms is denoted by $\mathrm{Te}_{\mathcal{L}}$. An **atomic $\mathcal{L}$-formula** is an expression of the form $r(t_0, \ldots, t_{\rho(r)-1})$, where $r \in R$ is a relation symbol of $\mathcal{L}$ and $t_0, \ldots, t_{\rho(r)-1}$ are $\mathcal{L}$-terms. Finally, similarly with the case of equality-free first-order logic, an **$\mathcal{L}$-formula** is built recursively out of atomic formulas as follows:

- An atomic $\mathcal{L}$-formula is an $\mathcal{L}$-formula.
- $(\alpha \wedge \beta), (\neg \alpha)$ are $\mathcal{L}$-formulas, for all $\mathcal{L}$-formulas α, β and
- $(\forall i)\alpha$ is an $\mathcal{L}$-formula, for every $i \in \omega$ and every $\mathcal{L}$-formula α.

The collection of all $\mathcal{L}$-formulas is denoted by $\mathrm{Fm}_{\mathcal{L}}$. Clearly, all other connectives, e.g., $\vee, \to, \leftrightarrow$, etc., may be defined in terms of these few basic connectives. We feel free to use the most convenient collection of connectives when a structural induction on the complexity of a formula is called for. Moreover, the usual metamathematical conventions in adding or omitting parentheses for clarity will be followed throughout.

Before introducing the concept of an $\mathcal{L}$-structure system, the definition of a relation system on a functor will be presented.

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a functor. An **n-ary relation system** R on SEN is a family $R = \{R_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, such that

- R_Σ is an n-ary relation on $\mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$, and
- $\mathrm{SEN}(f)^n(R_{\Sigma_1}) \subseteq R_{\Sigma_2}$, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$.

Sometimes, we write $\mathrm{SEN}(f)(R_{\Sigma_1}) \subseteq R_{\Sigma_2}$ instead of the more precise $\mathrm{SEN}(f)^n(R_{\Sigma_1}) \subseteq R_{\Sigma_2}$ to simplify notation.

An $\mathcal{L}$-**(structure) system** $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$ is a triple consisting of

- a functor $\mathrm{SEN}^{\mathfrak{A}} : \mathbf{Sign}^{\mathfrak{A}} \to \mathbf{Set}$,
- a category of natural transformations $\mathbf{N}^{\mathfrak{A}}$ on $\mathrm{SEN}^{\mathfrak{A}}$, such that $F : \mathbf{F} \to \mathbf{N}^{\mathfrak{A}}$ is a surjective functor that preserves all projections $p^{kl} : k \to 1, k \in \omega, l < k$, and
- $R^{\mathfrak{A}} = \{r^{\mathfrak{A}} : r \in R\}$ a family of relation systems on $\mathrm{SEN}^{\mathfrak{A}}$ indexed by R, such that $r^{\mathfrak{A}}$ is n-ary if $\rho(r) = n$.

Let t be an $\mathcal{L}$-term and $\mathfrak{A}$ an $\mathcal{L}$-system. Let also $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^{\omega}$. The **value of** t **at** $\langle \Sigma, \vec{\phi} \rangle$ **in the system** $\mathfrak{A}$, denoted by $t^{\mathfrak{A}}_{\Sigma}(\vec{\phi})$, is the value $F(t)_{\Sigma}(\vec{\phi}\restriction_n)$, where n is the domain of t:

$$t^{\mathfrak{A}}_{\Sigma}(\vec{\phi}) := F(t)_{\Sigma}(\vec{\phi}\restriction_n).$$

Finally, the satisfaction relation of $\mathcal{L}$-formulas by $\mathcal{L}$-systems will be defined.

Let α be an $\mathcal{L}$-formula, $\mathfrak{A}$ an $\mathcal{L}$-system, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^{\omega}$. $\mathfrak{A}$ **satisfies** α **at** $\langle \Sigma, \vec{\phi} \rangle$, written $\mathfrak{A} \models_{\Sigma} \alpha[\vec{\phi}]$, is defined by recursion on the structure of the $\mathcal{L}$-formula α as follows:

- If $\alpha = r(t_0, \ldots, t_{n-1})$ is atomic, then
$$\mathfrak{A} \models_{\Sigma} r(t_0, \ldots, t_{n-1})[\vec{\phi}] \quad \text{iff} \quad \langle t^{\mathfrak{A}}_{0_{\Sigma}}(\vec{\phi}), \ldots, t^{\mathfrak{A}}_{n-1_{\Sigma}}(\vec{\phi}) \rangle \in r^{\mathfrak{A}}_{\Sigma}.$$
- $\mathfrak{A} \models_{\Sigma} (\alpha \wedge \beta)[\vec{\phi}]$ iff $\mathfrak{A} \models_{\Sigma} \alpha[\vec{\phi}]$ and $\mathfrak{A} \models_{\Sigma} \beta[\vec{\phi}]$.
- $\mathfrak{A} \models_{\Sigma} (\neg\alpha)[\vec{\phi}]$ iff $\mathfrak{A} \not\models_{\Sigma} \alpha[\vec{\phi}]$ and, finally,
- $\mathfrak{A} \models_{\Sigma} (\forall i)\alpha[\vec{\phi}]$ iff $\mathfrak{A} \models_{\Sigma} \alpha[\vec{\psi}]$, for all $\vec{\psi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^{\omega}$, such that $\phi_j = \psi_j$, for all $j \neq i$.

These conditions clearly define the semantics of all other connectives in the first-order model theory of $\mathcal{L}$-systems.

If $\mathfrak{A} \models_{\Sigma} \alpha[\vec{\phi}]$ holds for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and all $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^{\omega}$, then we write $\mathfrak{A} \models \alpha$. The expressions $\mathfrak{A} \models \Gamma$ and $\mathtt{K} \models \alpha$, for Γ a set of $\mathcal{L}$-formulas and $\mathtt{K}$ a class of $\mathcal{L}$-systems are defined as usual. Finally, we denote by

$$\mathbf{Mod}(\Gamma) = \{\mathfrak{A} : \mathfrak{A} \models \Gamma\} \quad \text{and} \quad \mathbf{Th}(\mathtt{K}) = \{\alpha : \mathtt{K} \models \alpha\},$$

the collection of all $\mathcal{L}$-systems that are **models** of Γ and the $\mathcal{L}$-**theory** of the collection $\mathtt{K}$ of $\mathcal{L}$-systems, respectively.

3 Subsystems, Filter Extensions, Homomorphisms and Reduced Products

3.1 Subsystems and Filter extensions

Before proceeding to define subsystems of structure systems, we need to recall from Section 2 of [33] the definition of a subfunctor and that of an N-subfunctor.

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a functor. A functor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ is a **subfunctor** of SEN, if

- $\mathbf{Sign}'$ is a subcategory of $\mathbf{Sign}$,
- $\mathrm{SEN}'(\Sigma') \subseteq \mathrm{SEN}(\Sigma')$, for all $\Sigma' \in |\mathbf{Sign}'|$, and
- $\mathrm{SEN}'(f)(\phi) = \mathrm{SEN}(f)(\phi)$, for all $f \in \mathbf{Sign}'(\Sigma, \Sigma'), \phi \in \mathrm{SEN}'(\Sigma)$.

If N is a category of natural transformations on SEN, such that, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, all $\Sigma' \in |\mathbf{Sign}'|$ and all $\vec{\phi}' \in \mathrm{SEN}'(\Sigma')^n$, $\sigma_{\Sigma'}(\vec{\phi}') \in \mathrm{SEN}'(\Sigma')$, then SEN' will be said to be an N-**subfunctor** of SEN. If $\mathrm{SEN}' : \mathbf{Sign} \to \mathbf{Set}$ is a subfunctor of $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, with the same domain category, then SEN' is said to be a **simple subfunctor** of SEN.

Returning to the main developments, suppose, now, that $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$, $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}} \rangle, R^{\mathfrak{B}} \rangle$ are two $\mathcal{L}$-systems. We say that $\mathfrak{A}$ is a **(structure) subsystem** of $\mathfrak{B}$, in symbols $\mathfrak{A} \subseteq \mathfrak{B}$, if

- $\mathrm{SEN}^{\mathfrak{A}}$ is an $\mathbf{N}^{\mathfrak{B}}$-subfunctor of $\mathrm{SEN}^{\mathfrak{B}}$ and
- $r_\Sigma^{\mathfrak{A}} = r_\Sigma^{\mathfrak{B}} \cap \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^{\rho(r)}$, for all $r \in R$ and all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$.

We call $\mathfrak{A}$ a **simple subsystem** of $\mathfrak{B}$ if it is a subsystem of $\mathfrak{B}$, such that $\mathrm{SEN}^{\mathfrak{A}}$ is a simple subfunctor of $\mathrm{SEN}^{\mathfrak{B}}$. In this case, we write $\mathfrak{A} \subseteq^s \mathfrak{B}$.

Similarly, $\mathfrak{B}$ is a **filter extension** of $\mathfrak{A}$, written $\mathfrak{A} \sqsubseteq \mathfrak{B}$, if

- $\mathrm{SEN}^{\mathfrak{A}} = \mathrm{SEN}^{\mathfrak{B}}$, $\mathbf{N}^{\mathfrak{A}} = \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{A}} = F^{\mathfrak{B}}$ and
- $r^{\mathfrak{A}} \leq r^{\mathfrak{B}}$, for all $r \in R$, where, as usual, $\leq$ denotes signature-wise inclusion.

Let, now, $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$ be an $\mathcal{L}$-system. Suppose that $X = \{X_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ is an axiom system of $\mathrm{SEN}^{\mathfrak{A}}$, i.e., such that

- $X_\Sigma \subseteq \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, and
- $\mathrm{SEN}^{\mathfrak{A}}(f)(X_{\Sigma_1}) \subseteq X_{\Sigma_2}$, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}^{\mathfrak{A}}|, f \in \mathbf{Sign}^{\mathfrak{A}}(\Sigma_1, \Sigma_2)$.

Define the collection $[X] = \{[X]_\Sigma\}_{\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|}$ by letting, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$,

$$[X]_\Sigma = \{t_\Sigma^{\mathfrak{A}}(\vec{\phi}) : t \in \mathrm{Te}_{\mathcal{L}} \text{ and } \vec{\phi} \in X_\Sigma^\omega\}.$$

It is shown in the next proposition that, given an axiom system X, the collection $[X]$ is also an axiom system.

Proposition 1 *Suppose that* $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$ *is an* $\mathcal{L}$*-system. If* X *is an axiom system of* $\mathrm{SEN}^{\mathfrak{A}}$*, then* $[X]$ *is also an axiom system of* $\mathrm{SEN}^{\mathfrak{A}}$*.*

Proof:
It is clear, by definition, that $[X]_\Sigma \subseteq \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$. So it suffices to show that, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}^{\mathfrak{A}}|$, $f \in \mathbf{Sign}^{\mathfrak{A}}(\Sigma_1, \Sigma_2)$, we have that $\mathrm{SEN}^{\mathfrak{A}}(f)([X]_{\Sigma_1}) \subseteq [X]_{\Sigma_2}$. In fact, if $\phi \in [X]_{\Sigma_1}$, then, there exist $t \in \mathrm{Te}_{\mathcal{L}}, \vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma_1)^\omega$, such that $\phi = t^{\mathfrak{A}}_{\Sigma_1}(\vec{\phi})$, whence

$$\begin{array}{rcl}\mathrm{SEN}^{\mathfrak{A}}(f)(\phi) & = & \mathrm{SEN}^{\mathfrak{A}}(f)(t^{\mathfrak{A}}_{\Sigma_1}(\vec{\phi}))\\ & = & \mathrm{SEN}^{\mathfrak{A}}(f)(F^{\mathfrak{A}}(t)_{\Sigma_1}(\vec{\phi}\restriction_n))\\ & = & F^{\mathfrak{A}}(t)_{\Sigma_2}(\mathrm{SEN}^{\mathfrak{A}}(f)(\vec{\phi}\restriction_n))\\ & = & F^{\mathfrak{A}}(t)_{\Sigma_2}(\mathrm{SEN}^{\mathfrak{A}}(f)(\vec{\phi})\restriction_n),\end{array}$$

whence, since $\mathrm{SEN}^{\mathfrak{A}}(f)(\vec{\phi}) \in X^\omega_{\Sigma_2}$, we get that $\mathrm{SEN}^{\mathfrak{A}}(f)(\phi) \in [X]_{\Sigma_2}$, as was to be shown. ■

Now, given an axiom system $X = \{X_\Sigma\}_{\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|}$ on $\mathrm{SEN}^{\mathfrak{A}}$, as above, by $\mathfrak{A} \restriction X$ is denoted the **subsystem of $\mathfrak{A}$ generated by** X. This has

- the same signature category $\mathbf{Sign}^{\mathfrak{A}}$ as $\mathfrak{A}$,
- its sentence functor maps $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ to the set $[X]_\Sigma \subseteq \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$,
- it has the same pair $\langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle$ as $\mathfrak{A}$ and
- $r^{\mathfrak{A}\restriction X}_\Sigma = r^{\mathfrak{A}}_\Sigma \cap [X]^{\rho(r)}_\Sigma$.

It is not difficult to verify that $\mathfrak{A} \restriction X$, as defined above, is indeed a subsystem of $\mathfrak{A}$ and that, as a consequence, this definition makes sense.

Given $\mathcal{L}$-systems $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{A}$ is an **elementary subsystem** of $\mathfrak{B}$, in symbols $\mathfrak{A} \subseteq_e \mathfrak{B}$, iff $\mathfrak{A} \subseteq \mathfrak{B}$ and for all $\mathcal{L}$-formulas α, all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, and all $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^\omega$,

$$\mathfrak{A} \models_\Sigma \alpha[\vec{\phi}] \quad \text{iff} \quad \mathfrak{B} \models_\Sigma \alpha[\vec{\phi}].$$

Finally, $\mathfrak{A}$ and $\mathfrak{B}$ are **elementarily equivalent**, denoted $\mathfrak{A} \equiv \mathfrak{B}$, iff, for all $\mathcal{L}$-sentences ($\mathcal{L}$-formulas without any free variables) α,

$$\mathfrak{A} \models \alpha \quad \text{iff} \quad \mathfrak{B} \models \alpha.$$

It is clear that, if $\mathfrak{A}$ is a simple elementary subsystem of $\mathfrak{B}$, written $\mathfrak{A} \subseteq^s_e \mathfrak{B}$, then $\mathfrak{A} \equiv \mathfrak{B}$.

3.2 Homomorphisms

Suppose that $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle, \mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}} \rangle, R^{\mathfrak{B}} \rangle$ are two $\mathcal{L}$-structure systems. An $(\mathbf{N}^{\mathfrak{A}}, \mathbf{N}^{\mathfrak{B}})$-epimorphic translation $\langle F, \alpha \rangle : \mathrm{SEN}^{\mathfrak{A}} \rightarrow^{se} \mathrm{SEN}^{\mathfrak{B}}$ is said to be an **$\mathcal{L}$-morphism** $\langle F, \alpha \rangle : \mathfrak{A} \rightarrow \mathfrak{B}$ if

- the following triangle commutes:

$$\begin{tikzcd} & \mathbf{F} \arrow[dl, "F^{\mathfrak{A}}"'] \arrow[dr, "F^{\mathfrak{B}}"] & \\ \mathbf{N}^{\mathfrak{A}} \arrow[rr, dash, dashed] & & \mathbf{N}^{\mathfrak{B}} \end{tikzcd} \qquad (1)$$

  where the dashed line represents the two-way correspondence established by the ($\mathbf{N}^{\mathfrak{A}}$, $\mathbf{N}^{\mathfrak{B}}$)-epimorphic property, and

- for all $r \in R$, with $\rho(r) = n$, all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and all $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$,

$$\vec{\phi} \in r^{\mathfrak{A}}_{\Sigma} \quad \text{implies} \quad \alpha_{\Sigma}(\vec{\phi}) \in r^{\mathfrak{B}}_{F(\Sigma)}.$$

If $\langle F, \alpha\rangle$ is injective or surjective, then we write $\langle F, \alpha\rangle : \mathfrak{A} \rightarrowtail \mathfrak{B}$ and $\langle F, \alpha\rangle : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$, respectively. Finally, $\langle F, \alpha\rangle : \mathfrak{A} \cong \mathfrak{B}$ is an **isomorphism** if it is bijective and its inverse mapping is also an $\mathcal{L}$-morphism.

In the following lemma, we establish a property that will prove very useful in the sequel. It gives the analog of the usual universal algebraic homomorphism property in the context of $\mathcal{L}$-structures.

Lemma 2 *Let $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle, \mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle$ be two $\mathcal{L}$-systems and $\langle F, \alpha\rangle : \mathfrak{A} \to \mathfrak{B}$ an $\mathcal{L}$-morphism. Then, for every $t \in \mathrm{Te}_{\mathcal{L}}, \Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|, \vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^{\omega}$,*

$$\alpha_{\Sigma}(t^{\mathfrak{A}}_{\Sigma}(\vec{\phi})) = t^{\mathfrak{B}}_{F(\Sigma)}(\alpha_{\Sigma}(\vec{\phi})).$$

Proof:
Let $t \in \mathrm{Te}_{\mathcal{L}}, \Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|, \vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^{\omega}$. Then

$$\begin{array}{rcl} \alpha_{\Sigma}(t^{\mathfrak{A}}_{\Sigma}(\vec{\phi})) & = & \alpha_{\Sigma}(F^{\mathfrak{A}}(t)_{\Sigma}(\vec{\phi}\restriction_n)) \quad \text{(by definition)} \\ & = & F^{\mathfrak{B}}(t)_{F(\Sigma)}(\alpha_{\Sigma}(\vec{\phi}\restriction_n)) \quad \text{(by the commutativity of Diagram (1)} \\ & & \qquad\qquad \text{and the } (\mathbf{N}^{\mathfrak{A}}, \mathbf{N}^{\mathfrak{B}})\text{-epimorphic property)} \\ & = & t^{\mathfrak{B}}_{F(\Sigma)}(\alpha_{\Sigma}(\vec{\phi})) \quad \text{(again by definition).} \end{array}$$

■

An $\mathcal{L}$-morphism $\langle F, \alpha\rangle : \mathfrak{A} \to \mathfrak{B}$ is said to be a **strong $\mathcal{L}$-morphism**, denoted by $\langle F, \alpha\rangle : \mathfrak{A} \to_s \mathfrak{B}$, if, for all $r \in R$, with $\rho(r) = n$, all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and all $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$,

$$\vec{\phi} \in r^{\mathfrak{A}}_{\Sigma} \quad \text{if and only if} \quad \alpha_{\Sigma}(\vec{\phi}) \in r^{\mathfrak{B}}_{F(\Sigma)}.$$

$\mathcal{L}$-morphisms correspond to semi-interpretations, whereas strong $\mathcal{L}$-morphisms correspond to interpretations in the framework of categorical abstract algebraic logic.

A surjective strong $\mathcal{L}$-morphism is called a **reductive $\mathcal{L}$-morphism**. If $\langle F, \alpha\rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$ is a reductive $\mathcal{L}$-morphism, then $\mathfrak{B}$ is said to be a **reduction** of $\mathfrak{A}$ and $\mathfrak{A}$ an **expansion** of $\mathfrak{B}$, written $\mathfrak{B} \preccurlyeq \mathfrak{A}$ or $\mathfrak{A} \succcurlyeq \mathfrak{B}$.

Given a singleton translation $\langle F,\alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$ and r' an n-ary relation system on SEN', recall the standard convention of using the notation $\alpha^{-1}(r') = \{\alpha_\Sigma^{-1}(r'_{F(\Sigma)})\}_{\Sigma\in|\mathbf{Sign}|}$, for the n-ary relation system on SEN, generated by pulling back signature-wise the relation system r'. This notation will be used in the next lemma, which forms an analog for $\mathcal{L}$-systems of Lemma 1.1 of [13].

Lemma 3 *Suppose that $\langle F,\alpha\rangle : \mathrm{SEN}^{\mathfrak{A}} \to^{se} \mathrm{SEN}^{\mathfrak{B}}$ is an $(\mathbf{N}^{\mathfrak{A}},\mathbf{N}^{\mathfrak{B}})$-epimorphic translation, such that triangle (1) commutes. Then*

1. *$\langle F,\alpha\rangle : \mathfrak{A} \to \mathfrak{B}$ if and only if $r^{\mathfrak{A}} \leq \alpha^{-1}(r^{\mathfrak{B}})$, for all $r \in R$.*
2. *$\langle F,\alpha\rangle : \mathfrak{A} \to_s \mathfrak{B}$ if and only if $r^{\mathfrak{A}} = \alpha^{-1}(r^{\mathfrak{B}})$, for all $r \in R$.*
3. *$\langle F,\alpha\rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$ implies $r^{\mathfrak{A}} = \alpha^{-1}(r^{\mathfrak{B}})$ and $\alpha_\Sigma(r^{\mathfrak{A}}_\Sigma) = r^{\mathfrak{B}}_{F(\Sigma)}$, for all $r \in R$ and all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$.*

Proof: All three statements are easy consequences of the definitions involved. ■

Corollary 4

1. *A bijective strong $\mathcal{L}$-morphism $\langle F,\alpha\rangle : \mathfrak{A} \to \mathfrak{B}$ is an isomorphism.*
2. *If R contains the equality symbol, then reductive $\mathcal{L}$-morphisms coincide with isomorphisms.*

Proof:
The first statement is obvious. For the second, note that reductive $\mathcal{L}$-morphisms are surjective and strong and, moreover, if the language contains equality, then they are also injective. ■

Let $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$ and $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle$ be $\mathcal{L}$-systems and suppose that $\langle F,\alpha\rangle : \mathfrak{A} \to \mathfrak{B}$ is an $\mathcal{L}$-morphism.

Define the triple $\alpha^{-1}(\mathfrak{B}) = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\alpha^{-1}(\mathfrak{B})}\rangle$ by letting, for all $r \in R$, with $\rho(r) = n$, and all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, $r_\Sigma^{\alpha^{-1}(\mathfrak{B})} \subseteq \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$ be given by

$$r_\Sigma^{\alpha^{-1}(\mathfrak{B})} = \alpha_\Sigma^{-1}(r^{\mathfrak{B}}_{F(\Sigma)}).$$

Then, the following lemma, forming an analog of Lemma 1.2 of [13], asserts that, the restriction of the $\mathcal{L}$-morphism $\langle F,\alpha\rangle : \mathfrak{A} \to \mathfrak{B}$ to $\alpha^{-1}(\mathfrak{B})$ is a strong $\mathcal{L}$-morphism.

Lemma 5 *Let $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$ and $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle$ be $\mathcal{L}$-systems and $\langle F,\alpha\rangle : \mathfrak{A} \to \mathfrak{B}$ an $\mathcal{L}$-morphism.*

1. *$\langle F,\alpha\rangle \restriction_{\alpha^{-1}(\mathfrak{B})} : \alpha^{-1}(\mathfrak{B}) \to_s \mathfrak{B}$ is a strong $\mathcal{L}$-morphism.*
2. *If $\langle F,\alpha\rangle$ is surjective, then $\langle F,\alpha\rangle \restriction_{\alpha^{-1}(\mathfrak{B})} : \alpha^{-1}(\mathfrak{B}) \twoheadrightarrow_s \mathfrak{B}$ is a reductive $\mathcal{L}$-morphism.*

Proof:

1. The proof of this statement follows directly by the definition of $\alpha^{-1}(\mathfrak{B})$.

2. Just combine the statement of Part 1 with the hypothesis of Part 2.

■

Let $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$ and $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle$ be $\mathcal{L}$-systems and suppose that $\langle F, \alpha\rangle : \mathfrak{A} \to \mathfrak{B}$ is an $\mathcal{L}$-morphism. Assume that $\mathfrak{C} = \langle \mathbf{Sign}^{\mathfrak{C}}, \langle \mathbf{N}^{\mathfrak{C}}, F^{\mathfrak{C}}\rangle, R^{\mathfrak{C}}\rangle$ is a subsystem of $\mathfrak{A}$ and that $\mathfrak{D} = \langle \mathbf{Sign}^{\mathfrak{D}}, \langle \mathbf{N}^{\mathfrak{D}}, F^{\mathfrak{D}}\rangle, R^{\mathfrak{D}}\rangle$ is a subsystem of $\mathfrak{B}$.

Define, first, the triple $\alpha^{-1}(\mathfrak{D}) = \langle \mathrm{SEN}^{\alpha^{-1}(\mathfrak{D})}, \langle \mathbf{N}^{\alpha^{-1}(\mathfrak{D})}, F^{\alpha^{-1}(\mathfrak{D})}\rangle, R^{\alpha^{-1}(\mathfrak{D})}\rangle$ by setting:

- $\mathrm{SEN}^{\alpha^{-1}(\mathfrak{D})} : F^{-1}(\mathbf{Sign}^{\mathfrak{D}}) \to \mathbf{Set}$ be given by $\mathrm{SEN}^{\alpha^{-1}(\mathfrak{D})}(\Sigma) = \alpha_{\Sigma}^{-1}(\mathrm{SEN}^{\mathfrak{D}}(F(\Sigma)))$, for all $\Sigma \in |F^{-1}(\mathbf{Sign}^{\mathfrak{D}})|$, and, $\mathrm{SEN}^{\alpha^{-1}(\mathfrak{D})}(f) = \mathrm{SEN}^{\mathfrak{A}}(f)$, for all $f \in F^{-1}(\mathbf{Sign}^{\mathfrak{D}})(\Sigma, \Sigma')$,

- for all n-ary σ in $\mathbf{F}$, $F^{\alpha^{-1}(\mathfrak{D})}(\sigma)$ is the restriction of $F^{\mathfrak{A}}(\sigma)$ to $\alpha_{\Sigma}^{-1}(\mathrm{SEN}^{\mathfrak{D}}(F(\Sigma)))^n$, and

- for all $r \in R$, with $\rho(r) = n$, and all $\Sigma \in |\mathbf{Sign}^{\alpha^{-1}(\mathfrak{D})}|$, $r_{\Sigma}^{\alpha^{-1}(\mathfrak{D})} \subseteq \mathrm{SEN}^{\alpha^{-1}(\mathfrak{D})}(\Sigma)^n$, is given by
$$r_{\Sigma}^{\alpha^{-1}(\mathfrak{D})} = \alpha_{\Sigma}^{-1}(r_{F(\Sigma)}^{\mathfrak{D}}).$$

Now, if $F : \mathbf{Sign}^{\mathfrak{A}} \to \mathbf{Sign}^{\mathfrak{B}}$ is injective and $F(\mathbf{Sign}^{\mathfrak{C}})$ is a subcategory of $\mathbf{Sign}^{\mathfrak{B}}$, define the triple $\alpha(\mathfrak{C}) = \langle \mathrm{SEN}^{\alpha(\mathfrak{C})}, \langle \mathbf{N}^{\alpha(\mathfrak{C})}, F^{\alpha(\mathfrak{C})}\rangle, R^{\alpha(\mathfrak{C})}\rangle$ by setting:

- $\mathrm{SEN}^{\alpha(\mathfrak{C})} : F(\mathbf{Sign}^{\mathfrak{C}}) \to \mathbf{Set}$ be given by $\mathrm{SEN}^{\alpha(\mathfrak{C})}(F(\Sigma)) = \alpha_{\Sigma}(\mathrm{SEN}^{\mathfrak{C}}(\Sigma))$, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{C}}|$, and, given $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}^{\mathfrak{C}}|, f \in \mathbf{Sign}^{\mathfrak{C}}(\Sigma_1, \Sigma_2)$, $\mathrm{SEN}^{\alpha(\mathfrak{C})}(F(f)) = \mathrm{SEN}^{\mathfrak{B}}(F(f))$,

- for all n-ary σ in $\mathbf{F}$, $F^{\alpha(\mathfrak{C})}(\sigma)$ is the restriction of $F^{\mathfrak{B}}(\sigma)$ to $\alpha_{\Sigma}(\mathrm{SEN}^{\mathfrak{C}}(\Sigma))^n$ and

- for all $r \in R$, with $\rho(r) = n$, and all $F(\Sigma) \in |\mathbf{Sign}^{\alpha(\mathfrak{C})}|$, $r_{F(\Sigma)}^{\alpha(\mathfrak{C})} \subseteq \mathrm{SEN}^{\alpha(\mathfrak{C})}(F(\Sigma))^n$, is given by
$$r_{F(\Sigma)}^{\alpha(\mathfrak{C})} = \alpha_{\Sigma}(r_{\Sigma}^{\mathfrak{C}}).$$

It is now shown that if $\langle F, \alpha\rangle : \mathfrak{A} \to_s \mathfrak{B}$ is a strong $\mathcal{L}$-morphism, then, for all $\mathfrak{D} \subseteq \mathfrak{B}$, we have that $\alpha^{-1}(\mathfrak{D}) \subseteq \mathfrak{A}$ and that, if $\mathfrak{C} \subseteq \mathfrak{A}$, then $\alpha(\mathfrak{C}) \subseteq \mathfrak{B}$, when $\alpha(\mathfrak{C})$ is defined.

Lemma 6 *Let $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$ and $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle$ be $\mathcal{L}$-systems and $\langle F, \alpha\rangle : \mathfrak{A} \to_s \mathfrak{B}$ a strong $\mathcal{L}$-morphism.*

1. *If $\mathfrak{D} \subseteq \mathfrak{B}$, then $\alpha^{-1}(\mathfrak{D}) \subseteq \mathfrak{A}$.*

2. *If $\mathfrak{C} \subseteq \mathfrak{A}$, $F : \mathbf{Sign}^{\mathfrak{A}} \to \mathbf{Sign}^{\mathfrak{B}}$ is injective and $F(\mathbf{Sign}^{\mathfrak{C}})$ is a subcategory of $\mathbf{Sign}^{\mathfrak{B}}$, then $\alpha(\mathfrak{C}) \subseteq \mathfrak{B}$.*

Proof:

1. First, to see that $\mathrm{SEN}^{\alpha^{-1}(\mathfrak{D})}$ is well-defined at the morphism level, suppose that $\Sigma_1, \Sigma_2 \in |F^{-1}(\mathbf{Sign}^{\mathfrak{D}})|$, $f \in F^{-1}(\mathbf{Sign}^{\mathfrak{D}})(\Sigma_1, \Sigma_2)$ and $\phi \in \alpha_{\Sigma_1}^{-1}(\mathrm{SEN}^{\mathfrak{D}}(F(\Sigma_1)))$. Then we have
$$\begin{array}{rcl}\alpha_{\Sigma_2}(\mathrm{SEN}^{\alpha^{-1}(\mathfrak{D})}(f)(\phi)) & = & \alpha_{\Sigma_2}(\mathrm{SEN}^{\mathfrak{A}}(f)(\phi)) \\ & = & \mathrm{SEN}^{\mathfrak{B}}(F(f))(\alpha_{\Sigma_1}(\phi)) \\ & \in & \mathrm{SEN}^{\mathfrak{B}}(F(f))(\alpha_{\Sigma_1}(\alpha_{\Sigma_1}^{-1}(\mathrm{SEN}^{\mathfrak{D}}(F(\Sigma_1))))) \\ & \subseteq & \mathrm{SEN}^{\mathfrak{B}}(F(f))(\mathrm{SEN}^{\mathfrak{D}}(F(\Sigma_1))) \\ & \subseteq & \mathrm{SEN}^{\mathfrak{D}}(F(\Sigma_2)).\end{array}$$
Thus $\mathrm{SEN}^{\alpha^{-1}(\mathfrak{D})}(f)(\mathrm{SEN}^{\alpha^{-1}(\mathfrak{D})}(\Sigma_1)) \subseteq \mathrm{SEN}^{\alpha^{-1}(\mathfrak{D})}(\Sigma_2)$ and, hence, $\mathrm{SEN}^{\alpha^{-1}(\mathfrak{D})}$ is well-defined on morphisms.

   Next, to see that $\mathbf{N}^{\mathfrak{A}}$ restricts to a category of natural transformations on $\mathrm{SEN}^{\alpha^{-1}(\mathfrak{D})}$, suppose that $t \in \mathrm{Te}_{\mathcal{L}}$, $\Sigma \in |F^{-1}(\mathbf{Sign}^{\mathfrak{D}})|$ and $\vec{\phi} \in \alpha_{\Sigma}^{-1}(\mathrm{SEN}^{\mathfrak{D}}(F(\Sigma)))^{\omega}$. Then
$$\begin{array}{rcl}\alpha_{\Sigma}(t_{\Sigma}^{\alpha^{-1}(\mathfrak{D})}(\vec{\phi})) & = & \alpha_{\Sigma}(t_{\Sigma}^{\mathfrak{A}}(\vec{\phi})) \\ & = & t_{F(\Sigma)}^{\mathfrak{B}}(\alpha_{\Sigma}(\vec{\phi})) \\ & \in & \mathrm{SEN}^{\mathfrak{D}}(F(\Sigma)) \quad \text{(by the } \mathbf{N}^{\mathfrak{B}}\text{-subfunctor property)},\end{array}$$
whence $t_{\Sigma}^{\alpha^{-1}(\mathfrak{D})}(\vec{\phi}) \in \alpha_{\Sigma}^{-1}(\mathrm{SEN}^{\mathfrak{D}}(F(\Sigma)))$. Finally, the fact that $\langle F, \alpha\rangle$ strong implies that $\alpha^{-1}(\mathfrak{D})$ is an $\mathcal{L}$-subsystem of $\mathfrak{A}$ is fairly obvious.

2. We follow a similar order as in Part 1. To see that $\mathrm{SEN}^{\alpha(\mathfrak{C})}$ is well-defined at the morphism level, suppose that $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}^{\mathfrak{C}}|$, $f \in \mathbf{Sign}^{\mathfrak{C}}(\Sigma_1, \Sigma_2)$ and $\phi \in \mathrm{SEN}^{\mathfrak{C}}(\Sigma_1)$. Then we have
$$\begin{array}{rcl}\mathrm{SEN}^{\alpha(\mathfrak{C})}(F(f))(\alpha_{\Sigma_1}(\phi)) & = & \mathrm{SEN}^{\mathfrak{B}}(F(f))(\alpha_{\Sigma_1}(\phi)) \\ & = & \alpha_{\Sigma_2}(\mathrm{SEN}^{\mathfrak{A}}(f)(\phi)) \\ & = & \alpha_{\Sigma_2}(\mathrm{SEN}^{\mathfrak{C}}(f)(\phi)) \\ & \in & \alpha_{\Sigma_2}(\mathrm{SEN}^{\mathfrak{C}}(\Sigma_2)) \\ & = & \mathrm{SEN}^{\alpha(\mathfrak{C})}(F(\Sigma_2)).\end{array}$$
Thus $\mathrm{SEN}^{\alpha(\mathfrak{C})}(f)$ is well-defined on morphisms.

   Next, to see that $\mathbf{N}^{\mathfrak{B}}$ restricts to a category of natural transformations on $\mathrm{SEN}^{\alpha(\mathfrak{C})}$, suppose that $t \in \mathrm{Te}_{\mathcal{L}}$, $\Sigma \in |\mathbf{Sign}^{\mathfrak{C}}|$ and $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{C}}(\Sigma)^{\omega}$. Then
$$\begin{array}{rcl}t_{F(\Sigma)}^{\alpha(\mathfrak{C})}(\alpha_{\Sigma}(\vec{\phi})) & = & t_{F(\Sigma)}^{\mathfrak{B}}(\alpha_{\Sigma}(\vec{\phi})) \\ & = & \alpha_{\Sigma}(t_{\Sigma}^{\mathfrak{A}}(\vec{\phi})) \\ & = & \alpha_{\Sigma}(t_{\Sigma}^{\mathfrak{C}}(\vec{\phi})) \\ & \in & \alpha_{\Sigma}(\mathrm{SEN}^{\mathfrak{C}}(\Sigma)) \\ & = & \mathrm{SEN}^{\alpha(\mathfrak{C})}(F(\Sigma)),\end{array}$$
whence $t^{\alpha(\mathfrak{C})}$ is also well-defined. The fact that $\langle F, \alpha\rangle$ strong implies that $\alpha(\mathfrak{C})$ is an $\mathcal{L}$-subsystem of $\mathfrak{B}$ is also obvious.

■

An $\mathcal{L}$-morphism $\langle F, \alpha \rangle : \mathfrak{A} \to \mathfrak{B}$ is said to be **elementary** if, for every $\mathcal{L}$-formula γ, every $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and all $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^{\omega}$,

$$\mathfrak{A} \models_{\Sigma} \gamma[\vec{\phi}] \quad \text{iff} \quad \mathfrak{B} \models_{F(\Sigma)} \gamma[\alpha_{\Sigma}(\vec{\phi})].$$

In that case, we write $\langle F, \alpha \rangle : \mathfrak{A} \to_e \mathfrak{B}$. It is clear from the definitions involved that, if $\langle F, \alpha \rangle : \mathfrak{A} \twoheadrightarrow_e \mathfrak{B}$, then $\mathfrak{A} \equiv \mathfrak{B}$.

Proposition 7 *Every reductive $\mathcal{L}$-morphism is elementary.*

Proof:

Suppose that $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle, \mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}} \rangle, R^{\mathfrak{B}} \rangle$ are $\mathcal{L}$-structures and $\langle F, \alpha \rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$ is a reductive $\mathcal{L}$-morphism. It is shown that, for every $\mathcal{L}$-formula γ, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|, \vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^{\omega}$, we have that

$$\mathfrak{A} \models_{\Sigma} \gamma[\vec{\phi}] \quad \text{iff} \quad \mathfrak{B} \models_{F(\Sigma)} \gamma[\alpha_{\Sigma}(\vec{\phi})].$$

The proof goes by induction on the complexity of γ.

If $\gamma = r(t_0, \ldots, t_{n-1})$ is atomic, then

$$\begin{array}{lll} \mathfrak{A} \models_{\Sigma} \gamma[\vec{\phi}] & \text{iff} & \langle t^{\mathfrak{A}}_{0_{\Sigma}}(\vec{\phi}), \ldots, t^{\mathfrak{A}}_{n-1_{\Sigma}}(\vec{\phi}) \rangle \in r^{\mathfrak{A}}_{\Sigma} \quad \text{(by definition)} \\ & \text{iff} & \langle \alpha_{\Sigma}(t^{\mathfrak{A}}_{0_{\Sigma}}(\vec{\phi})), \ldots, \alpha_{\Sigma}(t^{\mathfrak{A}}_{n-1_{\Sigma}}(\vec{\phi})) \rangle \in r^{\mathfrak{B}}_{F(\Sigma)} \quad (\text{since } \langle F, \alpha \rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}) \\ & \text{iff} & \langle t^{\mathfrak{B}}_{0_{F(\Sigma)}}(\alpha_{\Sigma}(\vec{\phi})), \ldots, t^{\mathfrak{B}}_{n-1_{F(\Sigma)}}(\alpha_{\Sigma}(\vec{\phi})) \rangle \in r^{\mathfrak{B}}_{F(\Sigma)} \quad \text{(by Lemma 2)} \\ & \text{iff} & \mathfrak{B} \models_{F(\Sigma)} \gamma[\alpha_{\Sigma}(\vec{\phi})] \quad \text{(by definition).} \end{array}$$

If $\gamma = (\gamma_1 \wedge \gamma_2)$, then

$$\begin{array}{lll} \mathfrak{A} \models_{\Sigma} \gamma[\vec{\phi}] & \text{iff} & \mathfrak{A} \models_{\Sigma} \gamma_1[\vec{\phi}] \quad \text{and} \quad \mathfrak{A} \models_{\Sigma} \gamma_2[\vec{\phi}] \quad \text{(by definition)} \\ & \text{iff} & \mathfrak{B} \models_{F(\Sigma)} \gamma_1[\alpha_{\Sigma}(\vec{\phi})] \quad \text{and} \quad \mathfrak{B} \models_{F(\Sigma)} \gamma_2[\alpha_{\Sigma}(\vec{\phi})] \\ & & \hfill \text{(by the induction hypothesis)} \\ & \text{iff} & \mathfrak{B} \models_{F(\Sigma)} \gamma[\alpha_{\Sigma}(\vec{\phi})] \quad \text{(by definition).} \end{array}$$

The case of negation may be handled similarly. Suppose, now, that $\gamma = (\exists i)\gamma'$. Then

$$\begin{array}{lll} \mathfrak{A} \models_{\Sigma} (\exists i)\gamma'[\vec{\phi}] & \text{iff} & (\exists \vec{\psi} : \psi_j = \phi_j, j \neq i)(\mathfrak{A} \models_{\Sigma} \gamma'[\vec{\psi}]) \quad \text{(by definition)} \\ & \text{iff} & (\exists \vec{\psi} : \psi_j = \phi_j, j \neq i)(\mathfrak{B} \models_{F(\Sigma)} \gamma'[\alpha_{\Sigma}(\vec{\psi})]) \\ & & \hfill \text{(by the induction hypothesis)} \\ & \text{iff} & \mathfrak{B} \models_{F(\Sigma)} (\exists i)\gamma'[\alpha(\vec{\phi})] \quad \text{(by definition and} \\ & & \hfill \text{the surjectivity of } \langle F, \alpha \rangle). \end{array}$$

■

3.3 Reduced Products

Recall from Section 3 of [33] the definitions of a product functor, category of natural transformations on the product functor and product translation.

Suppose that $\mathfrak{A}^i = \langle \mathrm{SEN}^i, \langle \mathbf{N}^i, F^i\rangle, R^i\rangle$, $i \in I$, is a family of $\mathcal{L}$-systems. The **direct product** of the $\mathfrak{A}^i$ is defined by

$$\prod_{i\in I}\mathfrak{A}^i = \langle \prod_{i\in I}\mathrm{SEN}^i, \langle \prod_{i\in I}\mathbf{N}^i, \prod_{i\in I}F^i\rangle, \prod_{i\in I}R^i\rangle,$$

where $\prod_{i\in I}R^i = \{\prod_{i\in I}r^i : r \in R\}$ is defined, for every $r \in R$, with $\rho(r) = n$, by setting, for all $\Sigma_i \in |\mathbf{Sign}^i|, \phi^j_i \in \mathrm{SEN}^i(\Sigma_i), i \in I, j = 0,\dots,n-1$,

$$\langle \vec{\phi}^0,\dots,\vec{\phi}^{n-1}\rangle \in \prod_{i\in I} r^i_{\prod_{i\in I}\Sigma_i} \quad \text{iff} \quad \langle \phi^0_i,\dots,\phi^{n-1}_i\rangle \in r^i_{\Sigma_i}, \text{ for all } i \in I.$$

In case $I = \emptyset$, then the trivial system is obtained by taking the product $\prod\emptyset$.

Suppose now, that, in addition to the structure systems $\mathfrak{A}^i, i \in I$, we are given a filter $\mathcal{F}$ on I. Define the equivalence system $\equiv^{\mathcal{F}} = \{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i}\}_{\prod_{i\in I}\Sigma_i \in |\prod_{i\in I}\mathbf{Sign}^i|}$ on $\prod_{i\in I}\mathrm{SEN}^i$ by setting, for all $\Sigma_i \in |\mathbf{Sign}^i|, \phi_i, \psi_i \in \mathrm{SEN}^i(\Sigma_i), i \in I$,

$$\vec{\phi} \equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i} \vec{\psi} \quad \text{iff} \quad \{i \in I : \phi_i = \psi_i\} \in \mathcal{F}.$$

This is also a $\prod_{i\in I}\mathbf{N}^i$-congruence system on $\prod_{i\in I}\mathrm{SEN}^i$. We can thus define the reduced product functor $\prod^{\mathcal{F}}_{i\in I}\mathrm{SEN}^i = \prod_{i\in I}\mathrm{SEN}^i/{\equiv^{\mathcal{F}}}$, with category of natural transformations $\prod^{\mathcal{F}}_{i\in I}\mathbf{N}^i = \prod_{i\in I}\mathbf{N}^i/{\equiv^{\mathcal{F}}}$ via $\prod^{\mathcal{F}}_{i\in I}F^i : \mathbf{F} \to \prod_{i\in I}\mathbf{N}^i/{\equiv^{\mathcal{F}}}$, which is defined by composing the product functor $\prod_{i\in I}F^i : \mathbf{F} \to \prod_{i\in I}\mathbf{N}^i$ with the quotient functor $P^{\mathcal{F}} : \prod_{i\in I}\mathbf{N}^i \to \prod_{i\in I}\mathbf{N}^i/{\equiv^{\mathcal{F}}}$.

$$\mathbf{F} \xrightarrow{\prod_{i\in I}F^i} \prod_{i\in I}\mathbf{N}^i \xrightarrow{P^{\mathcal{F}}} \prod_{i\in I}\mathbf{N}^i/{\equiv^{\mathcal{F}}}$$

We may also define the relation system $\prod^{\mathcal{F}}_{i\in I}R^i$ on $\prod^{\mathcal{F}}_{i\in I}\mathrm{SEN}^i$ by setting, for all $r \in R$, with $\rho(r) = n$, and all $\Sigma_i \in |\mathbf{Sign}^i|, \phi^j_i \in \mathrm{SEN}^i(\Sigma_i), i \in I, j = 0,\dots,n-1$,

$$\langle \vec{\phi}^0/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i}},\dots,\vec{\phi}^{n-1}/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i}}\rangle \in \prod^{\mathcal{F}}_{i\in I} r^i_{\prod_{i\in I}\Sigma_i} \quad \text{iff} \quad \{i \in I : \langle \phi^0_i,\dots,\phi^{n-1}_i\rangle \in r^i_{\Sigma_i}\} \in \mathcal{F}.$$

It may be shown that this definition is independent of the choice of representatives and, thus, well-defines a relation system $\prod^{\mathcal{F}}_{i\in I}r^i$, for all $r \in R$, on $\prod^{\mathcal{F}}_{i\in I}\mathrm{SEN}^i$.

In fact, note that, if $\vec{\psi^0},\dots,\vec{\psi^{n-1}} \in \prod_{i\in I}\mathrm{SEN}^i(\Sigma_i)$ are such that $\vec{\phi^j} \equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i} \vec{\psi^j}$, for all $j = 0,\dots,n-1$, then we have that $\{i \in I : \phi^j_i = \psi^j_i\} \in \mathcal{F}$, for all $j = 0,\dots,n-1$, whence $\bigcap_{j=0}^{n-1}\{i \in I : \phi^j_i = \psi^j_i\} \in \mathcal{F}$. Therefore, if $\{i \in I : \langle \phi^0_i,\dots,\phi^{n-1}_i\rangle \in r^i_{\Sigma_i}\} \in \mathcal{F}$, then

$$\{i \in I : \langle \psi^0_i,\dots,\psi^{n-1}_i\rangle \in r^i_{\Sigma_i}\} \supseteq \{i \in I : \langle \phi^0_i,\dots,\phi^{n-1}_i\rangle \in r^i_{\Sigma_i}\} \cap \bigcap_{j=0}^{n-1}\{i \in I : \phi^j_i = \psi^j_i\} \in \mathcal{F}$$

and, hence, $\{i \in I : \langle \psi_i^0, \ldots, \psi_i^{n-1} \rangle \in r^i_{\Sigma_i}\} \in \mathcal{F}$. By symmetry, we get that

$$\{i \in I : \langle \psi_i^0, \ldots, \psi_i^{n-1} \rangle \in r^i_{\Sigma_i}\} \in \mathcal{F} \quad \text{iff} \quad \{i \in I : \langle \phi_i^0, \ldots, \phi_i^{n-1} \rangle \in r^i_{\Sigma_i}\} \in \mathcal{F},$$

i.e., that

$$\langle \vec{\phi}^0/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i}}, \ldots, \vec{\phi}^{n-1}/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i}} \rangle \in \textstyle\prod^{\mathcal{F}}_{i\in I} r^i_{\prod_{i\in I}\Sigma_i} \quad \text{iff} \quad \langle \vec{\psi}^0/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i}}, \ldots, \vec{\psi}^{n-1}/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i}} \rangle \in \textstyle\prod^{\mathcal{F}}_{i\in I} r^i_{\prod_{i\in I}\Sigma_i}.$$

Now, let

$$\prod_{i\in I}^{\mathcal{F}} \mathfrak{A}^i = \langle \prod_{i\in I}^{\mathcal{F}} \mathrm{SEN}^i, \langle \prod_{i\in I}^{\mathcal{F}} \mathbf{N}^i, \prod_{i\in I}^{\mathcal{F}} F^i \rangle, \prod_{i\in I}^{\mathcal{F}} R^i \rangle$$

be the **reduced product** of the structure systems $\mathfrak{A}^i, i \in I$, **by the filter** $\mathcal{F}$ on I. As is customary, the reduced product by an ultrafilter is termed an **ultraproduct**.

4 Łoś' Ultraproduct Theorem for $\mathcal{L}$-Systems

Recall the definition of an arbitrary first-order Horn formula. Theorem 8 is an extension of a classical theorem from first-order model theory to the model theory of $\mathcal{L}$-systems. It states, roughly speaking, that, for a given Horn formula, a given indexed collection of $\mathcal{L}$-systems and a given proper filter on the index set, if the set of all indices for which the formula is satisfied at a given tuple of elements in the corresponding structure belongs to the filter, then the formula is also satisfied by the equivalence class of the product tuple in the filtered product of the indexed family of the $\mathcal{L}$-systems by that filter.

Theorem 8 *Suppose that α is an arbitrary Horn $\mathcal{L}$-formula and let $\mathfrak{A}^i = \langle \mathrm{SEN}^i, \langle \mathbf{N}^i, F^i \rangle, R^i \rangle, i \in I$, be a family of $\mathcal{L}$-systems, with $I \neq \emptyset$. Let $\mathcal{F}$ be a proper filter on I. If $\Sigma_i \in |\mathbf{Sign}^i|, \phi_i^j \in \mathrm{SEN}^i(\Sigma_i), i \in I, j \in \omega$, then*

$$\{i \in I : \mathfrak{A}^i \models_{\Sigma_i} \alpha[\vec{\phi}_i]\} \in \mathcal{F} \quad \textit{implies} \quad \prod_{i\in I}^{\mathcal{F}} \mathfrak{A}^i \models_{\prod_{i\in I}\Sigma_i} \alpha[\vec{\phi}/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i}}].$$

The above implication becomes an equivalence in case α is an atomic $\mathcal{L}$-formula.

Proof:

Suppose, first, that $\alpha = \bigwedge_{j=0}^{n-1} r_j(t_0^j, \ldots, t_{k_j-1}^j) \to \bot$, where $r_j(t_0^j, \ldots, t_{k_j-1}^j), j < n$, are atomic. Then we have

$$\begin{array}{ll} & \{i \in I : \mathfrak{A}^i \models_{\Sigma_i} \bigwedge_{j=0}^{n-1} r_j(t_0^j, \ldots, t_{k_j-1}^j) \to \bot[\vec{\phi}_i]\} \in \mathcal{F} \\ \text{iff} & \{i \in I : \mathfrak{A}^i \models_{\Sigma_i} \bigvee_{j=0}^{n-1} \neg r_j(t_0^j, \ldots, t_{k_j-1}^j)[\vec{\phi}_i]\} \in \mathcal{F} \\ \text{iff} & \bigcup_{j=0}^{n-1}\{i \in I : \mathfrak{A}^i \models_{\Sigma_i} \neg r_j(t_0^j, \ldots, t_{k_j-1}^j)[\vec{\phi}_i]\} \in \mathcal{F} \\ \text{iff} & \bigcup_{j=0}^{n-1}\{i \in I : \mathfrak{A}^i \models_{\Sigma_i} r_j(t_0^j, \ldots, t_{k_j-1}^j)[\vec{\phi}_i]\}^c \in \mathcal{F} \\ \text{iff} & (\bigcap_{j=0}^{n-1}\{i \in I : \mathfrak{A}^i \models_{\Sigma_i} r_j(t_0^j, \ldots, t_{k_j-1}^j)[\vec{\phi}_i]\})^c \in \mathcal{F} \\ \text{implies} & \bigcap_{j=0}^{n-1}\{i \in I : \mathfrak{A}^i \models_{\Sigma_i} r_j(t_0^j, \ldots, t_{k_j-1}^j)[\vec{\phi}_i]\} \notin \mathcal{F} \end{array}$$

$$\begin{array}{ll}
\text{iff} & (\exists j<n)(\{i\in I:\mathfrak{A}^i\models_{\Sigma_i} r_j(t^j_0,\ldots,t^j_{k_j-1})[\vec{\phi}_i]\}\notin\mathcal{F})\\
\text{iff} & (\exists j<n)(\prod^{\mathcal{F}}_{i\in I}\mathfrak{A}^i\not\models_{\prod_{i\in I}\Sigma_i} r_j(t^j_0,\ldots,t^j_{k_j-1})[\vec{\phi}/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i}}])\\
\text{iff} & (\exists j<n)(\prod^{\mathcal{F}}_{i\in I}\mathfrak{A}^i\models_{\prod_{i\in I}\Sigma_i} \neg r_j(t^j_0,\ldots,t^j_{k_j-1})[\vec{\phi}/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i}}])\\
\text{iff} & \prod^{\mathcal{F}}_{i\in I}\mathfrak{A}^i\models_{\prod_{i\in I}\Sigma_i} \bigvee_{j=0}^{n-1}\neg r_j(t^j_0,\ldots,t^j_{k_j-1})[\vec{\phi}/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i}}]\\
\text{iff} & \prod^{\mathcal{F}}_{i\in I}\mathfrak{A}^i\models_{\prod_{i\in I}\Sigma_i} \bigwedge_{j=0}^{n-1} r_j(t^j_0,\ldots,t^j_{k_j-1})\to\bot[\vec{\phi}/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i}}].
\end{array}$$

Suppose, next that $\alpha=\bigwedge_{j=0}^{n-1} r_j(t^j_0,\ldots,t^j_{k_j-1})\to r_n(t^n_0,\ldots,t^n_{k_n-1})$ and that

$$\{i\in I:\mathfrak{A}^i\models_{\Sigma_i}\bigwedge_{j=0}^{n-1} r_j(t^j_0,\ldots,t^j_{k_j-1})\to r_n(t^n_0,\ldots,t^n_{k_n-1})[\vec{\phi}_i]\}\in\mathcal{F}. \tag{2}$$

Now, if $\prod^{\mathcal{F}}_{i\in I}\mathfrak{A}^i\models_{\prod_{i\in I}\Sigma_i} r_j(t^j_0,\ldots,t^j_{k_j-1})[\vec{\phi}/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i}}]$, for all $j=0,\ldots,n-1$, we get that, for all $j=0,\ldots,n-1$, $\{i\in I:\mathfrak{A}^i\models_{\Sigma_i} r_j(t^j_0,\ldots,t^j_{k_j-1})[\vec{\phi}_i]\}\in\mathcal{F}$. Therefore

$$\bigcap_{j=0}^{n-1}\{i\in I:\mathfrak{A}^i\models_{\Sigma_i} r_j(t^j_0,\ldots,t^j_{k_j-1})[\vec{\phi}_i]\}\in\mathcal{F}. \tag{3}$$

But note that

$$\begin{array}{r}
\{i\in I:\mathfrak{A}^i\models_{\Sigma_i} r_n(t^n_0,\ldots,t^n_{k_n-1})[\vec{\phi}_i]\}\supseteq\bigcap_{j=0}^{n-1}\{i\in I:\mathfrak{A}^i\models_{\Sigma_i} r_j(t^j_0,\ldots,t^j_{k_j-1})[\vec{\phi}_i]\}\cap\\
\{i\in I:\mathfrak{A}^i\models_{\Sigma_i}\bigwedge_{j=0}^{n-1} r_j(t^j_0,\ldots,t^j_{k_j-1})\to r_n(t^n_0,\ldots,t^n_{k_n-1})[\vec{\phi}_i]\},
\end{array}$$

whence, by Conditions (2) and (3), we obtain that $\{i\in I:\mathfrak{A}^i\models_{\Sigma_i} r_n(t^n_0,\ldots,t^n_{k_n-1})[\vec{\phi}_i]\}\in\mathcal{F}$, giving that $\prod^{\mathcal{F}}_{i\in I}\mathfrak{A}^i\models_{\prod_{i\in I}\Sigma_i}\bigwedge_{j=0}^{n-1} r_j(t^j_0,\ldots,t^j_{k_j-1})\to r_n(t^n_0,\ldots,t^n_{k_n-1})[\vec{\phi}/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i}}]$.

To finish the proof, it suffices now to show that, if the conclusion holds for the formulas α,α_1 and α_2, then it holds also for $(\alpha_1\wedge\alpha_2)$ and $(\exists k)\alpha$ and $(\forall k)\alpha$.

Suppose, first, that $\{i\in I:\mathfrak{A}^i\models_{\Sigma_i}(\alpha_1\wedge\alpha_2)[\vec{\phi}_i]\}\in\mathcal{F}$. Then $\bigcap_{j=1}^{2}\{i\in I:\mathfrak{A}^i\models_{\Sigma_i}\alpha_j[\vec{\phi}_i]\}\in\mathcal{F}$. This immediately yields that $\{i\in I:\mathfrak{A}^i\models_{\Sigma_i}\alpha_j[\vec{\phi}_i]\}\in\mathcal{F}$, for $j=1,2$, whence, by the induction hypothesis, $\prod_{i\in I}\mathfrak{A}^i\models_{\prod_{i\in I}\Sigma_i}\alpha_j[\vec{\phi}/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i}}]$, for $j=1,2$, which, finally, gives that $\prod_{i\in I}\mathfrak{A}^i\models_{\prod_{i\in I}\Sigma_i}(\alpha_1\wedge\alpha_2)[\vec{\phi}/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i}}]$.

For the existential quantification we have

$$\begin{array}{ll}
 & \{i\in I:\mathfrak{A}^i\models_{\Sigma_i}(\exists k)\alpha[\vec{\phi}_i]\}\in\mathcal{F}\\
\text{iff} & \{i\in I:\mathfrak{A}^i\models_{\Sigma_i}\alpha[\vec{\psi}_i],\ \text{for some } \vec{\psi}_i:\psi^j_i=\phi^j_i, j\neq k\}\in\mathcal{F}\\
\text{implies} & \{i\in I:\mathfrak{A}^i\models_{\Sigma_i}\alpha[\vec{\psi}_i]\}\in\mathcal{F},\\
 & \text{for some } \vec{\psi}\in(\prod_{i\in I}\mathrm{SEN}^i(\Sigma_i))^\omega:\vec{\psi}^j\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i}\vec{\phi}^j, j\neq k,\\
\text{iff} & \prod^{\mathcal{F}}_{i\in I}\mathfrak{A}^i\models_{\prod_{i\in I}\Sigma_i}\alpha[\vec{\psi}/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i}}],\\
 & \text{for some } \vec{\psi}\in(\prod_{i\in I}\mathrm{SEN}^i(\Sigma_i))^\omega:\vec{\psi}^j\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i}\vec{\phi}^j, j\neq k,\\
\text{iff} & \prod^{\mathcal{F}}_{i\in I}\mathfrak{A}^i\models_{\prod_{i\in I}\Sigma_i}(\exists k)\alpha[\vec{\phi}/{\equiv^{\mathcal{F}}_{\prod_{i\in I}\Sigma_i}}].
\end{array}$$

A similar reasoning works also for the universal quantification. ■

Finally, we present the main theorem of the paper, an analog of Łos' Ultraproduct Theorem for $\mathcal{L}$-systems. The original reference for Łos' Ultraproduct Theorem is Łos' 1955 paper [21]. See also Theorem 4.1.9 of [6] and Theorem 9.5.1 of [20].

Theorem 9 (Łoś' Ultraproduct Theorem) *Let $I \neq \emptyset$ be a set, $\mathfrak{A}^i = \langle \mathrm{SEN}^i, \langle \mathbf{N}^i, F^i \rangle, R^i \rangle, i \in I$, a collection of $\mathcal{L}$-systems, $\mathcal{U}$ an ultrafilter over I and α an arbitrary $\mathcal{L}$-formula. If $\Sigma_i \in |\mathbf{Sign}^i|, \phi_i^j \in \mathrm{SEN}^i(\Sigma_i), i \in I, j \in \omega$, then*

$$\{i \in I : \mathfrak{A}^i \models_{\Sigma_i} \alpha[\vec{\phi}_i]\} \in \mathcal{U} \quad \textit{iff} \quad \prod_{i\in I}^{\mathcal{U}} \mathfrak{A}^i \models_{\prod_{i\in I}\Sigma_i} \alpha[\vec{\phi}/{\equiv^{\mathcal{U}}_{\prod_{i\in I}\Sigma_i}}].$$

Proof:

Suppose, first, that $\alpha = r(t_0, \ldots, t_{n-1})$. Then we have

$$\begin{array}{ll} & \prod_{i\in I}^{\mathcal{U}} \mathfrak{A}^i \models_{\prod_{i\in I}\Sigma_i} r(t_0,\ldots,t_{n-1})[\vec{\phi}/{\equiv^{\mathcal{U}}_{\prod_{i\in I}\Sigma_i}}] \\ \text{iff} & \langle t_{0_{\prod_{i\in I}\Sigma_i}}^{\prod_{i\in I}^{\mathcal{U}}\mathfrak{A}^i}(\vec{\phi}/{\equiv^{\mathcal{U}}_{\prod_{i\in I}\Sigma_i}}), \ldots, t_{n-1_{\prod_{i\in I}\Sigma_i}}^{\prod_{i\in I}^{\mathcal{U}}\mathfrak{A}^i}(\vec{\phi}/{\equiv^{\mathcal{U}}_{\prod_{i\in I}\Sigma_i}})\rangle \in r_{\prod_{i\in I}\Sigma_i}^{\prod_{i\in I}^{\mathcal{U}}\mathfrak{A}^i} \\ \text{iff} & \{i \in I : \langle t^i_{0_{\Sigma_i}}(\vec{\phi}_i), \ldots, t^i_{n-1_{\Sigma_i}}(\vec{\phi}_i)\rangle \in r^i_{\Sigma_i}\} \in \mathcal{U} \\ \text{iff} & \{i \in I : \mathfrak{A}^i \models_{\Sigma_i} r(t_0,\ldots,t_{n-1})[\vec{\phi}_i]\} \in \mathcal{U}. \end{array}$$

Next, if $\alpha = (\alpha_1 \wedge \alpha_2)$, we have

$$\begin{array}{ll} & \prod_{i\in I}^{\mathcal{U}} \mathfrak{A}^i \models_{\prod_{i\in I}\Sigma_i} (\alpha_1 \wedge \alpha_2)[\vec{\phi}/{\equiv^{\mathcal{U}}_{\prod_{i\in I}\Sigma_i}}] \\ \text{iff} & \prod_{i\in I}^{\mathcal{U}} \mathfrak{A}^i \models_{\prod_{i\in I}\Sigma_i} \alpha_1[\vec{\phi}/{\equiv^{\mathcal{U}}_{\prod_{i\in I}\Sigma_i}}] \quad \text{and} \quad \prod_{i\in I}^{\mathcal{U}} \mathfrak{A}^i \models_{\prod_{i\in I}\Sigma_i} \alpha_2[\vec{\phi}/{\equiv^{\mathcal{U}}_{\prod_{i\in I}\Sigma_i}}] \\ \text{iff} & \{i \in I : \mathfrak{A}^i \models_{\Sigma_i} \alpha_1[\vec{\phi}_i]\} \in \mathcal{U} \quad \text{and} \quad \{i \in I : \mathfrak{A}^i \models_{\Sigma_i} \alpha_2[\vec{\phi}_i]\} \in \mathcal{U} \\ \text{iff} & \{i \in I : \mathfrak{A}^i \models_{\Sigma_i} \alpha_1[\vec{\phi}_i]\} \cap \{i \in I : \mathfrak{A}^i \models_{\Sigma_i} \alpha_2[\vec{\phi}_i]\} \in \mathcal{U} \\ \text{iff} & \{i \in I : \mathfrak{A}^i \models_{\Sigma_i} (\alpha_1 \wedge \alpha_2)[\vec{\phi}_i]\} \in \mathcal{U}. \end{array}$$

Similarly, if $\alpha = \neg\alpha'$, then ,we have

$$\begin{array}{lll} \prod_{i\in I}^{\mathcal{U}} \mathfrak{A}^i \models_{\prod_{i\in I}\Sigma_i} \neg\alpha'[\vec{\phi}/{\equiv^{\mathcal{U}}_{\prod_{i\in I}\Sigma_i}}] & \text{iff} & \prod_{i\in I}^{\mathcal{U}} \mathfrak{A}^i \not\models_{\prod_{i\in I}\Sigma_i} \alpha'[\vec{\phi}/{\equiv^{\mathcal{U}}_{\prod_{i\in I}\Sigma_i}}] \\ & \text{iff} & \{i \in I : \mathfrak{A}^i \models_{\Sigma_i} \alpha'[\vec{\phi}_i]\} \notin \mathcal{U} \\ & \text{iff} & \{i \in I : \mathfrak{A}^i \models_{\Sigma_i} \alpha'[\vec{\phi}_i]\}^c \in \mathcal{U} \\ & \text{iff} & \{i \in I : \mathfrak{A}^i \not\models_{\Sigma_i} \alpha'[\vec{\phi}_i]\} \in \mathcal{U} \\ & \text{iff} & \{i \in I : \mathfrak{A}^i \models_{\Sigma_i} \neg\alpha'[\vec{\phi}_i]\} \in \mathcal{U}. \end{array}$$

Finally, if $\alpha = (\exists k)\alpha'$, then ,we have

$$\begin{array}{ll} & \prod_{i\in I}^{\mathcal{U}} \mathfrak{A}^i \models_{\prod_{i\in I}\Sigma_i} (\exists k)\alpha'[\vec{\phi}/{\equiv^{\mathcal{U}}_{\prod_{i\in I}\Sigma_i}}] \\ \text{iff} & \prod_{i\in I}^{\mathcal{U}} \mathfrak{A}^i \models_{\prod_{i\in I}\Sigma_i} \alpha'[\vec{\psi}/{\equiv^{\mathcal{U}}_{\prod_{i\in I}\Sigma_i}}], \\ & \text{for some } \vec{\psi} \in (\prod_{i\in I}\mathrm{SEN}^i(\Sigma_i))^\omega, \text{ such that } \vec{\psi}^j \equiv^{\mathcal{U}}_{\prod_{i\in I}\Sigma_i} \vec{\phi}^j, j \neq k, \\ \text{iff} & \{i \in I : \mathfrak{A}^i \models_{\Sigma_i} \alpha'[\vec{\psi}_i]\} \in \mathcal{U} \\ & \text{for some } \vec{\psi} \in (\prod_{i\in I}\mathrm{SEN}^i(\Sigma_i))^\omega, \text{ such that } \vec{\psi}^j \equiv^{\mathcal{U}}_{\prod_{i\in I}\Sigma_i} \vec{\phi}^j, j \neq k, \end{array}$$

iff $\{i \in I : \mathfrak{A}^i \models_{\Sigma_i} \alpha'[\vec{\psi}_i],$ some $\vec{\psi}_i$ such that $\psi_i^j = \phi_i^j, j \neq k\} \in \mathcal{U}$
iff $\{i \in I : \mathfrak{A}^i \models_{\Sigma_i} (\exists k)\alpha'[\vec{\phi}_i]\} \in \mathcal{U}$.

■

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