Equational and Metaequational Theories and Semantic and Syntactic Varieties of Algebraic Systems

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To our colleagues in the biological sciences, and
to those Governments that did the Right Thing.
Primum non nocere

Abstract

Algebraic systems arise in categorical abstract algebraic logic and form a generalization of universal algebras. They allow multiple signatures and accommodate changes between signatures in the form of signature morphisms as well as natural transformations on signatures, which correspond to term operations in the universal algebraic context. In a way similar to ordinary equational logic and varieties of universal algebras, one may define equations and natural equations and the relation of satisfaction between algebraic systems, on the one hand, and equations or natural equations, on the other. They give rise, in the former case, to equational theories and semantic varieties, and, in the latter, to metaequational theories and syntactic varieties. We provide characterizations of these theories and of these classes of algebraic systems, and study various relationships among them. In the last section, we explore connections with equational classes of universal algebras.

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1 Introduction

Abstract algebraic logic is the area of mathematical logic that studies the interaction between logical systems, on the one hand, and classes of algebraic structures on the other. These studies incorporate three very closely related but distinct directions. In the first, which constitutes the backbone and unifying theme of the field, the process by which classes of algebraic structures are associated with given logical systems or classes of logical systems sharing some common properties is studied. In the second, the focus is shifted on the classes of algebraic structures and their properties, which are studied and analyzed by algebraic techniques or, sometimes, using model theoretic techniques, typically drawing on both logical and algebraic background and properties. The third direction establishes connections between properties of logical systems and corresponding algebraic properties of the classes of structures used for their algebraization, according to the general algebraization process. All three directions are expounded upon in greater or lesser detail in recent and relatively recent surveys, monographs and books on the field, e.g., [4, 15, 8, 16, 14].

The main underlying logical structure that is used to formalize logical systems in the classical (or universal algebraic) approach to the field is that of a sentential logic or deductive system. One fixes a logical (or algebraic, depending on the point of view) signature $\mathcal{L}$ and considers the free algebra of formulas (or terms, respectively) $\mathbf{Fm}_{\mathcal{L}}(V)$ ($\mathbf{Tm}_{\mathcal{L}}(V)$, respectively), generated by a countably infinite set V of variables. A *sentential logic* or *deductive system* over $\mathcal{L}$ is a pair $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$, where $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)) \times \mathrm{Fm}_{\mathcal{L}}(V)$ is a structural consequence relation on the set of $\mathcal{L}$-formulas, i.e., it satisfies, for all $\Gamma \cup \Delta \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$ and every substitution (endomorphism) $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$,

- $\Gamma \vdash_{\mathcal{S}} \gamma$, for all $\gamma \in \Gamma$;
- $\Gamma \vdash_{\mathcal{S}} \varphi$ implies $\Delta \vdash_{\mathcal{S}} \varphi$, if $\Gamma \subseteq \Delta$;
- $\Gamma \vdash_{\mathcal{S}} \varphi$ and $\Delta \vdash_{\mathcal{S}} \gamma$, for all $\gamma \in \Gamma$, imply $\Delta \vdash_{\mathcal{S}} \varphi$;
- $\Gamma \vdash_{\mathcal{S}} \varphi$ implies $\sigma(\Gamma) \vdash_{\mathcal{S}} \sigma(\varphi)$.

When the algebraization process is applied on a given deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$, a class of $\mathcal{L}$-algebras, in the sense of universal algebra [5, 21, 1], is obtained as its corresponding algebraic counterpart. As pointed out in

[4] (see also [14]), in general, this class is a generalized quasivariety of $\mathcal{L}$-algebraic systems, but, very often, it turns out that it is a variety, in which case the extensively developed theory of varieties from universal algebra can be brought to bear in the study of the original sentential logic or class of sentential logics. This short account gives a flavor of the importance of the theory of varieties in the study of logical systems in the framework of abstract algebraic logic.

From the early days of development, it became clear that the sentential framework was not well suited in handling logical systems that encompass multiple signatures and quantifiers. To deal with such logical systems one would have to first recast them as sentential systems, as was done in Appendix C of [4] and then use, e.g., in the case of first-order logic, cylindric [19] or polyadic [18] algebras to algebraize the sentential version of the system. This unappealing process had led Diskin (unpublished notes, but see, also, [12]) to consider using a categorical framework to incorporate changing of signatures and substitutions in the object language, rather than delegating their handling to the metalanguage. At around the same time, in the computer science domain of formal specification of data structures and programming languages, Goguen and Burstall [17] introduced the structure of an institution with a similar goal in mind, i.e., formalize multi-signature logics with quantifiers in an abstract way. For an extensive and thorough study of institutions from the model theoretic point of view, see [11]. Pigozzi, having pointed out in [4] the artificiality of using sentential logics in the handling of multi-signature systems, and being acquainted with both Diskin's and Goguen and Burstall's work, encouraged the author (his graduate student at the time) to look into the potential of using institutions in the algebraization process. Since the starting point and main inspiration stemmed from the extensive work that had already been accomplished in the sentential framework, it was natural to take the simpler step of incorporating signature changing morphisms and substitutions in the object language, but leaving the manipulation of the models (be it logical or algebraic structures) in the metalanguage. The appropriate structures that facilitated this transition were π-institutions [13], structures constituting modifications of institutions, that incorporate multiple signatures, but, instead of determining consequence model theoretically, adopt, as in deductive systems, an axiomatic viewpoint. Later, under the influence of Font and Jansana's work [15], it became clear to the author that an enriched version of π-institutions, where, in addition to signature changing morphisms, clones of operations were also incorporated in the object language, were even more suitable for the purposes of algebraization. These were first considered in [29].[1]

According to current understanding (expounded upon in [31]), the categorical side of abstract algebraic logic uses as its underlying structures these enhanced versions of π-institutions, which are based (as sentential logics are based on an algebraic signature and the free algebra of formulas) on algebraic systems, structures that capture both the logical and and algebraic fundamentals underlying the logical system under consideration.

An *algebraic system* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ consists of a category $\mathbf{Sign}^\flat$ of signatures, a sentence functor $\mathrm{SEN}^\flat : \mathbf{Sign}^\flat \to \mathbf{Set}$, giving, for each signature $\Sigma \in |\mathbf{Sign}^\flat|$, the set $\mathrm{SEN}^\flat(\Sigma)$ of Σ-sentences (and specifying how the signature changing morphisms in $\mathbf{Sign}^\flat$ transform sentences) and a category $N^\flat$ of natural transformations on $\mathrm{SEN}^\flat$, which formalizes the clone of algebraic operations and satisfies certain closure properties (contains all projections, is closed under generalized compositions and is closed under the formation of tuples).

A *π-institution* is a pair $\mathcal{I} = \langle \mathbf{F}, C \rangle$, where $\mathbf{F}$ is an algebraic system (called the base algebraic system of $\mathcal{I}$) and $C = \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ is a family of closure operators, one for each signature Σ, that, in addition to the standard axioms of closure operators (inflationarity, monotonicity and idempotence), satisfy the so-called structurality rule, which stipulates that, for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$ and all $\Phi \subseteq \mathrm{SEN}^\flat(\Sigma)$,

$$\mathrm{SEN}^\flat(f)(C_\Sigma(\Phi)) \subseteq C_{\Sigma'}(\mathrm{SEN}^\flat(f)(\Phi)).$$

If a process analogous to the one applied in the sentential logic framework, suitably modified, is now applied to π-institutions, one obtains a class or classes of algebraic structures that form the algebraic counterpart of the π-institution under consideration. In the same way that the ties between a sentential logic and the corresponding class of algebraic structures classifies logics into appropriate classes of an algebraic hierarchy, called the *Leibniz hierarchy* (see, e.g., [8] or Chapter 6 of [14]), a similar analysis classifies π-institutions into various classes depending on the strength of these ties (see [31]). The main or core classes in the Leibniz hierarchy of sentential logics are the protoalgebraic logics [3], the equivalential logics [6, 7], the truth equational logics [23], the weakly algebraizable [9] and the algebraizable logics

[1] Even though [29] is historically the first work written using this framework, it was published much later than, e.g., [25], which is the first work using the same framework that appeared in print (despite being written later and as one of the sequels to [29]).

[4, 20]. These classes are surrounded by various weakenings and strengthenings that contribute to the hierarchy pictured, e.g., in page 316 of [14] or page xviii of [22]. Corresponding classes have also been introduced in the hierarchy pertaining to logics formalized as π-institutions [26, 27, 28, 30, 24].[2]

But what are the algebraic structures that one considers in the π-institution framework in lieu of universal algebras, which are used in the algebraization of sentential logics? These are the so-called $\mathbf{F}$-algebraic systems, the study of whose classes forms the main object of the present work. An $\mathbf{F}$-*algebraic system* is a pair $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, where $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$ is an algebraic system, such that there exists a surjective functor $N^\flat \to N$ preserving all projection morphisms, and $\langle F, \alpha \rangle : \mathbf{F} \to \mathbf{A}$ is a surjective morphism, meaning that $F : \mathbf{Sign}^\flat \to \mathbf{Sign}$ is surjective on objects and full, and $\alpha_\Sigma : \mathrm{SEN}^\flat(\Sigma) \to \mathrm{SEN}(F(\Sigma))$ is surjective, for all $\Sigma \in |\mathbf{Sign}^\flat|$. The class of all $\mathbf{F}$-algebraic systems is denoted $\mathrm{AlgSys}(\mathbf{F})$.

When one wishes to study classes of $\mathbf{F}$-algebraic systems defined by objects playing the role of equations in the universal algebraic context, there are two possible choices. the first is to use pairs of Σ-sentences. These form the family of $\mathbf{F}$-*equations* defined by $\mathrm{Eq}(\mathbf{F}) = \{\mathrm{Eq}_\Sigma(\mathbf{F})\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, where, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{Eq}_\Sigma(\mathbf{F}) = \mathrm{SEN}^\flat(\Sigma)^2 = \{\phi \approx \psi : \phi, \psi \in \mathrm{SEN}^\flat(\Sigma)\}.$$

Here, the notation $\phi \approx \psi$ is considered interchangeable with $\langle \phi, \psi \rangle$ and will be used throughout as such. The second choice is to use pairs of natural transformations σ, τ in $N^\flat$. These are referred to as *natural* $\mathbf{F}$-*equations* and we define

$$\mathrm{NEq}(\mathbf{F}) = \{\sigma \approx \tau : \sigma, \tau \in N^\flat\}.$$

Having provided some motivation for studying classes of algebraic systems as a necessary component in the process of algebraization of logical systems and of its consequences, we now outline the contents of the present work.

In Section 2, we introduce the satisfaction relation of an $\mathbf{F}$-algebraic system by an $\mathbf{F}$-equation. This establishes in the ordinary way a Galois connection and defines a closure operator C on the equational side and a closure operator $\mathbb{V}^{\mathsf{Sem}}$ on the algebraic side. The former associates with a given collection X of $\mathbf{F}$-equations the equational theory consisting of all $\mathbf{F}$-equations

[2] The entire hierarchy constitutes the main subject of [31], in which many more classes are introduced, based on refinements of the various properties used to define the core classes.

that are satisfied by all $\mathbf{F}$-algebraic systems satisfying all $\mathbf{F}$-equations in X. The latter associates in a corresponding way with a given class K of $\mathbf{F}$-algebraic systems the so-called semantic variety generated by K.

The notion of a congruence system on an algebraic system is well known. It consists of a family of equivalence relations, indexed by the signatures of the algebraic system, that is invariant both under signature morphisms and under the natural transformations of the algebraic system. It corresponds to the notion of congruence in the framework of algebraic systems and, among other things, it is possible to consider quotients, which inherit many of the properties they possess in universal algebra [29]. Equational theories are characterized by showing (Proposition 3) that the notion of an equational theory is coextensive with that of a congruence system on $\mathbf{F}$.

Less well known are the operators $\mathbb{H}$ and $\overset{\triangleleft}{\mathbb{P}}$ on classes of $\mathbf{F}$-algebraic systems of taking morphic images and of closing under subdirect intersections, respectively. In an analog of Birkhoff's Theorem [2] (see, e.g., Theorem 11.9 of [5] or Theorem 4.41 of [1]), it is shown (Proposition 6) that a class K of $\mathbf{F}$-algebraic systems forms a semantic variety if and only if it is closed under the operators $\mathbb{H}$ and $\overset{\triangleleft}{\mathbb{P}}$.

In Section 3, we shift focus on the relation of satisfaction between $\mathbf{F}$-algebraic systems and natural $\mathbf{F}$-equations. This also establishes a Galois connection and gives rise to two closure operators C^N and $\mathbb{V}^{\mathsf{Syn}}$. C^N associates with a given collection E of natural $\mathbf{F}$-equations the equational metatheory consisting of all natural $\mathbf{F}$-equations that satisfy all $\mathbf{F}$-algebraic systems satisfying all natural $\mathbf{F}$-equations in E. $\mathbb{V}^{\mathsf{Syn}}$, on the other hand, associates with a given class K of $\mathbf{F}$-algebraic systems the syntactic variety generated by K, i.e., the class of all $\mathbf{F}$-algebraic systems satisfying all natural $\mathbf{F}$-equations satisfied by all systems in K.

In the context of natural $\mathbf{F}$-equations, the place of congruence systems is assumed by *metacongruences*, collections of natural $\mathbf{F}$-equations that form an equivalence relation on the set of natural transformations and, moreover, satisfy a natural substitution property. A metacongruence is called *feasible* if it arises as the collection of natural $\mathbf{F}$-equations satisfied by a quotient of $\mathbf{F}$ by some congruence system on $\mathbf{F}$. Equational metatheories (natural equational theories) are characterized (Proposition 12) as being exactly the feasible metacongruences on $\mathbf{F}$.

To characterize syntactic varieties, we establish a relationship with semantic varieties. We say that a given class K of $\mathbf{F}$-algebraic systems is *nat-*

ural if the family of $\mathbf{F}$-equations that it satisfies is induced by the natural $\mathbf{F}$-equations that it satisfies "by evaluation". It is then shown (Proposition 17) that a class K constitutes a syntactic variety if and only if it is a natural semantic variety. As a corollary of this connection and the characterization of semantic varieties, we get (Corollary 18) that K is a syntactic variety if and only if it is a natural class closed under morphic images and subdirect intersections.

In Section 4, we focus on the closure operators C and C^N, generating equational and metaequational theories, respectively, and the underlying equational and metaequational logics. We show how , starting from a collection X of $\mathbf{F}$-equations, the equational theory $C(X)$ is generated in a structured step-wise fashion, without reference to algebraic systems. Similarly, starting from a collection E of natural $\mathbf{F}$-equations, we show how $C^N(E)$ is generated, staying throughout on the logical side of the established Galois connections.

Symmetrically to Section 4, in Section 5, we concentrate on the operators $\mathbb{V}^{\mathsf{Sem}}$ and $\mathbb{V}^{\mathsf{Syn}}$ and provide for each a Birkhoff HSP-style characterization. Recalling that a class K of $\mathbf{F}$-algebraic systems is a semantic variety if and only if it is closed under morphic images and subdirect intersections, we show (Proposition 31) that $\mathbb{V}^{\mathsf{Sem}} = \mathbb{H}\overset{\triangleleft}{\mathbb{\Pi}}$. On the other hand, whenever there exists a morphism $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$, then the algebraic system $\mathcal{B}$ potentially satisfies more natural $\mathbf{F}$-equations than does $\mathcal{A}$. If, however, they satisfy exactly the same natural $\mathbf{F}$-equations, we say that $\mathcal{A}$ constitutes a *lifting* of $\mathcal{B}$. Recalling the characterization of syntactic varieties as natural classes closed under morphic images and subdirect intersections, we show (Proposition 37) that $\mathbb{V}^{\mathsf{Syn}} = \mathbb{H}\mathbb{L}\overset{\triangleleft}{\mathbb{\Pi}}$, $\mathbb{L}$ being the operator closing under liftings. Thus, in conjunction with $\mathbb{H}$ and $\overset{\triangleleft}{\mathbb{\Pi}}$, the operator $\mathbb{L}$ manages to capture the additional requirement of naturality placed on syntactic varieties.

Finally, in the closing Section 6, we establish connections of the framework relating to algebraic systems, developed in the preceding sections, with the classical framework of universal algebra. More precisely, modulo some cardinality issues (stemming from the stipulation of surjectivity of all morphisms in algebraic systems), we show (Proposition 39 and Corollary 43) that, by narrowing considerations on a special kind of base algebraic system $\mathbf{F}^{\mathcal{L}}$, associated with a given, but arbitrary, universal algebraic type $\mathcal{L}$, we can recover the traditional universal algebraic framework pertaining to $\mathcal{L}$-algebras from the framework of $\mathbf{F}^{\mathcal{L}}$-algebraic systems.

2 Equations and Semantic Varieties

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system. Define a binary relation

$$\vDash \subseteq \mathrm{AlgSys}(\mathbf{F}) \times \mathrm{Eq}(\mathbf{F})$$

by setting, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, every $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\mathcal{A} \vDash_\Sigma \phi \approx \psi \quad \text{iff} \quad \alpha_\Sigma(\phi) = \alpha_\Sigma(\psi).$$

We extend the notation to apply it to collections of $\mathbf{F}$-algebraic systems and families of $\mathbf{F}$-equations by setting, for all $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$ and all $X \leq \mathrm{Eq}(\mathbf{F})$ (meaning $X_\Sigma \subseteq \mathrm{Eq}_\Sigma(\mathbf{F})$, for all $\Sigma \in |\mathbf{Sign}^\flat|$),

$$\begin{array}{lll}\mathsf{K} \vDash X & \text{iff} & \text{for all } \mathcal{A} \in \mathsf{K}, \text{ all } \Sigma \in |\mathbf{Sign}^\flat| \text{ and all } \phi \approx \psi \in X_\Sigma,\\ & & \mathcal{A} \vDash_\Sigma \phi \approx \psi.\end{array}$$

It is clear that $\vDash$ determines a Galois connection between $\mathcal{P}(\mathrm{AlgSys}(\mathbf{F}))$ and $\mathcal{P}(\mathrm{Eq}(\mathbf{F}))$ (see, e.g., pages 232-233 of [10]). Related to this Galois connection, we use the following notational conventions.

First, given a class K of $\mathbf{F}$-algebraic systems, we define the collection $\mathrm{Eq}(\mathsf{K}) = \{\mathrm{Eq}_\Sigma(\mathsf{K})\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, where, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{Eq}_\Sigma(\mathsf{K}) = \{\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathbf{F}) : \mathsf{K} \vDash_\Sigma \phi \approx \psi\}.$$

Next, given a family $X = \{X_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|}$ of $\mathbf{F}$-equations, we define

$$\mathrm{Mod}(X) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : \mathcal{A} \vDash X\}.$$

Finally, for the closure operators associated with the Galois connection, we set, for all $X \leq \mathrm{Eq}(\mathbf{F})$ and all $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$,

$$\begin{array}{rcl} C(X) & = & \mathrm{Eq}(\mathrm{Mod}(X));\\ \mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) & = & \mathrm{Mod}(\mathrm{Eq}(\mathsf{K})).\end{array}$$

$C(X)$ is referred to as the **equational theory** generated by X and $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$ as the **semantic variety** generated by K. By the general theory of Galois connections, we know that the closed sets of the closure operator C are the ones of the form $\mathrm{Eq}(\mathsf{K})$ for a class K of $\mathbf{F}$-algebraic systems and those of the closure operator $\mathbb{V}^{\mathsf{Sem}}$ are those of the form $\mathrm{Mod}(X)$ for a family X of $\mathbf{F}$-equations.

We set out to provide intrinsic characterizations of those closed sets.

First, let us formulate two lemmas that will play a role in the subsequent characterizations. Recall that $\theta \leq \mathrm{Eq}(\mathbf{F})$ is called a **congruence system on F** if θ_Σ is an equivalence relation on $\mathrm{SEN}^\flat(\Sigma)$, for every $\Sigma \in |\mathbf{Sign}^\flat|$ and θ is invariant under both signature morphisms and under natural transformations in $N^\flat$, i.e., for all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, all $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, all $\sigma^\flat$ in $N^\flat$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$,

- $\mathrm{SEN}^\flat(f)(\theta_\Sigma) \subseteq \theta_{\Sigma'}$;
- $\vec{\phi}\ \theta_\Sigma\ \vec{\psi}$ implies $\sigma^\flat_\Sigma(\vec{\phi})\ \theta_\Sigma\ \sigma^\flat_\Sigma(\vec{\psi})$.

Recall, also, that, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$, the **kernel of $\mathcal{A}$** is the kernel of $\langle F, \alpha \rangle$, defined as $\mathrm{Ker}(\mathcal{A}) = \{\mathrm{Ker}_\Sigma(\mathcal{A})\}_{\Sigma \in |\mathbf{Sign}^\flat|}$, where, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\mathrm{Ker}_\Sigma(\mathcal{A}) = \{\phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathbf{F}) : \alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)\}.$$

Moreover, $\mathrm{Ker}(\mathcal{A})$ is a congruence system on $\mathbf{F}$ [29].

Lemma 1 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* $X \in \mathrm{ConSys}(\mathbf{F})$. *Then* $X = \mathrm{Eq}(\mathcal{F}/X)$, *where* $\mathcal{F} := \langle \mathbf{F}, \langle I, \iota \rangle \rangle$, *with* $\langle I, \iota \rangle : \mathbf{F} \to \mathbf{F}$ *the identity morphism.*

Proof: Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then, we have

$$\begin{array}{lll} \phi \approx \psi \in \mathrm{Eq}_\Sigma(\mathcal{F}/X) & \text{iff} & \iota_\Sigma(\phi)/X_\Sigma = \iota_\Sigma(\psi)/X_\Sigma \\ & \text{iff} & \phi/X_\Sigma = \psi/X_\Sigma \\ & \text{iff} & \langle \phi, \psi \rangle \in X_\Sigma. \end{array}$$

Since this holds for all $\Sigma \in |\mathbf{Sign}^\flat|$, we get $\mathrm{Eq}(\mathcal{F}/X) = X$. ∎

Lemma 2 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* $\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F})$. *Then* $\mathrm{Eq}(\mathcal{A}) \in \mathrm{ConSys}(\mathbf{F})$.

Proof: Let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ be an $\mathbf{F}$-algebraic system, where $\langle F, \alpha \rangle : \mathbf{F} \to \mathbf{A}$. The conclusion follows by noticing that $\mathrm{Eq}(\mathcal{A}) = \mathrm{Ker}(\langle F, \alpha \rangle)$ and the fact that the kernel system of any algebraic system morphism is a congruence system. ∎

First, we characterize the closed sets in $\mathcal{P}(\mathrm{Eq}(\mathbf{F}))$. Those turn out to be the congruence systems on $\mathbf{F}$.

Proposition 3 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $X \leq \mathrm{Eq}(\mathbf{F})$*. Then* $C(X) = X$ *if and only if* $X \in \mathrm{ConSys}(\mathbf{F})$

Proof: Let $X \leq \mathrm{Eq}(\mathbf{F})$, such that $C(X) = X$. Then, by the theory of Galois connections, we get that there exists $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$, such that $X = \mathrm{Eq}(\mathsf{K})$. Now we get

$$\begin{array}{rcll} X & = & \mathrm{Eq}(\mathsf{K}) & \\ & = & \bigcap_{\mathcal{A} \in \mathsf{K}} \mathrm{Eq}(\mathcal{A}) & \text{(definition of Eq)} \\ & \in & \mathrm{ConSys}(\mathbf{F}). & \text{(Lemma 2 and closure under } \bigcap) \end{array}$$

Suppose, conversely, that $X \in \mathrm{ConSys}(\mathbf{F})$. Then we have, by Lemma 1, $X = \mathrm{Eq}(\mathcal{F}/X)$. Since X is in the image of Eq, applying again the theory of Galois connections, we get $C(X) = X$. ■

Finally, we characterize the closed sets in $\mathcal{P}(\mathrm{AlgSys}(\mathbf{F}))$. They turn out to be those classes of $\mathbf{F}$-algebraic systems that are closed under morphic images and subdirect intersections. Briefly, given a class K of $\mathbf{F}$-algebraic systems, we write $\mathcal{B} \in \mathbb{H}(\mathsf{K})$ if there exists a morphism $\langle H, \gamma \rangle : \mathcal{A} \to \mathcal{B}$, with $\mathcal{A} \in \mathsf{K}$. We say that K is **closed under morphic images** if $\mathbb{H}(\mathsf{K}) \subseteq \mathsf{K}$. We write $\mathcal{A} \in \overset{\triangleleft}{\mathbb{II}}(\mathsf{K})$ if there exists a collection

$$\langle H^i, \gamma^i \rangle : \mathcal{A} \to \mathcal{A}^i, \quad i \in I,$$

of morphisms satisfying $\bigcap_{i \in I} \mathrm{Ker}(\langle H^i, \gamma^i \rangle) = \Delta^{\mathcal{A}}$, with $\mathcal{A}^i \in \mathsf{K}$, for all $i \in I$. Such a collection is called a **subdirect intersection**. The class K is **closed under subdirect intersections** if $\overset{\triangleleft}{\mathbb{II}}(\mathsf{K}) \subseteq \mathsf{K}$.

We shall rely on the following lemma:

Proposition 4 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$*. The class of morphisms*

$$\langle G, \beta^{\mathsf{K}} \rangle : \mathcal{F} / \bigcap_{\mathcal{B} \in \mathsf{K}} \mathrm{Ker}(\langle G, \beta \rangle) \to \mathcal{B}, \quad \mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle \in \mathsf{K},$$

where, for all $\Sigma \in |\mathbf{Sign}^\flat|$ *and all* $\phi \in \mathrm{SEN}^\flat(\Sigma)$*,*

$$\beta^{\mathsf{K}}_{\Sigma}(\phi / \bigcap_{\mathcal{B} \in \mathsf{K}} \mathrm{Ker}_{\Sigma}(\langle G, \beta \rangle)) = \beta_{\Sigma}(\phi),$$

forms a subdirect intersection.

Proof: It is not difficult to see that β^{K} is well defined and forms a natural transformation. Moreover, $\langle G, \beta^{\mathsf{K}}\rangle$ is an $\mathbf{F}$-morphism. Letting $\mathrm{Ker}(\mathsf{K}) = \bigcap_{\mathcal{B}\in\mathsf{K}} \mathrm{Ker}(\langle G, \beta\rangle)$, we have

$$\begin{tikzcd} & \mathbf{F} \arrow[dl, "\langle I, \pi^{\mathrm{Ker}(\mathsf{K})}\rangle"'] \arrow[dr, "\langle G, \beta\rangle"] & \\ \mathbf{F}/\mathrm{Ker}(\mathsf{K}) \arrow[rr, "\langle G, \beta^{\mathsf{K}}\rangle"'] & & \mathbf{B} \end{tikzcd}$$

To show that the displayed family forms a subdirect intersection, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$. Then, we get

$$\begin{array}{ll} \multicolumn{2}{l}{\langle \phi/\mathrm{Ker}_\Sigma(\mathsf{K}), \psi/\mathrm{Ker}_\Sigma(\mathsf{K})\rangle \in \bigcap_{\mathcal{B}\in\mathsf{K}} \mathrm{Ker}_\Sigma(\langle G, \beta^{\mathsf{K}}\rangle)} \\ \text{iff} & \beta^{\mathsf{K}}_\Sigma(\phi/\mathrm{Ker}_\Sigma(\mathsf{K})) = \beta^{\mathsf{K}}_\Sigma(\psi/\mathrm{Ker}_\Sigma(\mathsf{K})), \quad \mathcal{B} \in \mathsf{K}, \\ \text{iff} & \beta_\Sigma(\phi) = \beta_\Sigma(\psi), \quad \mathcal{B} \in \mathsf{K}, \\ \text{iff} & \phi/\mathrm{Ker}_\Sigma(\mathsf{K}) = \psi/\mathrm{Ker}_\Sigma(\mathsf{K}). \end{array}$$

Thus, $\bigcap_{\mathcal{B}\in\mathsf{K}} \mathrm{Ker}(\langle G, \beta^{\mathsf{K}}\rangle) = \Delta^{\mathcal{F}/\mathrm{Ker}(\mathsf{K})}$, showing that

$$\langle G, \beta^{\mathsf{K}}\rangle : \mathcal{F}/\bigcap_{\mathcal{B}\in\mathsf{K}} \mathrm{Ker}(\langle G, \beta\rangle) \to \mathcal{B}, \quad \mathcal{B} = \langle \mathbf{B}, \langle G, \beta\rangle\rangle \in \mathsf{K},$$

constitutes indeed a subdirect intersection. ∎

In the next lemma, it is shown that a semantic variety is closed under morphic images and subdirect intersections. It provides the necessity part of the main characterization that follows.

Lemma 5 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be an algebraic system and* $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$. *If* $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$*, then* K *is closed under* $\mathbb{H}$ *and* $\overset{\triangleleft}{\mathbb{II}}$.

Proof: Let $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$. Suppose that $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$. By the theory of Galois connections, there exists $X \leq \mathrm{Eq}(\mathbf{F})$, such that $\mathsf{K} = \mathrm{Mod}(X)$.

To show closure under $\mathbb{H}$, suppose that $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta\rangle\rangle \in \mathbb{H}(\mathsf{K})$. Thus, there exists $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle \in \mathsf{K}$, such that $\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$ is an $\mathbf{F}$-morphism.

$$\begin{tikzcd} & \mathbf{F} \arrow[dl, "\langle F, \alpha\rangle"'] \arrow[dr, "\langle G, \beta\rangle"] & \\ \mathbf{A} \arrow[rr, "\langle H, \gamma\rangle"'] & & \mathbf{B} \end{tikzcd}$$

Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \approx \psi \in X_\Sigma$. Since $\mathcal{A} \in \mathsf{K} = \mathrm{Mod}(X)$, we get that $\mathcal{A} \vDash_\Sigma \phi \approx \psi$, whence $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$. Thus, $\gamma_{F(\Sigma)}(\alpha_\Sigma(\phi)) = \gamma_{F(\Sigma)}(\alpha_\Sigma(\psi))$, i.e., $\beta_\Sigma(\phi) = \beta_\Sigma(\psi)$, showing that $\mathcal{B} \vDash_\Sigma \phi \approx \psi$. Therefore, $\mathcal{B} \in \mathrm{Mod}(X) = \mathsf{K}$ and $\mathbb{H}(\mathsf{K}) \subseteq \mathsf{K}$.

To show closure under $\overset{\triangleleft}{\mathbb{\Pi}}$, let $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle \in \overset{\triangleleft}{\mathbb{\Pi}}(\mathsf{K})$. Then, there exists a subdirect intersection

$$\langle H^i, \gamma^i\rangle : \mathcal{A} \to \mathcal{A}^i, \quad i \in I,$$

where, for all $i \in I$, $\mathcal{A}^i = \langle \mathbf{A}^i, \langle F^i, \alpha^i\rangle\rangle \in \mathsf{K}$.

$$\begin{tikzcd}
 & \mathbf{F} \arrow[dl, "\langle F,\alpha\rangle"'] \arrow[dr, "\langle F^i,\alpha^i\rangle"] & \\
\mathbf{A} \arrow[rr, "\langle H^i,\gamma^i\rangle"'] & & \mathbf{A}^i
\end{tikzcd}$$

Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \approx \psi \in X_\Sigma$. Since $\mathcal{A}^i \in \mathsf{K} = \mathrm{Mod}(X)$, we get that $\mathcal{A}^i \vDash_\Sigma \phi \approx \psi$, i.e., $\alpha^i_\Sigma(\phi) = \alpha^i_\Sigma(\psi)$. This gives $\gamma^i_{F(\Sigma)}(\alpha_\Sigma(\phi)) = \gamma^i_{F(\Sigma)}(\alpha_\Sigma(\psi))$. Since this holds for all $i \in I$, we get that $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in \bigcap_{i \in I} \mathrm{Ker}_{F(\Sigma)}(\langle H^i, \gamma^i\rangle) = \Delta^{\mathcal{A}}_{F(\Sigma)}$, where the last equation follows by the definition of a subdirect intersection. Therefore, $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$, showing that $\mathcal{A} \vDash_\Sigma \phi \approx \psi$. Hence, $\mathcal{A} \in \mathrm{Mod}(X) = \mathsf{K}$, and, thus, $\overset{\triangleleft}{\mathbb{\Pi}}(\mathsf{K}) \subseteq \mathsf{K}$. ∎

Proposition 6 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be an algebraic system and $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$. Then $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$ if and only if K is closed under $\mathbb{H}$ and $\overset{\triangleleft}{\mathbb{\Pi}}$.*

Proof: Let $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$. The necessity of the given condition is by Lemma 5. Conversely, suppose that $\mathbb{H}(\mathsf{K}) \subseteq \mathsf{K}$ and $\overset{\triangleleft}{\mathbb{\Pi}}(\mathsf{K}) \subseteq \mathsf{K}$. It suffices to show that $\mathsf{K} = \mathrm{Mod}(\mathrm{Eq}(\mathsf{K}))$. The left to right inclusion is obvious. For the converse, consider $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle \in \mathrm{Mod}(\mathrm{Eq}(\mathsf{K}))$. By Proposition 4,

$$\mathcal{F}/\mathrm{Eq}(\mathsf{K}) = \mathcal{F}/\bigcap_{\mathcal{B} \in \mathsf{K}} \mathrm{Ker}(\langle G, \beta\rangle) \in \overset{\triangleleft}{\mathbb{\Pi}}(\mathsf{K}) = \mathsf{K}.$$

But, since $\mathcal{A} \in \mathrm{Mod}(\mathrm{Eq}(\mathsf{K}))$, we may define a morphism

$$\langle F, \alpha^*\rangle : \mathcal{F}/\mathrm{Eq}(\mathsf{K}) \to \mathcal{A},$$

by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\alpha^*_\Sigma(\phi/\mathrm{Eq}_\Sigma(\mathsf{K})) = \alpha_\Sigma(\phi).$$

$$\begin{tikzcd}
 & \mathbf{F} \arrow[dl, "\langle I, \pi^{\mathrm{Eq}(\mathsf{K})}\rangle"'] \arrow[dr, "\langle F, \alpha\rangle"] & \\
\mathbf{F}/\mathrm{Eq}(\mathsf{K}) \arrow[rr, "\langle F, \alpha^*\rangle"'] & & \mathbf{A}
\end{tikzcd}$$

Thus, we get $\mathcal{A} \in \mathbb{H}(\mathsf{K}) = \mathsf{K}$. We conclude that $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$. ∎

3 Natural Equations and Syntactic Varieties

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system. Define a binary relation

$$\vDash \subseteq \mathrm{AlgSys}(\mathbf{F}) \times \mathrm{NEq}(\mathbf{F})$$

by setting, for every $\mathbf{F}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N\rangle$, every $\sigma \approx \tau \in \mathrm{NEq}(\mathbf{F})$,

$$\mathcal{A} \vDash \sigma \approx \tau \quad \text{iff} \quad \begin{array}{l}\text{for all } \Sigma \in |\mathbf{Sign}| \text{ and all } \vec{\phi} \in \mathrm{SEN}(\Sigma),\\ \sigma^{\mathcal{A}}_\Sigma(\vec{\phi}) = \tau^{\mathcal{A}}_\Sigma(\vec{\phi}).\end{array}$$

Note that, because of the surjectivity of $\langle F, \alpha\rangle : \mathbf{F} \to \mathbf{A}$, the condition above may be equivalently expressed by saying that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\sigma^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})) = \tau^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})).$$

We extend the notation to collections of $\mathbf{F}$-algebraic systems and families of natural $\mathbf{F}$-equations by setting, for all $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$ and all $E \subseteq \mathrm{NEq}(\mathbf{F})$,

$$\mathsf{K} \vDash E \quad \text{iff} \quad \begin{array}{l}\text{for all } \mathcal{A} \in \mathsf{K} \text{ and all } \sigma \approx \tau \in E,\\ \mathcal{A} \vDash \sigma \approx \tau.\end{array}$$

It is clear that $\vDash$ determines a Galois connection between $\mathcal{P}(\mathrm{AlgSys}(\mathbf{F}))$ and $\mathcal{P}(\mathrm{NEq}(\mathbf{F}))$. Related to this Galois connection, we use the following notational conventions.

First, given a class K of $\mathbf{F}$-algebraic systems, we define the collection

$$\mathrm{NEq}(\mathsf{K}) = \{\sigma \approx \tau \in \mathrm{NEq}(\mathbf{F}) : \mathsf{K} \vDash \sigma \approx \tau\}.$$

Next, given a collection E of natural $\mathbf{F}$-equations, we define

$$\mathrm{NMod}(E) = \{\mathcal{A} \in \mathrm{AlgSys}(\mathbf{F}) : \mathcal{A} \vDash E\}.$$

Finally, for the closure operators associated with the Galois connection, we set, for all $E \subseteq \mathrm{NEq}(\mathbf{F})$ and all $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$,

$$\begin{array}{rcl} C^N(E) & = & \mathrm{NEq}(\mathrm{NMod}(E)); \\ \mathbb{V}^{\mathsf{Syn}}(\mathsf{K}) & = & \mathrm{NMod}(\mathrm{NEq}(\mathsf{K})). \end{array}$$

We call $C^N(E)$ the **metaequational theory** or the **natural equational theory** generated by E and $\mathbb{V}^{\mathsf{Syn}}(\mathsf{K})$ the **syntactic variety** generated by K. By the general theory of Galois connections, we know that the closed sets of the closure operator C^N are the ones of the form $\mathrm{NEq}(\mathsf{K})$ for a class K of $\mathbf{F}$-algebraic systems and those of the closure operator $\mathbb{V}^{\mathsf{Syn}}$ are those of the form $\mathrm{NMod}(E)$ for a collection E of natural $\mathbf{F}$-equations.

We set out to provide intrinsic characterizations of those closed sets.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. A binary relation R on $N^\flat$ is called a **metacongruence** on $\mathbf{F}$ if it is an equivalence relation and, in addition, satisfies the **substitution property**:

For all $o, \rho : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$ and all $\sigma^i, \tau^i : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, $i < \omega$,

$$\langle o, \rho \rangle \in R \text{ and } \langle \sigma^i, \tau^i \rangle \in R,\ i < \omega, \quad \text{imply} \quad \langle o \circ \vec{\sigma}, \rho \circ \vec{\tau} \rangle \in R.$$

We must recall, at this point, that all natural transformations in $N^\flat$ are assumed finitary. The seemingly infinitary notation above is adopted for convenience, since, even though the arities of o and ρ are finite, they are not a priori bounded.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system, R a metacongruence on $\mathbf{F}$ and $\theta \in \mathrm{ConSys}(\mathbf{F})$. R is said to be **compatible with** θ if it satisfies the θ-**compatibility property**:

For all $\sigma, \tau, \sigma', \tau' : (\mathrm{SEN}^\flat)^\omega \to \mathrm{SEN}^\flat$ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{tikzcd} \sigma \arrow[r, dash, "R"] \arrow[d, dash, "\sigma_\Sigma(\vec{\phi})\ \theta_\Sigma\ \sigma'_\Sigma(\vec{\psi})"'] & \tau \arrow[d, dash, dotted, "\tau_\Sigma(\vec{\phi})\ \theta_\Sigma\ \tau'_\Sigma(\vec{\psi})"] \\ \sigma' \arrow[r, dash, "R"'] & \tau' \end{tikzcd}$$

$$\begin{array}{l}\langle\sigma,\tau\rangle \in R \text{ and } \langle\sigma',\tau'\rangle \in R \text{ and } \sigma_\Sigma(\vec{\phi})\ \theta_\Sigma\ \sigma'_\Sigma(\psi)\\ \qquad\qquad\qquad\qquad \text{imply} \quad \tau_\Sigma(\vec{\phi})\ \theta_\Sigma\ \tau'_\Sigma(\vec{\psi}).\end{array}$$

Let $\mathrm{MetCon}(\mathbf{F})$ stand for the collection of all metacongruences on $\mathbf{F}$. It is clear that it forms a complete lattice under ordinary inclusion, which is denoted by $\mathbf{MetCon}(\mathbf{F}) = \langle \mathrm{MetCon}(\mathbf{F}), \subseteq\rangle$. We denote by $\mathrm{MetCon}^\theta(\mathbf{F})$ the collection of all metacongruences on $\mathbf{F}$ that are compatible with a given congruence system θ on $\mathbf{F}$.

As was, perhaps, to be expected, metacongruences on $\mathbf{F}$ and congruence systems on $\mathbf{F}$ are very closely related.

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system and $R \in \mathrm{MetCon}(\mathbf{F})$. Define the binary relation family $\theta^R = \{\theta^R_\Sigma\}_{\Sigma\in|\mathbf{Sign}^\flat|}$ on $\mathbf{F}$ by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\begin{array}{lll}\langle\phi,\psi\rangle \in \theta^R_\Sigma & \text{iff} & \text{there exist } \langle\sigma,\tau\rangle \in R,\ \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma),\\ & & \text{such that } \phi = \sigma_\Sigma(\vec{\chi}) \text{ and } \psi = \tau_\Sigma(\vec{\chi}).\end{array}$$

I.e., we have, for all $\Sigma \in |\mathbf{Sign}^\flat|$,

$$\theta^R_\Sigma = \{\langle\sigma_\Sigma(\vec{\chi}),\tau_\Sigma(\vec{\chi})\rangle : \langle\sigma,\tau\rangle \in R, \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)\}.$$

It is not difficult to show that, if R is a metaconguence on $\mathbf{F}$ compatible with $\Delta^{\mathbf{F}}$, then θ^R, thus defined, is a congruence system on $\mathbf{F}$.

Proposition 7 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system and $R \in \mathrm{MetCon}^\Delta(\mathbf{F})$. Then $\theta^R \in \mathrm{ConSys}(\mathbf{F})$. Moreover, R is compatible with θ^R.*

Proof: Let us show, first, that θ^R is an equivalence family on $\mathbf{F}$. To this end, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi,\psi,\chi \in \mathrm{SEN}^\flat(\Sigma)$.

- From the fact that $\langle\iota,\iota\rangle \in R$, we get that $\langle\phi,\phi\rangle = \langle\iota_\Sigma(\phi),\iota_\Sigma(\phi)\rangle \in \theta^R_\Sigma$, whence θ^R is reflexive.
- Suppose $\langle\phi,\psi\rangle \in \theta^R_\Sigma$. Then, there exist $\langle\sigma,\tau\rangle \in R$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi = \sigma_\Sigma(\vec{\chi})$ and $\psi \in \tau_\Sigma(\vec{\chi})$. Then, the pair $\langle\tau,\sigma\rangle \in R$ and the tuple $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$ witness that $\langle\psi,\phi\rangle \in \theta^R_\Sigma$ and, hence, θ^R is also symmetric.

- Finally, suppose that $\langle\phi,\psi\rangle,\langle\psi,\chi\rangle \in \theta^R_\Sigma$. Then, there exist $\langle\sigma,\tau\rangle \in R$, $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$ and $\langle\sigma',\tau'\rangle \in R$ and $\vec{\chi}' \in \mathrm{SEN}^\flat(\Sigma)$, such that
$$\phi = \sigma_\Sigma(\vec{\chi}),\ \psi = \tau_\Sigma(\vec{\chi}),\ \psi = \sigma'_\Sigma(\vec{\chi}'),\ \chi = \tau'_\Sigma(\vec{\chi}').$$
Taking into account the compatibility property of R, together with the equation $\tau_\Sigma(\vec{\chi}) = \sigma'_\Sigma(\vec{\chi}')$, we conclude that $\sigma_\Sigma(\vec{\chi}) = \tau'_\Sigma(\vec{\chi}')$, i.e., that $\langle\phi,\chi\rangle \in \Delta^{\mathbf{F}}_\Sigma \subseteq \theta^R_\Sigma$. Hence θ^R is also transitive.

Next, we show that θ^R is a system, i.e., invariant under signature morphisms. To this end, let $\Sigma,\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma,\Sigma')$ and $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle\phi,\psi\rangle \in R^\theta_\Sigma$. Thus, there exist $\langle\sigma,\tau\rangle \in R$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi = \sigma_\Sigma(\vec{\chi})$ and $\psi = \tau_\Sigma(\vec{\chi})$. Thus, we get
$$\mathrm{SEN}^\flat(f)(\phi) = \mathrm{SEN}^\flat(f)(\sigma_\Sigma(\vec{\chi})) = \sigma_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\chi}))$$
and, similarly, $\mathrm{SEN}^\flat(f)(\psi) = \tau_{\Sigma'}(\mathrm{SEN}^\flat(f)(\vec{\chi}))$. Thus, the pair $\langle\sigma,\tau\rangle \in R$ and the tuple $\mathrm{SEN}^\flat(f)(\vec{\chi}) \in \mathrm{SEN}^\flat(\Sigma')$ witness $\langle\mathrm{SEN}^\flat(f)(\phi),\mathrm{SEN}^\flat(f)(\psi)\rangle \in \theta^R_{\Sigma'}$, showing that θ^R is indeed an equivalence system on $\mathbf{F}$.

Next, we must show that θ^R is a congruence system. Let $\rho : (\mathrm{SEN}^\flat)^k \to \mathrm{SEN}^\flat$ in $N^\flat$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi},\vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle\phi_i,\psi_i\rangle \in \theta^R_\Sigma$, for all $i < k$. Thus, there exist $\langle\sigma^i,\tau^i\rangle \in R$ and $\vec{\chi}^i \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi_i = \sigma^i_\Sigma(\vec{\chi}^i)$ and $\psi_i = \tau^i_\Sigma(\vec{\chi}^i)$, for all $i < k$. Now, taking into account the substitution property of R, we obtain
$$\begin{array}{rcl}\rho_\Sigma(\vec{\phi}) & = & \rho_\Sigma(\sigma^0_\Sigma(\vec{\chi}^0),\ldots,\sigma^{k-1}_\Sigma(\vec{\chi}^{k-1}))\\ & \theta^R_\Sigma & \rho_\Sigma(\tau^0_\Sigma(\vec{\chi}^0),\ldots,\tau^{k-1}_\Sigma(\vec{\chi}^{k-1}))\\ & = & \rho_\Sigma(\vec{\psi}).\end{array}$$
Thus, θ^R is a congruence system on $\mathbf{F}$.

Finally, we show that R is compatible with θ^R. To this end, let $\langle\sigma,\tau\rangle \in R$, $\langle\sigma',\tau'\rangle \in R$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi},\vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$, such that $\sigma_\Sigma(\vec{\phi})\ \theta^R_\Sigma\ \sigma'_\Sigma(\vec{\psi})$. Since $\langle\sigma,\tau\rangle \in R$, $\sigma_\Sigma(\vec{\phi})\ \theta^R_\Sigma\ \tau_\Sigma(\vec{\phi})$. Similarly, since $\langle\sigma',\tau'\rangle \in R$, we get $\sigma'_\Sigma(\vec{\psi})\ \theta^R_\Sigma\ \tau'_\Sigma(\vec{\psi})$. Since θ^R is a congruence system, we now conclude that $\tau_\Sigma(\vec{\phi})\ \theta^R_\Sigma\ \tau'_\Sigma(\vec{\psi})$, which proves that R is compatible with θ^R. ∎

On the other hand, let $\mathbf{F} = \langle\mathbf{Sign}^\flat,\mathrm{SEN}^\flat,N^\flat\rangle$ be a base algebraic system and $\theta \in \mathrm{ConSys}(\mathbf{F})$. Define a binary relation R^θ on $N^\flat$ by setting, for all σ,τ in $N^\flat$,
$$\langle\sigma,\tau\rangle \in R^\theta \quad \text{iff} \quad \begin{array}{l}\text{for all } \Sigma \in |\mathbf{Sign}^\flat| \text{ and all } \vec{\chi} \in \mathrm{SEN}^\flat(\Sigma),\\ \langle\sigma_\Sigma(\vec{\chi}),\tau_\Sigma(\vec{\chi})\rangle \in \theta_\Sigma.\end{array}$$

In other words, using some obvious abbreviations,

$$R^\theta = \{\langle\sigma,\tau\rangle : (\forall\Sigma)(\forall\vec{\chi})(\sigma_\Sigma(\vec{\chi})\ \theta_\Sigma\ \tau_\Sigma(\vec{\chi}))\}.$$

Again, it is not difficult to show that R^θ is a metacongruence on $\mathbf{F}$, which is compatible with θ.

Proposition 8 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system and* $\theta \in \mathrm{ConSys}(\mathbf{F})$. *Then* $R^\theta \in \mathrm{MetCon}^\theta(\mathbf{F})$.

Proof: Let $\theta \in \mathrm{ConSys}(\mathbf{F})$. Clearly, R^θ is an equivalence relation on $N^\flat$. To see that it also satisfies the substitution property, suppose $\langle o,\rho\rangle \in R^\theta$ and $\langle\sigma^i,\tau^i\rangle \in R^\theta$, for $i < \omega$. Then, since $\langle\sigma^i,\tau^i\rangle \in R^\theta$, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, we have,

$$\langle\sigma^i_\Sigma(\vec{\chi}),\tau^i_\Sigma(\vec{\chi})\rangle \in \theta_\Sigma,\ i < \omega.$$

Since $\theta \in \mathrm{ConSys}(\mathbf{F})$, we now get

$$\langle o_\Sigma(\vec{\sigma}_\Sigma(\vec{\chi})), o_\Sigma(\vec{\tau}_\Sigma(\vec{\chi}))\rangle \in \theta_\Sigma.$$

As $\langle o,\rho\rangle \in R^\theta$, we get

$$\langle o_\Sigma(\vec{\tau}_\Sigma(\vec{\chi})), \rho_\Sigma(\vec{\tau}_\Sigma(\vec{\chi}))\rangle \in \theta_\Sigma.$$

Using the fact that θ is a congruence system, we now get

$$\langle o_\Sigma(\vec{\sigma}_\Sigma(\vec{\chi})), \rho_\Sigma(\vec{\tau}_\Sigma(\vec{\chi}))\rangle \in \theta_\Sigma.$$

Since $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$ were arbitrary, we conclude that $\langle o\circ\vec{\sigma}, \rho\circ\vec{\tau}\rangle \in R^\theta$. Hence, R^θ satisfies the substitution property and, therefore, it is a metacongruence on $\mathbf{F}$.

Finally, we show that R^θ is compatible with θ. To this end, assume that $\langle\sigma,\tau\rangle \in R^\theta$, $\langle\sigma',\tau'\rangle \in R^\theta$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\phi},\vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle\sigma_\Sigma(\vec{\phi}),\sigma'_\Sigma(\vec{\psi})\rangle \in \theta_\Sigma$. From the hypothesis $\langle\sigma,\tau\rangle \in R^\theta$, we get $\langle\sigma_\Sigma(\vec{\phi}),\tau_\Sigma(\vec{\phi})\rangle \in \theta_\Sigma$ and from the hypothesis $\langle\sigma',\tau'\rangle \in R^\theta$, we get $\langle\sigma'_\Sigma(\vec{\psi}),\tau'_\Sigma(\vec{\psi})\rangle \in \theta_\Sigma$. Hence, since θ is a congruence system, we get $\langle\tau_\Sigma(\vec{\phi}),\tau'_\Sigma(\vec{\psi})\rangle \in \theta_\Sigma$. This proves that R^θ is compatible with θ. ∎

We now characterize the closed sets in $\mathcal{P}(\mathrm{NEq}(\mathbf{F}))$. They turn out to be those metacongruences on $\mathbf{F}$ satisfying an additional property.

Lemma 9 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $R \subseteq \mathrm{NEq}(\mathbf{F})$. *If* $C^N(R) = R$, *then* $R \in \mathrm{MetCon}(\mathbf{F})$.

Proof: Let $R \subseteq \mathrm{NEq}(\mathbf{F})$ and $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$, such that $R = \mathrm{NEq}(\mathsf{K})$.

For σ in $N^\flat$, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle \in \mathsf{K}$, with $\mathbf{A} = \langle \mathbf{Sign}, \mathrm{SEN}, N \rangle$, $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi} \in \mathrm{SEN}(\Sigma)$, $\sigma^{\mathcal{A}}_\Sigma(\vec{\phi}) = \sigma^{\mathcal{A}}_\Sigma(\vec{\phi})$. Hence $\mathcal{A} \vDash \sigma \approx \sigma$, showing that $\sigma \approx \sigma \in R$.

Suppose that $\sigma \approx \tau \in R$ and let $\mathcal{A} \in \mathsf{K}$, $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi} \in \mathrm{SEN}(\Sigma)$. Then, since $R = \mathrm{NEq}(\mathsf{K})$, $\sigma^{\mathcal{A}}_\Sigma(\vec{\phi}) = \tau^{\mathcal{A}}_\Sigma(\vec{\phi})$. Thus, $\tau^{\mathcal{A}}_\Sigma(\vec{\phi}) = \sigma^{\mathcal{A}}_\Sigma(\vec{\phi})$. This shows that $\mathcal{A} \vDash \tau \approx \sigma$ and, since $R = \mathrm{NEq}(\mathsf{K})$, we get $\tau \approx \sigma \in R$.

Similarly, we see that R is transitive. Therefore, R is an equivalence relation on $\mathrm{NEq}(\mathbf{F})$. To conclude the proof, we show that R also satisfies the substitution property. To this end, let $\langle o, \rho \rangle \in R$ and $\langle \sigma^i, \tau^i \rangle \in R$, $i < \omega$. Then, for all $\mathcal{A} \in \mathsf{K}$, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)$, since $R = \mathrm{NEq}(\mathsf{K})$, $\sigma^{i\mathcal{A}}_\Sigma(\vec{\phi}) = \tau^{i\mathcal{A}}_\Sigma(\vec{\phi})$, for all $i < \omega$. Therefore,

$$\begin{array}{rcll} o^{\mathcal{A}}_\Sigma(\vec{\sigma}^{\mathcal{A}}_\Sigma(\vec{\phi})) & = & \rho^{\mathcal{A}}_\Sigma(\vec{\sigma}^{\mathcal{A}}_\Sigma(\vec{\phi})) & (\langle o, \rho \rangle \in R = \mathrm{NEq}(\mathsf{K})) \\ & = & \rho^{\mathcal{A}}_\Sigma(\vec{\tau}^{\mathcal{A}}_\Sigma(\vec{\phi})). & (\sigma^{i\mathcal{A}}_\Sigma(\vec{\phi}) = \tau^{i\mathcal{A}}_\Sigma(\vec{\phi}),\ i < \omega) \end{array}$$

Thus, $\mathcal{A} \vDash o \circ \vec{\sigma} \approx \rho \circ \vec{\tau}$ and, hence, since $R = \mathrm{NEq}(\mathsf{K})$, we get $o \circ \vec{\sigma} \approx \rho \circ \vec{\tau} \in R$. We now conclude that R also satisfies the substitution property and, therefore, it is a metacongruence on $\mathbf{F}$. ■

There is one additional property, however, that, by necessity, all metacongruences on $\mathbf{F}$ of the form $\mathrm{NEq}(\mathsf{K})$, for some class K of $\mathbf{F}$-algebraic systems, must satisfy. Recall that, given $\theta \in \mathrm{ConSys}(\mathbf{F})$, we defined

$$\begin{array}{rcl} R^\theta & = & \{\langle \sigma, \tau \rangle : (\forall \Sigma \in |\mathbf{Sign}^\flat|)(\forall \vec{\phi} \in \mathrm{SEN}^\flat(\Sigma))(\langle \sigma_\Sigma(\vec{\phi}), \tau_\Sigma(\vec{\phi}) \rangle \in \theta_\Sigma)\} \\ & = & \mathrm{NEq}(\mathcal{F}/\theta). \end{array}$$

Definition 10 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system. A metacongruence* $R \in \mathrm{MetCon}(\mathbf{F})$ *is called* **feasible** *if there exists a congruence system* $\theta \in \mathrm{ConSys}(\mathbf{F})$, *such that* $R = R^\theta$.

Lemma 11 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $R \subseteq \mathrm{NEq}(\mathbf{F})$. *If* $C^N(R) = R$, *then* R *is a feasible metacongruence on* $\mathbf{F}$.

Proof: Let $R \in \mathrm{NEq}(\mathbf{F})$, such that $R = C^N(R)$. By Lemma 9, $R \in \mathrm{MetCon}(\mathbf{F})$. To see that R is feasible, let $\mathsf{K} = \mathrm{NMod}(R)$ and define $\theta = \mathrm{Eq}(\mathsf{K})$. It suffices to show that $R = R^\theta$.

Suppose, first, that $\sigma \approx \tau \in R$. Let $\mathcal{A} \in \mathsf{K}$, $\Sigma \in |\mathbf{Sign}^\flat|$, $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$. Since $\sigma \approx \tau \in R$ and $\mathcal{A} \in \mathsf{K} = \mathrm{NMod}(R)$, $\mathcal{A} \models \sigma \approx \tau$, whence $\sigma^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})) = \tau^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi}))$. Equivalently, $\alpha_\Sigma(\sigma_\Sigma(\vec{\phi})) = \alpha_\Sigma(\tau_\Sigma(\vec{\phi}))$. Since this holds for all $\mathcal{A} \in \mathsf{K}$, $\langle \sigma_\Sigma(\vec{\phi}), \tau_\Sigma(\vec{\phi})\rangle \in \mathrm{Eq}_\Sigma(\mathsf{K}) = \theta_\Sigma$. Since this holds for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$, $\sigma \approx \tau \in R^\theta$.

Suppose, conversely, that $\sigma \approx \tau \in R^\theta$. Then, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$, $\langle \sigma_\Sigma(\vec{\phi}), \tau_\Sigma(\vec{\phi})\rangle \in \theta_\Sigma = \mathrm{Eq}_\Sigma(\mathsf{K})$. That is, for all $\mathcal{A} \in \mathsf{K}$, $\mathcal{A} \models \sigma \approx \tau$ and, hence, $\sigma \approx \tau \in \mathrm{NEq}(\mathsf{K}) = \mathrm{NEq}(\mathrm{NMod}(R)) = R$. ■

We are now ready for the promised characterization of the closed sets of natural equations under C^N.

Proposition 12 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system and $R \subseteq \mathrm{NEq}(\mathbf{F})$. Then $C^N(R) = R$ if and only if R is a feasible metacongruence on $\mathbf{F}$.*

Proof: If $R = C^N(R)$, then, by Lemma 11, R is a feasible metacongruence on $\mathbf{F}$. Suppose, conversely, that R is a feasible metacongruence on $\mathbf{F}$. Then, by definition, there exists $\theta \in \mathrm{ConSys}(\mathbf{F})$, such that $R = R^\theta$. By the theory of Galois connections, to show that R is closed under C^N, it suffices to show that it is in the image of NEq. In fact, it is not difficult to see that $R = \mathrm{NEq}(\mathcal{F}/\theta) \coloneqq \mathrm{NEq}(\{\mathcal{F}/\theta\})$, where $\mathcal{F} = \langle \mathbf{F}, \langle I, \iota\rangle\rangle$:

$$\begin{array}{rll}
\sigma \approx \tau \in R & \text{iff} & \sigma \approx \tau \in R^\theta \\
 & \text{iff} & \text{for all } \Sigma \in |\mathbf{Sign}^\flat| \text{ and all } \vec{\phi} \in \mathrm{SEN}^\flat(\Sigma), \\
 & & \langle \sigma_\Sigma(\vec{\phi}), \tau_\Sigma(\vec{\phi})\rangle \in \theta_\Sigma \\
 & \text{iff} & \text{for all } \Sigma \in |\mathbf{Sign}^\flat| \text{ and all } \vec{\phi} \in \mathrm{SEN}^\flat(\Sigma), \\
 & & \sigma^\theta_\Sigma(\vec{\phi}/\theta_\Sigma) = \tau^\theta_\Sigma(\vec{\phi}/\theta_\Sigma) \\
 & \text{iff} & \sigma \approx \tau \in \mathrm{NEq}(\mathcal{F}/\theta).
\end{array}$$

Therefore, $R = C^N(R)$, as was to be shown. ■

Finally, we characterize the closed sets in $\mathcal{P}(\mathrm{AlgSys}(\mathbf{F}))$ under $\mathbb{V}^{\mathsf{Syn}}$, i.e., those that constitute syntactic varieties of $\mathbf{F}$-algebraic systems. Similarly to the case of semantic varieties, they turn out to be those classes of $\mathbf{F}$-algebraic systems that are closed under morphic images and subdirect intersections and, in addition, are related in a specific way to semantics.

Proposition 13 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system. For all $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$, $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathbb{V}^{\mathsf{Syn}}(\mathsf{K})$.*

Proof: Let K be a class of $\mathbf{F}$-algebraic systems and $\mathcal{A}$ an $\mathbf{F}$-algebraic system. Suppose $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$ and let $\sigma \approx \tau \in \mathrm{NEq}(\mathsf{K})$. This means that, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$, $\sigma_\Sigma(\vec{\phi}) \approx \tau_\Sigma(\vec{\phi}) \in \mathrm{Eq}_\Sigma(\mathsf{K})$. Since $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathrm{Mod}(\mathrm{Eq}(\mathsf{K}))$, we get that $\mathcal{A} \vDash_\Sigma \sigma_\Sigma(\vec{\phi}) \approx \tau_\Sigma(\vec{\phi})$. Since this holds for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi} \in \mathrm{SEN}^\flat(\Sigma)$, we conclude that $\mathcal{A} \vDash \sigma \approx \tau$. But $\sigma \approx \tau \in \mathrm{NEq}(\mathsf{K})$ was arbitrary, whence $\mathcal{A} \in \mathrm{NMod}(\mathrm{NEq}(\mathsf{K})) = \mathbb{V}^{\mathsf{Syn}}(\mathsf{K})$. We conclude that $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathbb{V}^{\mathsf{Syn}}(\mathsf{K})$. ■

Corollary 14 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be an algebraic system and* $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$. *If* $\mathbb{V}^{\mathsf{Syn}}(\mathsf{K}) = \mathsf{K}$, *then* $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathsf{K}$.

Proof: If $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$, such that $\mathbb{V}^{\mathsf{Syn}}(\mathsf{K}) = \mathsf{K}$, then

$$\begin{array}{lcll} \mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) & \subseteq & \mathbb{V}^{\mathsf{Syn}}(\mathsf{K}) & \text{(by Proposition 13)} \\ & = & \mathsf{K}. & \text{(by hypothesis)} \end{array}$$

Since the reverse inclusion always holds, we get the conclusion. ■

But, if K is a syntactic variety, it has to satisfy an additional condition.

Definition 15 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$ *a class of* $\mathbf{F}$*-algebraic systems. The class* K *is called* **natural** *if* $\mathrm{Eq}(\mathsf{K}) = \theta^{\mathrm{NEq}(\mathsf{K})}$.

Proposition 16 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system. For all* $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$, *if* $\mathbb{V}^{\mathsf{Syn}}(\mathsf{K}) = \mathsf{K}$, *then* K *is a natural class.*

Proof: Let $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F})$, such that $\mathbb{V}^{\mathsf{Syn}}(\mathsf{K}) = \mathsf{K}$. We must show that $\mathrm{Eq}(\mathsf{K}) = \theta^{\mathrm{NEq}(\mathsf{K})}$.

Suppose, first, that $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in \mathrm{Eq}_\Sigma(\mathsf{K})$. Note that, since $\mathsf{K} = \mathbb{V}^{\mathsf{Syn}}(\mathsf{K}) := \mathrm{NMod}(\mathrm{NEq}(\mathsf{K}))$, we obtain that $\mathcal{F}/\theta^{\mathrm{NEq}(\mathsf{K})} \in \mathsf{K}$. Since $\langle \phi, \psi \rangle \in \mathrm{Eq}_\Sigma(\mathsf{K})$, we now get $\mathcal{F}/\theta^{\mathrm{NEq}(\mathsf{K})} \vDash_\Sigma \phi \approx \psi$. But this means that $\langle \phi, \psi \rangle \in \theta_\Sigma^{\mathrm{NEq}(\mathsf{K})}$. Therefore, $\mathrm{Eq}(\mathsf{K}) \leq \theta^{\mathrm{NEq}(\mathsf{K})}$.

Assume, conversely, that $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in \theta_\Sigma^{\mathrm{NEq}(\mathsf{K})}$. By definition, there exists $\sigma \approx \tau \in \mathrm{NEq}(\mathsf{K})$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi = \sigma_\Sigma(\vec{\chi})$ and $\psi = \tau_\Sigma(\vec{\chi})$. Thus, since $\sigma \approx \tau \in \mathrm{NEq}(\mathsf{K})$, if $\mathcal{A} \in \mathsf{K}$, we get $\mathcal{A} \vDash_\Sigma \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi})$, i.e., $\mathcal{A} \vDash_\Sigma \phi \approx \psi$. Since this holds for all $\mathcal{A} \in \mathsf{K}$, we conclude that $\langle \phi, \psi \rangle \in \mathrm{Eq}_\Sigma(\mathsf{K})$. Hence, $\theta^{\mathrm{NEq}(\mathsf{K})} \leq \mathrm{Eq}(\mathsf{K})$. ■

Now, we are ready to provide a characterization of syntactic varieties.

Proposition 17 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system. A class* K *of* $\mathbf{F}$*-algebraic systems is a syntactic variety if and only if it is a natural semantic variety.*

Proof: If K is a syntactic variety, then, by Corollary 14, it is a semantic variety and, moreover, by Proposition 16, it is a natural class. Suppose, conversely, that K is a natural semantic variety. We must show that $\mathsf{K} = \mathbb{V}^{\mathsf{Syn}}(\mathsf{K}) = \mathrm{NMod}(\mathrm{NEq}(\mathsf{K}))$. Since the left to right inclusion always holds, we focus on showing the reverse. To this end, assume $\mathcal{A} \in \mathrm{NMod}(\mathrm{NEq}(\mathsf{K}))$. Since, by hypothesis, $\mathsf{K} = \mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathrm{Mod}(\mathrm{Eq}(\mathsf{K}))$, we have $\mathcal{F}/\mathrm{Eq}(\mathsf{K}) \in \mathsf{K}$. Since, by hypothesis, K is natural, we have $\mathrm{Eq}(\mathsf{K}) = \theta^{\mathrm{NEq}(\mathsf{K})}$, whence, $\mathcal{F}/\theta^{\mathrm{NEq}(\mathsf{K})} \in \mathsf{K}$. The proof will be completed if we show that $\mathcal{A}$ is a morphic image of $\mathcal{F}/\theta^{\mathrm{NEq}(\mathsf{K})}$, since, then, by Proposition 6 and the hypothesis, we will have $\mathcal{A} \in \mathbb{H}(\mathsf{K}) = \mathsf{K}$. In fact, we show that

$$\langle F, \alpha^{\theta^{\mathrm{NEq}(\mathsf{K})}} \rangle : \mathcal{F}/\theta^{\mathrm{NEq}(\mathsf{K})} \to \mathcal{A},$$

$$\begin{tikzcd}
 & \mathbf{F} \arrow[dl, "\langle I, \pi^{\theta^{\mathrm{NEq}(\mathsf{K})}} \rangle"'] \arrow[dr, "\langle F, \alpha \rangle"] & \\
\mathbf{F}/\theta^{\mathrm{NEq}(\mathsf{K})} \arrow[rr, "\langle F, \alpha^{\theta^{\mathrm{NEq}(\mathsf{K})}} \rangle"'] & & \mathbf{A}
\end{tikzcd}$$

defined, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, by

$$\alpha_\Sigma^{\theta^{\mathrm{NEq}(\mathsf{K})}}(\phi/\theta_\Sigma^{\mathrm{NEq}(\mathsf{K})}) = \alpha_\Sigma(\phi),$$

is an algebraic system morphism. To verify this, let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in \theta_\Sigma^{\mathrm{NEq}(\mathsf{K})}$. Then, by definition, there exists $\sigma \approx \tau \in \mathrm{NEq}(\mathsf{K})$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi = \sigma_\Sigma(\vec{\chi})$ and $\psi = \tau_\Sigma(\vec{\chi})$. As, by hypothesis, $\mathcal{A} \in \mathrm{NMod}(\mathrm{NEq}(\mathsf{K}))$, we get that $\mathcal{A} \vDash \sigma \approx \tau$. In particular, $\mathcal{A} \vDash_\Sigma \sigma_\Sigma(\vec{\chi}) \approx \tau_\Sigma(\vec{\chi})$, i.e., $\mathcal{A} \vDash_\Sigma \phi \approx \psi$. But this means that $\alpha_\Sigma(\phi) = \alpha_\Sigma(\psi)$ and proves that $\alpha^{\theta^{\mathrm{Neq}(\mathsf{K})}}$ and, hence $\langle F, \alpha^{\theta^{\mathrm{NEq}(\mathsf{K})}} \rangle : \mathcal{F}/\theta^{\mathrm{NEq}(\mathsf{K})} \to \mathcal{A}$ is a well-defined $\mathbf{F}$-morphism. ∎

Corollary 18 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system. A class* K *of* $\mathbf{F}$*-algebraic systems is a syntactic variety if and only if* K *is natural and closed under* $\mathbb{H}$ *and* $\overset{\triangleleft}{\mathbb{P}}$.

Proof: We have K is a syntactic variety if and only if, by Proposition 17, it is a natural semantic variety, if and only if, by Proposition 6, it is a natural class closed under $\mathbb{H}$ and $\overset{\triangleleft}{\mathbb{P}}$. ∎

4 The Closures C, C^N

In this section, we characterize $C : \mathcal{P}\mathrm{Eq}(\mathbf{F}) \to \mathcal{P}\mathrm{Eq}(\mathbf{F})$ and $C^N : \mathcal{P}\mathrm{NEq}(\mathbf{F}) \to \mathcal{P}\mathrm{NEq}(\mathbf{F})$ as closure operators, by showing how to obtain the closure of given $X \leq \mathrm{Eq}(\mathbf{F})$ and $E \subseteq \mathrm{NEq}(\mathbf{F})$ in a step-wise fashion. In particular, our processes will show that both operators are finitary closure operators.

Let $X \leq \mathrm{Eq}(\mathbf{F})$. We define, for all $k < \omega$, by induction on k, the family $X^k = \{X^k_\Sigma\}_{\Sigma \in |\mathbf{Sign}^\flat|} \leq \mathrm{Eq}(\mathbf{F})$ by letting, for all $\Sigma \in |\mathbf{Sign}^\flat|$, X^k_Σ be given by

$$\begin{array}{rcl}
X^0_\Sigma & = & X_\Sigma \cup \{\phi \approx \phi : \phi \in \mathrm{SEN}^\flat(\Sigma)\};\\
X^{k+1}_\Sigma & = & X^k_\Sigma \cup \{\psi \approx \phi : \phi \approx \psi \in X^k_\Sigma\}\\
& & \cup \{\phi \approx \chi : \phi \approx \psi, \psi \approx \chi \in X^k_\Sigma\}\\
& & \cup \{\sigma_\Sigma(\vec{\phi}) \approx \sigma_\Sigma(\vec{\psi}) : \vec{\phi} \approx \vec{\psi} \in X^k_\Sigma \text{ and } \sigma \in N^\flat\}\\
& & \cup \{\mathrm{SEN}^\flat(f)(\phi \approx \psi) : \phi \approx \psi \in X^k_{\Sigma'} \text{ and } f \in \mathbf{Sign}^\flat(\Sigma', \Sigma)\}.
\end{array}$$

Proposition 19 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system and $X \leq \mathrm{Eq}(\mathbf{F})$. Then*

$$C(X) = \bigcup_{k=0}^{\infty} X^k.$$

Proof: Note that, for all $\Sigma \in |\mathbf{Sign}^\flat|$, $X_\Sigma \subseteq X^0_\Sigma \subseteq \bigcup_{k=0}^\infty X^k_\Sigma$. So to see that $C(X) \leq \bigcup_{k=0}^\infty X^k$, it suffices to show, by Proposition 3, that $\bigcup_{k=0}^\infty X^k$ is a congruence system on $\mathbf{F}$.

- For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \approx \phi \in X^0_\Sigma \subseteq \bigcup_{k=0}^\infty X^k_\Sigma$, whence $\bigcup_{k=0}^\infty X^k$ is reflexive;
- For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\psi \approx \phi \in \bigcup_{k=0}^\infty X^k_\Sigma$, we get that $\psi \approx \phi \in X^k_\Sigma$, for some $k < \infty$, whence $\phi \approx \psi \in X^{k+1}_\Sigma \subseteq \bigcup_{k=0}^\infty X^k_\Sigma$, showing that $\bigcup_{k=0}^\infty X^k$ is symmetric;
- For all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi, \psi, \chi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \approx \psi, \psi \approx \chi \in \bigcup_{k=0}^\infty X^k_\Sigma$, we get that $\phi \approx \psi \in X^k_\Sigma$ and $\psi \approx \chi \in X^\ell_\Sigma$, for some $k, \ell < \infty$, whence $\phi \approx \chi \in X^{\max\{k,\ell\}+1}_\Sigma \subseteq \bigcup_{k=0}^\infty X^k_\Sigma$, showing that $\bigcup_{k=0}^\infty X^k$ is also transitive;
- For all $\Sigma, \Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma, \Sigma')$, and all $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \approx \psi \in \bigcup_{k=0}^\infty X^k_\Sigma$, we get that $\phi \approx \psi \in X^k_\Sigma$, for some $k < \infty$, whence $\mathrm{SEN}(f)(\phi \approx \psi) \in X^{k+1}_{\Sigma'} \subseteq \bigcup_{k=0}^\infty X^k_{\Sigma'}$, showing that $\bigcup_{k=0}^\infty X^k$ is a system;

- For all n-ary σ in $N^\flat$, all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^\flat(\Sigma)$, such that $\vec{\phi} \approx \vec{\psi} \in \bigcup_{k=0}^\infty X_\Sigma^k$, we get that $\phi_i \approx \psi_i \in X_\Sigma^{k_i}$, for some $k_i < \infty$, $i < n$, whence $\sigma_\Sigma(\vec{\phi}) \approx \sigma_\Sigma(\vec{\psi}) \in X_\Sigma^{\max_{i<n}\{k_i\}+1} \subseteq \bigcup_{k=0}^\infty X_\Sigma^k$, showing that $\bigcup_{k=0}^\infty X^k$ also satisfies the congruence property.

Thus, $\bigcup_{k=0}^\infty X^k$ is indeed a congruence system on $\mathbf{F}$ and, hence, since $X \leq \bigcup_{k=0}^\infty X^k$, we get $C(X) \leq \bigcup_{k=0}^\infty X^k$.

Conversely, we work by induction on k to show that, for all $k \geq 0$, $X^k \leq C(X)$. Clearly, $X \leq C(X)$ and, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$, $\phi \approx \phi \in C_\Sigma(X)$, by Proposition 3. Thus, $X^0 \leq C(X)$. Suppose that, for some $k < \infty$, we have $X^k \leq C(X)$. Let $\Sigma \in |\mathbf{Sign}^\flat|$, $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \approx \psi \in X_\Sigma^{k+1}$. Then one of the following must hold:

- $\phi \approx \psi \in X_\Sigma^k$; Then $\phi \approx \psi \in C_\Sigma(X)$, by the induction hypothesis;
- $\psi \approx \phi \in X_\Sigma^k$; Then $\psi \approx \phi \in C_\Sigma(X)$, by the induction hypothesis. Since, by Proposition 3, $C(X) \in \mathrm{ConSys}(\mathbf{F})$, we get that $\phi \approx \psi \in C_\Sigma(X)$;
- $\phi \approx \chi, \chi \approx \psi \in X_\Sigma^k$, for some $\chi \in \mathrm{SEN}^\flat(\Sigma)$; Then $\phi \approx \chi, \chi \approx \psi \in C_\Sigma(X)$, by the induction hypothesis. Since, by Proposition 3, $C(X) \in \mathrm{ConSys}(\mathbf{F})$, we get that $\phi \approx \psi \in C_\Sigma(X)$;
- There exist σ in $N^\flat$, $\vec{\phi}, \vec{\psi} \in X_\Sigma^k$, such that $\vec{\phi} \approx \vec{\psi} \in X_\Sigma^k$ and $\phi = \sigma_\Sigma(\vec{\phi})$, $\psi = \sigma_\Sigma(\vec{\psi})$; Then $\vec{\phi} \approx \vec{\psi} \in C_\Sigma(X)$, by the induction hypothesis. Since, by Proposition 3, $C(X) \in \mathrm{ConSys}(\mathbf{F})$, we get that $\phi \approx \psi = \sigma_\Sigma(\vec{\phi}) \approx \sigma_\Sigma(\vec{\psi}) \in C_\Sigma(X)$;
- There exist $\Sigma' \in |\mathbf{Sign}^\flat|$, $f \in \mathbf{Sign}^\flat(\Sigma', \Sigma)$ and $\phi', \psi' \in \mathrm{SEN}^\flat(\Sigma')$, such that $\phi' \approx \psi' \in X_{\Sigma'}^k$, and $\phi = \mathrm{SEN}^\flat(f)(\phi')$, $\psi = \mathrm{SEN}^\flat(f)(\psi')$; Then $\phi' \approx \psi' \in C_{\Sigma'}(X)$, by the induction hypothesis. Since, by Proposition 3, $C(X) \in \mathrm{ConSys}(\mathbf{F})$, we get that $\phi \approx \psi = \mathrm{SEN}^\flat(f)(\phi' \approx \psi') \in C_\Sigma(X)$.

We conclude that $X^{k+1} \leq C(X)$, which finishes the induction and shows that $\bigcup_{k=0}^\infty X^k \leq C(X)$. Equality now follows. ■

Corollary 20 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system. The closure operator $C : \mathcal{P}\mathrm{Eq}(\mathbf{F}) \to \mathcal{P}\mathrm{Eq}(\mathbf{F})$ is finitary.*

Proof: Let $X \leq \mathrm{Eq}(\mathbf{F})$, $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi \approx \psi \in C_\Sigma(X)$. By Proposition 19, $\phi \approx \psi \in \bigcup_{k=0}^\infty X_\Sigma^k$. Thus, there exists $k < \omega$, such that $\phi \approx \psi \in X^k_\Sigma$. It is not difficult to show, by induction on k, that this implies that, there exists a locally finite subfamily $Y \leq_{lf} X$ of X, such that $\phi \approx \psi \in Y^k_\Sigma \subseteq C_\Sigma(Y)$. This proves that C is indeed finitary. ■

We work analogously regarding C^N. However, to provide an elegant, relatively simple characterization of $C^N(E)$, for every collection E of natural equations, and, importantly, one referring only to syntax, we opt to incorporate the unavoidable reference to semantics in the general hypotheses underlying our adopted framework. Accordingly, it is assumed that the base algebraic system $\mathbf{F}$ is such that all its metacongruences are feasible. Such an algebraic system is termed *metafeasible*.

Definition 21 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be an algebraic system. $\mathbf{F}$ is called* **metafeasible** *if every metacongruence R on $\mathbf{F}$ is feasible, i.e., if, for all $R \in \mathrm{MetCon}(\mathbf{F})$, there exists $\theta \in \mathrm{ConSys}(\mathbf{F})$, such that $R = R^\theta$.*

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system and $E \subseteq \mathrm{NEq}(\mathbf{F})$. We define, for all $k < \omega$, by induction on k, the family $E^k \subseteq \mathrm{NEq}(\mathbf{F})$ as follows:

$$\begin{array}{rcl}
E^0 & = & E \cup \{\sigma \approx \sigma : \sigma \in N^\flat\};\\
E^{k+1} & = & E^k \cup \{\tau \approx \sigma : \sigma \approx \tau \in E^k\}\\
& & \quad \cup \{\sigma \approx \rho : \sigma \approx \tau, \tau \approx \rho \in E^k\}\\
& & \quad \cup \{o \circ \vec{\sigma} \approx \rho \circ \vec{\tau} : o \approx \rho, \vec{\sigma} \approx \vec{\tau} \in E^k\}.
\end{array}$$

Moreover, we set

$$E^\cup = \bigcup_{k=0}^\infty E^k.$$

Our eventual goal is to show that, under the hypothesis that $\mathbf{F}$ is metafeasible, for all $E \subseteq \mathrm{NEq}(\mathbf{F})$, $E^\cup$ is exactly $C^N(E)$. We proceed in steps by formulating some lemmas, forming parts of the proof of Proposition 26.

We start by showing that each level E^k, $k \geq 0$, is contained in $C^N(E)$.

Lemma 22 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ be a base algebraic system and $E \subseteq \mathrm{NEq}(\mathbf{F})$. For all $k \geq 0$, $E^k \subseteq C^N(E)$.*

Proof: Let $E \subseteq \mathrm{NEq}(\mathbf{F})$. We work by induction on $k \geq 0$.

Since C^N is a closure operator, $E \subseteq C^N(E)$. Moreover, since, for all $\mathcal{A} \in \mathrm{NMod}(E)$, it is obvious that $\mathcal{A} \vDash \sigma \approx \sigma$, for all σ in $N^\flat$, we get that $\{\sigma \approx \sigma : \sigma \text{ in } N^\flat\} \subseteq C^N(E)$. This proves that $E^0 \subseteq C^N(E)$.

Assume, now, that, for some $k \geq 0$, $E^k \subseteq C^N(E)$.

Suppose, first, that $\tau \approx \sigma \in E^k$. Then, by the induction hypothesis, $\tau \approx \sigma \in C^N(E)$. Thus, by Proposition 12, $\sigma \approx \tau \in C^N(E)$.

Suppose, next, that $\sigma \approx \tau, \tau \approx \rho \in E^k$. Then, by the induction hypothesis, $\sigma \approx \tau, \tau \approx \rho \in C^N(E)$. Thus, by Proposition 12, $\sigma \approx \rho \in C^N(E)$.

Suppose, finally, that $o \approx \rho \in E^k$ and $\vec{\sigma} \approx \vec{\tau} \subseteq E^k$, Once more applying the induction hypothesis, we get $o \approx \rho \in C^N(E)$ and $\vec{\sigma} \approx \vec{\tau} \subseteq C^N(E)$. Hence, by Proposition 12, $o \circ \vec{\sigma} \approx \rho \circ \vec{\tau} \in C^N(E)$.

This proves that $E^{k+1} \subseteq C^N(E)$, and concludes the induction.

Thus, for all $k \geq 0$, $E^k \subseteq C^N(E)$. ■

Immediately from Lemma 22 and the definition of $E^\cup$, we get

Corollary 23 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* $E \subseteq \mathrm{NEq}(\mathbf{F})$. *Then* $E^\cup \subseteq C^N(E)$.

The next lemma asserts that $E^\cup$ is a metacongruence on $\mathbf{F}$.

Lemma 24 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* $E \subseteq \mathrm{NEq}(\mathbf{F})$. $E^\cup \in \mathrm{MetCon}(\mathbf{F})$.

Proof: We show that $E^\cup$ satisfies all required properties.

- For all σ in $N^\flat$, $\sigma \approx \sigma \in E^0 \subseteq E^\cup$. So $E^\cup$ is reflexive;
- Let $\sigma \approx \tau \in E^\cup = \bigcup_{k=0}^\infty E^k$. Then, there exists $k < \omega$, such that $\sigma \approx \tau \in E^k$, whence, by definition, $\tau \approx \sigma \in E^{k+1} \subseteq \bigcup_{k=0}^\infty E^k = E^\cup$. Thus, $E^\cup$ is also symmetric;
- Suppose $\sigma \approx \tau, \tau \approx \rho \in E^\cup = \bigcup_{k=0}^\infty E^k$. Then, there exist $k, \ell < \omega$, such that $\sigma \approx \tau \in E^k$ and $\tau \approx \rho \in E^\ell$. Thus, by definition, $\sigma \approx \rho \in E^{\max\{k,\ell\}+1} \subseteq \bigcup_{k=0}^\infty E^k = E^\cup$. Therefore, $E^\cup$ is transitive;
- Finally, suppose that $o \approx \rho \in E^\cup = \bigcup_{k=0}^\infty E^k$ and $\vec{\sigma} \approx \vec{\tau} \in E^\cup = \bigcup_{k=0}^\infty E^k$ and assume that $n = \max\{\mathrm{ar}(o), \mathrm{ar}(\rho)\}$, where $\mathrm{ar}(o)$ and $\mathrm{ar}(\rho)$ denote, respectively, the arities of the natural transformations o and ρ. Then, there exist $k, \ell_0, \ldots, \ell_{n-1} < \omega$, such that $o \approx \rho \in E^k$ and $\sigma^i \approx \tau^i \in E^{\ell_i}$, $i < n$. Thus, taking again $m = \max\{k, \ell_0, \ldots, \ell_{n-1}\}$, we get, by definition, $o \circ \vec{\sigma} \approx \rho \circ \vec{\tau} \in E^{m+1} \subseteq \bigcup_{k=0}^\infty E^k = E^\cup$.

We now conclude that $E^\cup \in \mathrm{MetCon}(\mathbf{F})$. ■

At this point, to use Lemma 24 advantageously, we resort to our preannounced hypothesis that the base algebraic system $\mathbf{F}$ be metafeasible.

Corollary 25 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a metafeasible base algebraic system and* $E \subseteq \mathrm{NEq}(\mathbf{F})$*. Then* $C^N(E) \subseteq E^\cup$*.*

Proof: By Lemma 24, $E^\cup$ is a metacongruence on $\mathbf{F}$ and, by definition, it contains E. By the metafeasibility of $\mathbf{F}$, $E^\cup$ is a feasible metacongruence on $\mathbf{F}$ containing E. By Proposition 12, $C^N(E)$ is the least feasible metacongruence on $\mathbf{F}$ containing E. We conclude that $C^N(E) \subseteq E^\cup$. ∎

We are now ready for the main

Proposition 26 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a metafeasible algebraic system and* $E \subseteq \mathrm{NEq}(\mathbf{F})$*. Then*

$$C^N(E) = E^\cup.$$

Proof: By Corollary 23, $E^\cup \subseteq C^N(E)$, and, by Corollary 25, we have $C^N(E) \subseteq E^\cup$. ∎

Finally, we use Proposition 26 to show that, under the hypothesis of metafeasibility of $\mathbf{F}$, the closure operator C^N on $\mathrm{NEq}(\mathbf{F})$ is finitary.

Lemma 27 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a metafeasible algebraic system. For all* $E \subseteq \mathrm{NEq}(\mathbf{F})$ *and all* $k \geq 0$*,*

$$E^k \subseteq \bigcup \{E'^k : E' \subseteq_f E\},$$

where $\subseteq_f$ *denotes the finite subset relation.*

Proof: We use induction on $k \geq 0$.

If $k = 0$ and $\sigma \approx \tau \in E^0$, then $\sigma \approx \tau \in E$ or $\sigma = \tau$. In the first case, $\sigma \approx \tau \in \{\sigma \approx \tau\}^0$ and, in the second, $\sigma \approx \sigma \in \varnothing^0$.

Suppose, now, that the claim holds for some $k \geq 0$.

If $\tau \approx \sigma \in E^k$, then, by the induction hypothesis, $\tau \approx \sigma \in E'^k$, for some $E' \subseteq_f E$. Therefore, we get $\sigma \approx \tau \in E'^{k+1}$.

Similarly, if $\sigma \approx \tau, \tau \approx \rho \in E^k$, then, by the induction hypothesis, there exist $E', E'' \subseteq_f E$, such that $\sigma \approx \tau \in E'^k$ and $\tau \approx \rho \in E''^k$. Thus, we get $\sigma \approx \tau, \tau \approx \rho \in (E' \cup E'')^k$ and, therefore, $\sigma \approx \rho \in (E' \cup E'')^{k+1}$.

Finally, assume that $o \approx \rho \in E^k$ and $\vec{\sigma} \approx \vec{\tau} \subseteq E^k$. Then, by the induction hypothesis, for $n = \max\{\mathrm{ar}(o), \mathrm{ar}(\rho)\}$, there exist $E_0, \ldots, E_n \subseteq_f E$, such that $o \approx \rho \in E_0^k$ and $\sigma^i \approx \tau^i \in E_{i+1}^k$, for $i \leq n$. Therefore, we get that $o \approx \rho \in (\bigcup_{i=0}^n E_i)^k$ and $\vec{\sigma} \approx \vec{\tau} \subseteq (\bigcup_{i=0}^n E_i)^k$, and this yields $o \circ \vec{\sigma} \approx \rho \circ \vec{\tau} \in (\bigcup_{i=0}^n E_i)^{k+1}$. This concludes the proof of the induction step.

We now have that, for all $k \geq 0$, $E^k \subseteq \bigcup \{E'^k : E' \subseteq_f E\}$. ∎

Corollary 28 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a metafeasible algebraic system. The closure operator* $C^N : \mathcal{P}\mathrm{NEq}(\mathbf{F}) \to \mathcal{P}\mathrm{NEq}(\mathbf{F})$ *is finitary.*

Proof: By Proposition 26 and Lemma 27. ∎

5 The Closures $\mathbb{V}^{\mathsf{Sem}}$ and $\mathbb{V}^{\mathsf{Syn}}$

In this section we show how to obtain the semantic and the syntactic varieties generated by a given class K of $\mathbf{F}$-algebraic systems by applying on K a series of class operators. Of course, we shall take advantage and make use of the relevant results obtained in Sections 2 and 3.

We start by looking at two lemmas that we help us show that $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathbb{H}\overset{\lhd}{\mathbb{IP}}(\mathsf{K})$. They are, however, interesting in their own right.

Lemma 29 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems. If* $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$*, then* $\mathcal{A} \in \mathbb{H}\overset{\lhd}{\mathbb{IP}}(\mathsf{K})$.

Proof: Suppose that $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$.

$$\begin{tikzcd}
\mathbf{F}/\bigcap_{\mathcal{B}\in\mathsf{K}} \mathrm{Ker}(\mathcal{B}) \arrow[r] \arrow[d] & \mathcal{B}, \quad \mathcal{B} \in \mathsf{K} \\
\mathcal{A} &
\end{tikzcd}$$

By Proposition 4, $\mathcal{F}/\bigcap_{\mathcal{B}\in\mathsf{K}} \mathrm{Ker}(\mathcal{B}) \in \overset{\lhd}{\mathbb{IP}}(\mathsf{K})$. Since, by hypothesis, $\mathcal{A} \in \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$, we have, by the definition of a semantic variety, $\bigcap_{\mathcal{B}\in\mathsf{K}} \mathrm{Ker}(\mathcal{B}) \le \mathrm{Ker}(\mathcal{A})$. Therefore, $\mathcal{A} \in \mathbb{H}(\mathcal{F}/\bigcap_{\mathcal{B}\in\mathsf{K}} \mathrm{Ker}(\mathcal{B}))$. We conclude that $\mathcal{A} \in \mathbb{H}\overset{\lhd}{\mathbb{IP}}(\mathsf{K})$. ∎

Lemma 30 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems. Then* $\overset{\lhd}{\mathbb{IP}}\mathbb{H}(\mathsf{K}) \subseteq \mathbb{H}\overset{\lhd}{\mathbb{IP}}(\mathsf{K})$.

Proof: Suppose that $\mathcal{A} = \langle \mathbf{F}, \langle F, \alpha \rangle \rangle \in \overset{\lhd}{\mathbb{IP}}\mathbb{H}(\mathsf{K})$. Thus, there exist $\mathcal{B}^i = \langle \mathbf{B}^i, \langle G^i, \beta^i \rangle \rangle \in \mathbb{H}(\mathsf{K})$, $i \in I$, and a subdirect intersection $\{\langle H^i, \gamma^i \rangle : \mathcal{A} \to \mathcal{B}^i, i \in$

$I\}$. Hence, for every $i \in I$, there exists a morphism $\langle D^i, \delta^i\rangle : \mathcal{A}^i \to \mathcal{B}^i$, where $\mathcal{A}^i = \langle \mathbf{A}^i, \langle F^i, \alpha^i\rangle\rangle \in \mathsf{K}$.

$$\begin{tikzcd}
 & \mathbf{F} \arrow[dl, "\langle F^i,\alpha^i\rangle"'] \arrow[d, "\langle G^i,\beta^i\rangle"] \arrow[dr, "\langle F,\alpha\rangle"] & \\
\mathcal{A}^i \arrow[r, "\langle D^i,\delta^i\rangle"'] & \mathcal{B}^i & \mathcal{A} \arrow[l, "\langle H^i,\gamma^i\rangle"]
\end{tikzcd}$$

By Proposition 4, $\mathcal{F}/\bigcap_{i\in I}\mathrm{Ker}(\mathcal{A}^i) \in \mathbb{I}\overset{\triangleleft}{\mathbb{P}}(\mathsf{K})$. So to see that $\mathcal{A} \in \mathbb{H}\mathbb{I}\overset{\triangleleft}{\mathbb{P}}(\mathsf{K})$, it suffices to show that there exists a morphism $\mathcal{F}/\bigcap_{i\in I}\mathrm{Ker}(\mathcal{A}^i) \to \mathcal{A}$ and for this, in turn, it suffices to show that $\bigcap_{i\in I}\mathrm{Ker}(\mathcal{A}^i) \leq \mathrm{Ker}(\mathcal{A})$. Indeed, we have

$$\begin{array}{lll}
\bigcap_{i\in I}\mathrm{Ker}(\mathcal{A}^i) & \leq & \bigcap_{i\in I}\mathrm{Ker}(\mathcal{B}^i) \quad (\langle G^i,\beta^i\rangle = \langle D^i,\delta^i\rangle \circ \langle F^i,\alpha^i\rangle) \\
 & = & \bigcap_{i\in I}\mathrm{Ker}(\langle H^i \circ \gamma^i\rangle \circ \langle F,\alpha\rangle) \quad (\langle G^i,\beta^i\rangle = \langle H^i,\gamma^i\rangle \circ \langle F,\alpha\rangle) \\
 & = & \alpha^{-1}(\bigcap_{i\in I}\mathrm{Ker}(\langle H^i \circ \gamma^i\rangle)) \quad (\text{set theory}) \\
 & = & \alpha^{-1}(\Delta^{\mathcal{A}}) \quad (\{\langle H^i,\gamma^i\rangle : i \in I\} \text{ subdirect intersection}) \\
 & = & \mathrm{Ker}(\mathcal{A}).
\end{array}$$

This shows that $\mathbb{I}\overset{\triangleleft}{\mathbb{P}}\mathbb{H}(\mathsf{K}) \leq \mathbb{H}\mathbb{I}\overset{\triangleleft}{\mathbb{P}}(\mathsf{K})$. ■

Proposition 31 *Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system and K a class of $\mathbf{F}$-algebraic systems. Then $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) = \mathbb{H}\mathbb{I}\overset{\triangleleft}{\mathbb{P}}(\mathsf{K})$.*

Proof: On the one hand, since $\mathsf{K} \subseteq \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$ and, by Proposition 6, $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$ is closed under $\mathbb{H}$ and $\mathbb{I}\overset{\triangleleft}{\mathbb{P}}$, we get that $\mathbb{H}\mathbb{I}\overset{\triangleleft}{\mathbb{P}}(\mathsf{K}) \subseteq \mathbb{V}^{\mathsf{Sem}}(\mathsf{K})$.

For the converse, we can either appeal directly to Lemma 29 or, as was done above, notice that $\mathsf{K} \subseteq \mathbb{H}\mathbb{I}\overset{\triangleleft}{\mathbb{P}}(\mathsf{K})$. But $\mathbb{H}\mathbb{I}\overset{\triangleleft}{\mathbb{P}}(\mathsf{K})$ is obviously closed under $\mathbb{H}$ and, by Lemma 30, is also closed under $\mathbb{I}\overset{\triangleleft}{\mathbb{P}}$. Hence, by Proposition 6, it forms, a semantic variety and, therefore, $\mathbb{V}^{\mathsf{Sem}}(\mathsf{K}) \subseteq \mathbb{H}\mathbb{I}\overset{\triangleleft}{\mathbb{P}}(\mathsf{K})$. ■

Let $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ be a base algebraic system, $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha\rangle\rangle$ and $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta\rangle\rangle$ two $\mathbf{F}$-algebraic systems and $\langle H, \gamma\rangle : \mathcal{A} \to \mathcal{B}$ an $\mathbf{F}$-morphism. Clearly, $\mathrm{NEq}(\mathcal{A}) \subseteq \mathrm{NEq}(\mathcal{B})$. If $\mathrm{NEq}(\mathcal{A}) = \mathrm{NEq}(\mathcal{B})$, then we call $\mathcal{A}$ a (**natural**) **lifting** of $\mathcal{B}$. Given a class K of $\mathbf{F}$-algebraic systems, we let $\mathbb{L}(\mathsf{K})$ denote the **class of all natural liftings** of members of K.

Almost by definition, we get

Lemma 32 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems. If* K *is a syntactic variety, then* $\mathbb{L}(\mathsf{K}) \subseteq \mathsf{K}$.

Proof: Suppose $\mathbb{V}^{\mathsf{Syn}}(\mathsf{K}) \subseteq \mathsf{K}$. Let $\mathcal{A} \in \mathbb{L}(\mathsf{K})$ and $\sigma \approx \tau \in \mathrm{NEq}(\mathsf{K})$. By hypothesis, there exists $\mathcal{B} \in \mathsf{K}$, such that $\mathcal{A}$ is a lifting of $\mathcal{B}$, whence $\sigma \approx \tau \in \mathrm{NEq}(\mathcal{B}) = \mathrm{NEq}(\mathcal{A})$. We conclude that $\mathrm{NEq}(\mathsf{K}) \subseteq \mathrm{NEq}(\mathcal{A})$ and, therefore, $\mathcal{A} \in \mathbb{V}^{\mathsf{Syn}}(\mathsf{K}) = \mathsf{K}$. ■

The addition of closure under the $\mathbb{L}$ operator to those under the $\mathbb{H}$ and the $\overset{\triangleleft}{\mathbb{P}}$ operators guarantees that a class K of $\mathbf{F}$-algebraic systems is a syntactic variety.

Lemma 33 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems. If* K *is closed under* $\mathbb{H}$*,* $\mathbb{L}$ *and* $\overset{\triangleleft}{\mathbb{P}}$*, then it forms a syntactic variety, i.e.,* $\mathbb{V}^{\mathsf{Syn}}(\mathsf{K}) = \mathsf{K}$.

Proof: By Corollary 18, it suffices to show that K is natural, i.e., that $\mathrm{Eq}(\mathsf{K}) = \theta^{\mathrm{NEq}(\mathsf{K})}$. Since $\theta^{\mathrm{NEq}(\mathsf{K})} \leq \mathrm{Eq}(\mathsf{K})$ always holds, it suffices to show the reverse inclusion. We take advantage of the postulated closure of K under the three operators in the following way: First, by closure under $\overset{\triangleleft}{\mathbb{P}}$ (see Proposition 4), $\mathcal{F}/\mathrm{Eq}(\mathsf{K}) \in \mathsf{K}$. Next, since $\theta^{\mathrm{NEq}(\mathsf{K})} \leq \mathrm{Eq}(\mathsf{K})$, there exists a natural morphism

$$\mathcal{F}/\theta^{\mathrm{NEq}(\mathsf{K})} \to \mathcal{F}/\mathrm{Eq}(\mathsf{K}).$$

Moreover, it is not difficult to see that

$$\mathrm{NEq}(\mathcal{F}/\theta^{\mathrm{NEq}(\mathsf{K})}) = \mathrm{NEq}(\mathcal{F}/\mathrm{Eq}(\mathsf{K})) = \mathrm{NEq}(\mathsf{K}).$$

Thus, by closure under $\mathbb{L}$, $\mathcal{F}/\theta^{\mathrm{NEq}(\mathsf{K})} \in \mathsf{K}$. This now ensures that $\mathrm{Eq}(\mathsf{K}) \leq \mathrm{Eq}(\mathcal{F}/\theta^{\mathrm{NEq}(\mathsf{K})}) = \theta^{\mathrm{NEq}(\mathsf{K})}$. Hence, K is indeed natural and, as it is closed under $\mathbb{H}$ and $\overset{\triangleleft}{\mathbb{P}}$, we get, by Corollary 18, that it is a syntactic variety. ■

We now get

Proposition 34 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems.* K *is a syntactic variety if and only if it is closed under* $\mathbb{H}$*,* $\mathbb{L}$ *and* $\overset{\triangleleft}{\mathbb{P}}$.

Proof: If K is a syntactic variety, then, by Corollary 18, it is closed under $\mathbb{H}$ and $\overset{\triangleleft}{\mathbb{P}}$. Finally, by Lemma 32, it is also closed under $\mathbb{L}$. Conversely, if K is closed under the three operators, then it is, by Lemma 33, a syntactic variety. ∎

Finally, we prove that, for a class K of $\mathbf{F}$-algebraic systems, the syntactic variety generated by K is formed by applying on K the operator $\mathbb{H}\mathbb{L}\overset{\triangleleft}{\mathbb{P}}$.

Lemma 35 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems. Then* $\mathbb{L}\mathbb{H}(\mathsf{K}) \subseteq \mathbb{H}\mathbb{L}(\mathsf{K})$.

Proof: Suppose that $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle \in \mathbb{L}\mathbb{H}(\mathsf{K})$. Thus, there exists $\mathcal{B} = \langle \mathbf{B}, \langle G, \beta \rangle \rangle \in \mathbb{H}(\mathsf{K})$ and $\langle J, \delta \rangle : \mathcal{A} \to \mathcal{B}$, such that $\mathrm{NEq}(\mathcal{A}) = \mathrm{NEq}(\mathcal{B})$ and, in turn, $\mathcal{C} = \langle \mathbf{C}, \langle H, \gamma \rangle \rangle \in \mathsf{K}$ and $\langle K, \epsilon \rangle : \mathcal{C} \to \mathcal{B}$.

$$\begin{tikzcd}
 & \mathbf{F} \arrow[dl, "\langle F,\alpha\rangle"'] \arrow[d, "\langle G,\beta\rangle"] \arrow[dr, "\langle H,\gamma\rangle"] & \\
\mathbf{A} \arrow[r, "\langle J,\delta\rangle"'] & \mathbf{B} & \mathbf{C} \arrow[l, "\langle K,\epsilon\rangle"]
\end{tikzcd}$$

Based on these data, we can now define two new morphisms

$$\langle F, \alpha^* \rangle : \mathcal{F}/\theta^{\mathrm{NEq}(\mathcal{C})} \to \mathcal{A} \quad \text{and} \quad \langle H, \gamma^* \rangle : \mathcal{F}/\theta^{\mathrm{NEq}(\mathcal{C})} \to \mathcal{C},$$

as shown in the diagram

$$\begin{tikzcd}
 & \mathbf{F} \arrow[dl, "\langle F,\alpha\rangle"'] \arrow[d, "\langle I,\pi\rangle"] \arrow[dr, "\langle H,\gamma\rangle"] & \\
\mathbf{A} & \mathbf{F}/\theta^{\mathrm{NEq}(\mathcal{C})} \arrow[l, "\langle F,\alpha^*\rangle"] \arrow[r, "\langle H,\gamma^*\rangle"'] & \mathbf{C}
\end{tikzcd}$$

by setting, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi \in \mathrm{SEN}^\flat(\Sigma)$,

$$\alpha^*_\Sigma(\phi/\theta^{\mathrm{NEq}(\mathcal{C})}_\Sigma) = \alpha_\Sigma(\phi) \quad \text{and} \quad \gamma^*_\Sigma(\phi/\theta^{\mathrm{NEq}(\mathcal{C})}_\Sigma) = \gamma_\Sigma(\phi).$$

First, we show that $\langle F, \alpha^* \rangle$ is well-defined. Let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\phi, \psi \in \mathrm{SEN}^\flat(\Sigma)$, such that $\langle \phi, \psi \rangle \in \theta^{\mathrm{NEq}(\mathcal{C})}_\Sigma$. Since $\mathcal{C} \xrightarrow{\langle K,\epsilon\rangle} \mathcal{B}$, we get $\langle \phi, \psi \rangle \in \theta^{\mathrm{NEq}(\mathcal{B})}_\Sigma$.

But, by hypothesis, $\mathrm{NEq}(\mathcal{B}) = \mathrm{NEq}(\mathcal{A})$, whence, $\langle\phi,\psi\rangle \in \theta_\Sigma^{\mathrm{NEq}(\mathcal{A})}$. Thus, there exists $\sigma \approx \tau \in \mathrm{NEq}(\mathcal{A})$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi = \sigma_\Sigma(\vec{\chi})$ and $\psi = \tau_\Sigma(\vec{\chi})$. Thus, we finally get

$$\alpha_\Sigma(\phi) = \alpha_\Sigma(\sigma_\Sigma(\vec{\chi})) = \sigma_\Sigma^{\mathcal{A}}(\alpha_\Sigma(\vec{\chi})) = \tau_\Sigma^{\mathcal{A}}(\alpha_\Sigma(\vec{\chi})) = \alpha_\Sigma(\tau_\Sigma(\vec{\chi})) = \alpha_\Sigma(\psi).$$

This shows that $\langle F,\alpha^*\rangle$ is well-defined. Similarly, for all $\Sigma \in |\mathbf{Sign}^\flat|$ and all $\phi,\psi \in \mathrm{SEN}^\flat(\Sigma)$, if $\langle\phi,\psi\rangle \in \theta_\Sigma^{\mathrm{NEq}(\mathcal{C})}$, there exists $\sigma \approx \tau \in \mathrm{NEq}(\mathcal{C})$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$, such that $\phi = \sigma_\Sigma(\vec{\chi})$ and $\psi = \tau_\Sigma(\vec{\chi})$. But then

$$\gamma_\Sigma(\phi) = \gamma_\Sigma(\sigma_\Sigma(\vec{\chi})) = \sigma_\Sigma^{\mathcal{C}}(\gamma_\Sigma(\vec{\chi})) = \tau_\Sigma^{\mathcal{C}}(\gamma_\Sigma(\vec{\chi})) = \gamma_\Sigma(\tau_\Sigma(\vec{\chi})) = \gamma_\Sigma(\psi)$$

and $\langle H,\gamma^*\rangle$ is also well-defined.

In the next step, we show that $\mathrm{NEq}(\mathcal{F}/\theta^{\mathrm{NEq}(\mathcal{C})}) = \mathrm{NEq}(\mathcal{C})$, which will conclude the proof. First, since $\mathcal{F}/\theta^{\mathrm{NEq}(\mathcal{C})} \xrightarrow{\langle H,\gamma^*\rangle} \mathcal{C}$, we have $\mathrm{NEq}(\mathcal{F}/\theta^{\mathrm{NEq}(\mathcal{C})}) \subseteq \mathrm{NEq}(\mathcal{C})$. For the reverse inclusion, suppose $\sigma \approx \tau \in \mathrm{NEq}(\mathcal{C})$ and let $\Sigma \in |\mathbf{Sign}^\flat|$ and $\vec{\chi} \in \mathrm{SEN}^\flat(\Sigma)$. Then we get

$$\begin{array}{rcll}
\sigma_\Sigma^{\theta^{\mathrm{NEq}(\mathcal{C})}}(\vec{\chi}/\theta_\Sigma^{\mathrm{NEq}(\mathcal{C})}) & = & \sigma_\Sigma(\vec{\chi})/\theta_\Sigma^{\mathrm{NEq}(\mathcal{C})} & \\
 & = & \tau_\Sigma(\vec{\chi})/\theta_\Sigma^{\mathrm{NEq}(\mathcal{C})} & (\sigma \approx \tau \in \mathrm{NEq}(\mathcal{C})) \\
 & = & \tau_\Sigma^{\theta^{\mathrm{NEq}(\mathcal{C})}}(\vec{\chi}/\theta_\Sigma^{\mathrm{NEq}(\mathcal{C})}). &
\end{array}$$

Thus, $\sigma \approx \tau \in \mathrm{NEq}(\mathcal{F}/\theta^{\mathrm{NEq}(\mathcal{C})})$. Since $\mathcal{C} \in \mathsf{K}$, we now conclude that $\mathcal{A} \in \mathbb{H}\mathbb{L}(\mathsf{K})$ and this proves that $\mathbb{L}\mathbb{H}(\mathsf{K}) \subseteq \mathbb{H}\mathbb{L}(\mathsf{K})$. ■

Lemma 36 *Let* $\mathbf{F} = \langle\mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat\rangle$ *be a base algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems. Then* $\overset{\triangleleft}{ℿ}\mathbb{L}(\mathsf{K}) \subseteq \mathbb{L}\overset{\triangleleft}{ℿ}(\mathsf{K})$.

Proof: Let $\mathcal{A} \in \overset{\triangleleft}{ℿ}\mathbb{L}(\mathsf{K})$. Thus, there exist $\mathcal{A}^i \in \mathbb{L}(\mathsf{K})$ and $\langle H^i,\gamma^i\rangle : \mathcal{A} \to \mathcal{A}^i$, $i \in I$, such that $\bigcap_{i\in I} \mathrm{Ker}(\langle H^i,\gamma^i\rangle) = \Delta^{\mathcal{A}}$. Moreover, there exist $\mathcal{B}^i \in \mathsf{K}$ and $\langle L^i,\lambda^i\rangle : \mathcal{A}^i \to \mathcal{B}^i$, $i \in I$, such that $\mathrm{NEq}(\mathcal{A}^i) = \mathrm{NEq}(\mathcal{B}^i)$. So we have the configuration

$$\begin{tikzcd}
 & \mathbf{F} \arrow[dl, "{\langle F,\alpha\rangle}"'] \arrow[d, "{\langle F^i,\alpha^i\rangle}"] \arrow[dr, "{\langle G^i,\beta^i\rangle}"] & \\
\mathbf{A} \arrow[r, "{\langle H^i,\gamma^i\rangle}"'] & \mathbf{A}^i \arrow[r, "{\langle L^i,\lambda^i\rangle}"'] & \mathbf{B}^i
\end{tikzcd}$$

Denote, for the sake of brevity

$$\theta^i = \mathrm{Ker}(\langle L^i,\lambda^i\rangle \circ \langle H^i,\gamma^i\rangle), \quad i \in I.$$

Define, for all $i \in I$,

$$\langle J^i,\delta^i\rangle : \mathcal{A}/\bigcap_{i\in I}\theta^i \to \mathcal{B}^i,$$

by setting $J^i = L^i \circ H^i$ and, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$,

$$\delta^i_\Sigma(\phi/\bigcap_{i\in I}\theta^i_\Sigma) = \lambda^i_{H^i(\Sigma)}(\gamma^i_\Sigma(\phi)).$$

The definition is clearly sound because of the meaning of θ^i, $i \in I$. Moreover, the collection

$$\mathcal{A}/\bigcap_{i\in I}\theta^i \xrightarrow{\langle J^i,\delta^i\rangle} \mathcal{B}^i, \quad i \in I,$$

forms a subdirect intersection. In fact, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi,\psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{ll}
\multicolumn{2}{l}{\langle \phi/\bigcap_{i\in I}\theta^i_\Sigma, \psi/\bigcap_{i\in I}\theta^i_\Sigma\rangle \in \bigcap_{i\in I}\mathrm{Ker}_\Sigma(\langle J^i,\delta^i\rangle)}\\
\text{iff} & \delta^i_\Sigma(\phi/\bigcap_{i\in I}\theta^i_\Sigma) = \delta^i_\Sigma(\psi/\bigcap_{i\in I}\theta^i_\Sigma),\ i \in I,\\
\text{iff} & \lambda^i_{H^i(\Sigma)}(\gamma^i_\Sigma(\phi)) = \lambda^i_{H^i(\Sigma)}(\gamma^i_\Sigma(\psi)),\ i \in I,\\
\text{iff} & \langle\phi,\psi\rangle \in \mathrm{Ker}_\Sigma(\langle L^i,\lambda^i\rangle \circ \langle H^i,\gamma^i\rangle),\ i \in I,\\
\text{iff} & \langle\phi,\psi\rangle \in \theta^i_\Sigma,\ i \in I,\\
\text{iff} & \langle\phi,\psi\rangle \in \bigcap_{i\in I}\theta^i_\Sigma\\
\text{iff} & \langle \phi/\bigcap_{i\in I}\theta^i_\Sigma, \psi/\bigcap_{i\in I}\theta^i_\Sigma\rangle \in \Delta^{\mathcal{A}/\bigcap_{i\in I}\theta^i}_\Sigma.
\end{array}$$

Since $\mathcal{B}^i \in \mathsf{K}$, for all $i \in I$, we now get $\mathcal{A}/\bigcap_{i\in I}\theta^i \in \overset{\triangleleft}{\mathbb{P}}(\mathsf{K})$. Thus, it suffices to show that the quotient morphism

$$\mathcal{A} \xrightarrow{\langle I,\pi\rangle} \mathcal{A}/\bigcap_{i\in I}\theta^i$$

is a lifting, i.e., that $\mathrm{NEq}(\mathcal{A}/\bigcap_{i\in I}\theta^i) = \mathrm{NEq}(\mathcal{A})$. Suppose that $\sigma \approx \tau \in \mathrm{NEq}(\mathcal{A}/\bigcap_{i\in I}\theta^i)$. Then, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma)$, we have the following equivalent statements

$$\begin{array}{c}
\sigma^{\mathcal{A}/\bigcap_{i\in I}\theta^i}_\Sigma(\vec{\chi}/\bigcap_{i\in I}\theta^i_\Sigma) = \tau^{\mathcal{A}/\bigcap_{i\in I}\theta^i}_\Sigma(\vec{\chi}/\bigcap_{i\in I}\theta^i_\Sigma)\\
\langle\sigma^{\mathcal{A}}_\Sigma(\vec{\chi}),\tau^{\mathcal{A}}_\Sigma(\vec{\chi})\rangle \in \bigcap_{i\in I}\theta^i_\Sigma\\
\lambda^i_{H^i(\Sigma)}(\gamma^i_\Sigma(\sigma^{\mathcal{A}}_\Sigma(\vec{\chi}))) = \lambda^i_{H^i(\Sigma)}(\gamma^i_\Sigma(\tau^{\mathcal{A}}_\Sigma(\vec{\chi}))),\ i \in I,\\
\lambda^i_{H^i(\Sigma)}(\sigma^{\mathcal{A}^i}_{H^i(\Sigma)}(\gamma^i_\Sigma(\vec{\chi}))) = \lambda^i_{H^i(\Sigma)}(\tau^{\mathcal{A}^i}_{H^i(\Sigma)}(\gamma^i_\Sigma(\vec{\chi}))),\ i \in I,\\
\sigma^{\mathcal{B}^i}_{L^i(H^i(\Sigma))}(\lambda^i_{L^i(H^i(\Sigma))}(\gamma^i_\Sigma(\vec{\chi}))) = \tau^{\mathcal{B}^i}_{L^i(H^i(\Sigma))}(\lambda^i_{L^i(H^i(\Sigma))}(\gamma^i_\Sigma(\vec{\chi}))),\ i \in I.
\end{array}$$

Since, by hypothesis, $\langle L^i, \lambda^i \rangle$, $i \in I$, are liftings, we now get, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma)$, $\sigma^{\mathcal{A}^i}_{H^i(\Sigma)}(\gamma^i_\Sigma(\vec{\chi})) = \tau^{\mathcal{A}^i}_{H^i(\Sigma)}(\gamma^i_\Sigma(\vec{\chi}))$, $i \in I$, i.e., $\gamma^i_\Sigma(\sigma^{\mathcal{A}}_\Sigma(\vec{\chi})) = \gamma^i_\Sigma(\tau^{\mathcal{A}}_\Sigma(\vec{\chi}))$, $i \in I$. But, also by hypothesis, the collection $\{\langle H^i, \gamma^i \rangle : i \in I\}$ forms a subdirect intersection. Therefore, $\sigma^{\mathcal{A}}_\Sigma(\vec{\chi}) = \tau^{\mathcal{A}}_\Sigma(\vec{\chi})$. Since this holds for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma)$, we conclude that $\sigma \approx \tau \in \mathrm{NEq}(\mathcal{A})$ and, hence, $\langle I, \pi \rangle$ is a lifting, showing that $\mathcal{A} \in \mathbb{L}\overset{\triangleleft}{\mathbb{P}}(\mathsf{K})$. ∎

Now we obtain

Proposition 37 *Let* $\mathbf{F} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ *be a base algebraic system and* K *a class of* $\mathbf{F}$*-algebraic systems. Then* $\mathbb{V}^{\mathsf{Syn}}(\mathsf{K}) = \mathbb{H}\mathbb{L}\overset{\triangleleft}{\mathbb{P}}(\mathsf{K})$.

Proof: On the one hand, since $\mathsf{K} \subseteq \mathbb{V}^{\mathsf{Syn}}(\mathsf{K})$ and, by Proposition 34, $\mathbb{V}^{\mathsf{Syn}}(\mathsf{K})$ is closed under $\mathbb{H}$, $\mathbb{L}$ and $\overset{\triangleleft}{\mathbb{P}}$, we get that $\mathbb{H}\mathbb{L}\overset{\triangleleft}{\mathbb{P}}(\mathsf{K}) \subseteq \mathbb{V}^{\mathsf{Syn}}(\mathsf{K})$.

For the converse, we can provide a direct proof, analogous to that of Lemma 29, which is done in the proof of Lemma 33. Alternatively, we may notice that $\mathsf{K} \subseteq \mathbb{H}\mathbb{L}\overset{\triangleleft}{\mathbb{P}}(\mathsf{K})$ and show that $\mathbb{H}\mathbb{L}\overset{\triangleleft}{\mathbb{P}}(\mathsf{K})$ is closed under $\mathbb{H}$, $\mathbb{L}$ and $\overset{\triangleleft}{\mathbb{P}}$. It is obviously closed under $\mathbb{H}$, by Lemma 35, it is closed under $\mathbb{L}$, and, by Lemmas 30 and 36, it is also closed under $\overset{\triangleleft}{\mathbb{P}}$. Hence, it forms, by Lemma 33, a syntactic variety, and, therefore, $\mathbb{V}^{\mathsf{Syn}}(\mathsf{K}) \subseteq \mathbb{H}\mathbb{L}\overset{\triangleleft}{\mathbb{P}}(\mathsf{K})$. ∎

6 Relations with Universal Algebra

In this section, we explain some of the connections of the framework developed so far with that of varieties (or equational classes) of universal algebras. On the one hand, the framework presented in this work is more general, since it allows considerations of more complex structures. On the other hand, the restriction imposed in the case of algebraic systems of considering only surjective morphisms, imposes a certain cardinality restriction on the algebras considered, when we specialize the current framework to universal algebras. So, when adapting the current framework to compare it to varieties of universal algebras, we consider only algebras whose cardinalities do not surpass a given fixed but arbitrary cardinality, determined by the base algebraic system $\mathbf{F}$. We embark on some of the details.

For the purpose of comparing the two frameworks, we consider an algebraic signature $\mathcal{L} = \langle \Lambda, \mathrm{ar} \rangle$, where Λ is a set of operation symbols, $\mathrm{ar} : \Lambda \to \omega$

is the arity function, and a denumerable collection V of variables. Let us denote by $\mathbf{Tm}_{\mathcal{L}}(V) = \langle \mathrm{Tm}_{\mathcal{L}}(V), \mathcal{L} \rangle$, the free algebra of $\mathcal{L}$-terms with variables in V. Based on the well-known duality between terms and formulas, we will use lower case Greek letters, $\phi, \psi, \ldots$ to denote $\mathcal{L}$-terms and corresponding starred versions $\phi^*, \psi^*, \ldots$ to denote the corresponding term operations in the clone $\mathrm{Clo}_{\mathcal{L}}$ of $\mathcal{L}$-operations on $\mathrm{Tm}_{\mathcal{L}}(V)$.

Define $\mathbf{F}^{\mathcal{L}} = \langle \mathbf{Sign}^\flat, \mathrm{SEN}^\flat, N^\flat \rangle$ as follows:

- $\mathbf{Sign}^\flat$ is a trivial category with object, say, V;
- $\mathrm{SEN}^\flat$ gives the set $\mathrm{SEN}^\flat(V) = \mathrm{Tm}_{\mathcal{L}}(V)$;
- $N^\flat = \mathrm{Clo}_{\mathcal{L}}$.

Then an $\mathbf{F}^{\mathcal{L}}$-algebraic system $\mathcal{A} = \langle \mathbf{A}, \langle F, \alpha \rangle \rangle$ may be more simply specified as a pair $\langle \mathbf{A}, h \rangle$, where

- $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$ is an $\mathcal{L}$-algebra;
- $h : \mathbf{Tm}_{\mathcal{L}}(V) \to \mathbf{A}$ is a surjective $\mathcal{L}$-homomorphism.

Let $\mathrm{AlgSys}(\mathbf{F}^{\mathcal{L}})$ denote the class of all $\mathbf{F}^{\mathcal{L}}$-algebraic systems and by $\mathrm{Alg}(\mathcal{L})$ the class of all $\mathcal{L}$-algebras of cardinality not exceeding that of $\mathrm{Tm}_{\mathcal{L}}(V)$.

Classes of $\mathbf{F}^{\mathcal{L}}$-algebraic systems give rise to classes of $\mathcal{L}$-algebras and vice versa via the two operators ${}^{\downarrow} : \mathcal{P}(\mathrm{AlgSys}(\mathbf{F}^{\mathcal{L}})) \to \mathcal{P}(\mathrm{Alg}(\mathcal{L}))$ and ${}^{\uparrow} : \mathcal{P}(\mathrm{Alg}(\mathcal{L})) \to \mathcal{P}(\mathrm{AlgSys}(\mathbf{F}^{\mathcal{L}}))$ specified as follows:

- First, given a class K of $\mathbf{F}$-algebraic systems, define
$$\mathsf{K}^{\downarrow} = \{\mathbf{A} \in \mathrm{Alg}(\mathcal{L}) : (\exists h)(\langle \mathbf{A}, h \rangle \in \mathsf{K})\}.$$
- Secondly, given a class A of $\mathcal{L}$-algebras, define
$$\mathsf{A}^{\uparrow} = \{\langle \mathbf{A}, h \rangle \in \mathrm{AlgSys}(\mathbf{F}^{\mathcal{L}}) : \mathbf{A} \in \mathsf{A}\}.$$

To formalize some connections between these operators, let us call a class K of $\mathbf{F}^{\mathcal{L}}$-algebraic systems **universal** if, for all $\langle \mathbf{A}, k \rangle, \langle \mathbf{A}, h \rangle \in \mathrm{AlgSys}(\mathbf{F}^{\mathcal{L}})$,
$$\langle \mathbf{A}, k \rangle \in \mathsf{K} \quad \text{iff} \quad \langle \mathbf{A}, h \rangle \in \mathsf{K},$$
i.e., K is universal if and only if membership in K is determined by the $\mathcal{L}$-algebra component of an $\mathbf{F}^{\mathcal{L}}$-algebraic system.

Lemma 38 *Let $\mathcal{L}$ be an algebraic signature and $\mathbf{F}^{\mathcal{L}}$ the corresponding algebraic system.*

(a) If $\mathsf{A} \subseteq \mathrm{Alg}(\mathcal{L})$, then $\mathsf{A}^{\uparrow}$ is universal and $\mathsf{A}^{\uparrow\downarrow} = \mathsf{A}$;

(b) If $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F}^{\mathcal{L}})$, then $\mathsf{K}^{\downarrow\uparrow} = \mathsf{K}$ if and only if K is universal.

Proof:

(a) Suppose $\langle \mathbf{A}, k\rangle, \langle \mathbf{A}, h\rangle \in \mathrm{AlgSys}(\mathbf{F}^{\mathcal{L}})$, such that $\langle \mathbf{A}, k\rangle \in \mathsf{A}^{\uparrow}$. Then, by the definition of $^{\uparrow}$, $\mathbf{A} \in \mathsf{A}$, whence, again by the definition of $^{\uparrow}$, $\langle \mathbf{A}, h\rangle \in \mathsf{A}^{\uparrow}$. By symmetry, we have $\langle \mathbf{A}, k\rangle \in \mathsf{A}^{\uparrow}$ iff $\langle \mathbf{A}, h\rangle \in \mathsf{A}^{\uparrow}$, whence $\mathsf{A}^{\uparrow}$ is universal. Moreover, for all $\mathbf{A} \in \mathrm{Alg}(\mathcal{L})$, we get

$$\begin{array}{lll} \mathbf{A} \in \mathsf{A}^{\uparrow\downarrow} & \text{iff} & (\exists h)(\langle \mathbf{A}, h\rangle \in \mathsf{A}^{\uparrow}) \quad (\text{definition of } {}^{\downarrow}) \\ & \text{iff} & (\forall h)(\langle \mathbf{A}, h\rangle \in \mathsf{A}^{\uparrow}) \quad (\mathsf{A}^{\uparrow} \text{ universal}) \\ & \text{iff} & \mathbf{A} \in \mathsf{A}. \quad (\text{definition of } {}^{\uparrow}) \end{array}$$

(b) Let $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F}^{\mathcal{L}})$. Suppose, first, that K is a universal class. Then we get, for all $\langle \mathbf{A}, h\rangle \in \mathrm{AlgSys}(\mathbf{F}^{\mathcal{L}})$,

$$\begin{array}{lll} \langle \mathbf{A}, h\rangle \in \mathsf{K}^{\downarrow\uparrow} & \text{iff} & \mathbf{A} \in \mathsf{K}^{\downarrow} \quad (\text{definition of } {}^{\uparrow}) \\ & \text{iff} & \langle \mathbf{A}, h\rangle \in \mathsf{K}. \quad (\text{definition of } {}^{\downarrow} \text{ and universality}). \end{array}$$

Thus, if K is universal, $\mathsf{K}^{\downarrow\uparrow} = \mathsf{K}$. Conversely, if $\mathsf{K}^{\downarrow\uparrow} = \mathsf{K}$, then K happens to be a class of $\mathbf{F}^{\mathcal{L}}$-algebraic systems of the form $\mathsf{A}^{\uparrow}$ for a class $\mathsf{A} := \mathsf{K}^{\downarrow}$ of $\mathcal{L}$-algebras, whence, by Part (a), K is universal.

■

It turns out that the notion of universal class is closely related to one we have encountered before in a more general context; namely, that of a natural class of $\mathbf{F}^{\mathcal{L}}$-algebraic systems in the sense of Definition 15, i.e., a class K, such that $\mathrm{Eq}(\mathsf{K}) = \theta^{\mathrm{NEq}(\mathsf{K})}$.

Proposition 39 *Let $\mathcal{L}$ be an algebraic signature and $\mathbf{F}^{\mathcal{L}}$ the corresponding algebraic system. If a class K of $\mathbf{F}^{\mathcal{L}}$-algebraic systems is universal, then it is natural.*

Proof: Suppose $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F}^{\mathcal{L}})$ is universal. We must show that $\mathrm{Eq}(\mathsf{K}) = \theta^{\mathrm{NEq}(\mathsf{K})}$. The right to left inclusion always holds, but let us present the proof as a warm up for the reverse inclusion. Suppose $\phi, \psi \in \theta^{\mathrm{NEq}(\mathsf{K})}$. Then, by definition, there exist $\phi'^{*} \approx \psi'^{*} \in \mathrm{NEq}(\mathsf{K})$ and $\vec{\chi} \in \mathrm{Tm}_{\mathcal{L}}(V)$, such that $\phi = \phi'^{*}(\vec{\chi})$ and $\psi = \psi'^{*}(\vec{\chi})$. Since $\phi'^{*} \approx \psi'^{*} \in \mathrm{NEq}(\mathsf{K})$, we get, for all $\langle \mathbf{A}, k\rangle \in \mathsf{K}$ and all $h : \mathbf{Tm}_{\mathcal{L}}(V) \to \mathbf{A}$, $\phi'^{*\mathbf{A}}(h(\vec{v})) = \psi'^{*\mathbf{A}}(h(\vec{v}))$. In particular, since this holds for all h, we can substitute in the values $h(\vec{\chi})$, getting $\phi'^{*\mathbf{A}}(h(\vec{\chi})) = \psi'^{*\mathbf{A}}(h(\vec{\chi}))$, which yields $h(\phi'^{*}(\vec{\chi})) = h(\psi'^{*}(\vec{\chi}))$, i.e., $h(\phi) = h(\psi)$. Thus, we get, for k in place of h, $\langle \mathbf{A}, k\rangle \vDash \phi \approx \psi$. Since $\langle \mathbf{A}, k\rangle \in \mathsf{K}$ was arbitrary, $\phi \approx \psi \in \mathrm{Eq}(\mathsf{K})$. So $\theta^{\mathrm{NEq}(\mathsf{K})} \subseteq \mathrm{Eq}(\mathsf{K})$.

Assume, conversely, that $\phi \approx \psi \in \mathrm{Eq}(\mathsf{K})$. Consider $\phi^{*} \approx \psi^{*} \in \mathrm{NEq}(\mathbf{F}^{\mathcal{L}})$ and $\vec{v} \in \mathrm{Tm}_{\mathcal{L}}(V)$ consisting of the variables in their natural order. Since, in this case, we have $\phi^{*}(\vec{v}) = \phi$ and $\psi^{*}(\vec{v}) = \psi$, to see that $\psi \approx \psi \in \theta^{\mathrm{NEq}(\mathsf{K})}$, it suffices to show that $\phi^{*} \approx \psi^{*} \in \mathrm{NEq}(\mathsf{K})$. To this end, let $\langle \mathbf{A}, k\rangle \in \mathsf{K}$ and $h : \mathbf{Tm}_{\mathcal{L}}(V) \to \mathbf{A}$. We have the following series of implications, where the initial equality is based on the assumptions that $\langle \mathbf{A}, k\rangle \in \mathsf{K}$, $\phi \approx \psi \in \mathrm{Eq}(\mathsf{K})$ and the hypothesis that K is universal:

$$\begin{gathered} h(\phi) = h(\psi) \\ h(\phi^{*}(\vec{v})) = h(\psi^{*}(\vec{v})) \\ \phi^{*}(h(\vec{v})) = \psi^{*}(h(\vec{v})) \\ \phi^{*}(k(\vec{\chi})) = \psi^{*}(k(\vec{\chi})),\ \text{all } \vec{\chi} \in \mathrm{Tm}_{\mathcal{L}}(V) \\ k(\phi^{*}(\vec{\chi})) = k(\psi^{*}(\vec{\chi})),\ \text{all } \vec{\chi} \in \mathrm{Tm}_{\mathcal{L}}(V) \\ \langle \mathbf{A}, k\rangle \vDash \phi^{*} \approx \psi^{*}. \end{gathered}$$

We conclude that $\phi^{*} \approx \psi^{*} \in \mathrm{NEq}(\mathsf{K})$, whence, $\phi \approx \psi \in \theta^{\mathrm{NEq}(\mathsf{K})}$. ∎

It is also not very difficult to show that a syntactic variety of $\mathbf{F}^{\mathcal{L}}$-algebraic systems must be a universal class.

Lemma 40 *Let $\mathcal{L}$ be an algebraic signature and $\mathbf{F}^{\mathcal{L}}$ the corresponding algebraic system. If K is a syntactic variety of $\mathbf{F}^{\mathcal{L}}$-algebraic systems, then K is a universal class.*

Proof: Suppose that K is a syntactic variety and consider $\langle \mathbf{A}, k\rangle, \langle \mathbf{A}, h\rangle \in \mathrm{AlgSys}(\mathbf{F}^{\mathcal{L}})$, such that $\langle \mathbf{A}, k\rangle \in \mathsf{K} = \mathrm{NMod}(\mathrm{NEq}(\mathsf{K}))$. Then, for all $\phi^{*} \approx \psi^{*} \in \mathrm{NEq}(\mathsf{K})$, we have

$$\begin{array}{lll} \langle \mathbf{A}, h\rangle \vDash \phi^{*} \approx \psi^{*} & \text{iff} & \phi^{*\mathbf{A}}(\vec{a}) = \psi^{*\mathbf{A}}(\vec{a}),\ \text{all } \vec{a} \in A, \\ & \text{iff} & \langle \mathbf{A}, k\rangle \vDash \phi^{*} \approx \psi^{*}. \end{array}$$

Since the latter holds, by hypothesis, so does the former, and, therefore, $\langle \mathbf{A}, h\rangle \in \mathrm{NMod}(\mathrm{NEq}(\mathsf{K})) = \mathsf{K}$. We conclude that K is indeed universal. ∎

Thus, we get the following relation between semantic and syntactic varieties and universal classes when referring to $\mathbf{F}^{\mathcal{L}}$-algebraic systems.

Corollary 41 *Let $\mathcal{L}$ be an algebraic signature and $\mathbf{F}^{\mathcal{L}}$ the corresponding algebraic system. A class K of $\mathbf{F}^{\mathcal{L}}$-algebraic systems is a syntactic variety if and only if it is a universal semantic variety.*

Proof: If K is a syntactic variety, then, by Lemma 40, it is a universal class and, by Proposition 17, it is a semantic variety. Conversely, if K is a universal semantic variety, then, by Proposition 39, it is a natural semantic variety, whence, by Proposition 17, it is a syntactic variety. ■

Finally, a result relating varieties of $\mathcal{L}$-algebras with syntactic varieties of $\mathbf{F}^{\mathcal{L}}$-algebraic systems may be formalized as follows (note that, when writing $\mathbf{A} \vDash \phi \approx \psi$ for $\mathbf{A} \in \mathrm{Alg}(\mathcal{L})$, we refer to the ordinary satisfaction relation of universal algebra):

Proposition 42 *Let $\mathcal{L}$ be an algebraic signature, $\mathbf{F}^{\mathcal{L}}$ the corresponding algebraic system, $\mathsf{A} \subseteq \mathrm{Alg}(\mathcal{L})$ and $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F}^{\mathcal{L}})$.*

(a) If A is an equational class, then $\mathsf{A}^{\uparrow}$ is a syntactic variety of $\mathbf{F}^{\mathcal{L}}$-algebraic systems;

(b) If K is a syntactic variety, then $\mathsf{K}^{\downarrow}$ is an equational class of $\mathcal{L}$-algebras.

Proof: The proof of both parts relies on the fact that, for every $\mathcal{L}$-algebra $\mathbf{A} \in \mathrm{Alg}(\mathcal{L})$, every $h : \mathbf{Tm}_{\mathcal{L}}(V) \twoheadrightarrow \mathbf{A}$ and every $\mathcal{L}$-equation (or $\mathbf{F}^{\mathcal{L}}$-equation) $\phi \approx \psi$, it holds that

$$\mathbf{A} \vDash \phi \approx \psi \quad \text{iff} \quad \langle \mathbf{A}, h \rangle \vDash \phi^* \approx \psi^*.$$

Given $X \subseteq \mathrm{Eq}(\mathcal{L})$ and $E \subseteq \mathrm{NEq}(\mathbf{F}^{\mathcal{L}})$, denote

$$X^* = \{\phi^* \approx \psi^* : \phi \approx \psi \in X\}$$

and

$$E^- = \{\phi \approx \psi : \phi^* \approx \psi^* \in E\}.$$

(a) Assume $\mathsf{A} = \mathrm{Alg}(X)$, for some collection $X \subseteq \mathrm{Eq}(\mathcal{L})$. Then $\mathsf{A}^{\uparrow} = \mathrm{NMod}(X^*)$. Indeed, for all $\mathbf{A} \in \mathrm{Alg}(\mathcal{L})$ and all $h : \mathbf{Tm}_{\mathcal{L}}(V) \twoheadrightarrow \mathbf{A}$,

$$
\begin{array}{rll}
\langle \mathbf{A}, h\rangle \in \mathsf{A}^{\uparrow} & \text{iff} & \mathbf{A} \in \mathsf{A} \quad (\text{definition of } {}^{\uparrow})\\
& \text{iff} & \mathbf{A} \vDash X \quad (\mathsf{A} = \mathrm{Alg}(X))\\
& \text{iff} & \langle \mathbf{A}, h\rangle \vDash X^* \quad (\text{displayed equivalence above})\\
& \text{iff} & \langle \mathbf{A}, h\rangle \in \mathrm{NMod}(X^*). \quad (\text{definition of NMod})
\end{array}
$$

Therefore, $\mathsf{A}^{\uparrow} = \mathrm{NMod}(X^*)$ and $\mathsf{A}^{\uparrow}$ is a syntactic variety of $\mathbf{F}^{\mathcal{L}}$-algebraic systems.

(b) Assume $\mathsf{K} = \mathrm{NMod}(E)$, for some collection $E \subseteq \mathrm{NEq}(\mathbf{F}^{\mathcal{L}})$. Then $\mathsf{K}^{\downarrow} = \mathrm{Alg}(E^-)$. Indeed, for all $\mathbf{A} \in \mathrm{Alg}(\mathcal{L})$ and all $h : \mathbf{Tm}_{\mathcal{L}}(V) \twoheadrightarrow \mathbf{A}$,

$$
\begin{array}{rll}
\mathbf{A} \in \mathsf{K}^{\downarrow} & \text{iff} & \langle \mathbf{A}, h\rangle \in \mathsf{K} \quad (\text{definition of } {}^{\downarrow})\\
& \text{iff} & \langle \mathbf{A}, h\rangle \vDash E \quad (\mathsf{K} = \mathrm{NMod}(E))\\
& \text{iff} & \mathbf{A} \vDash E^- \quad (\text{displayed equivalence above})\\
& \text{iff} & \mathbf{A} \in \mathrm{Alg}(E^-). \quad (\text{definition of Alg})
\end{array}
$$

Therefore, $\mathsf{K}^{\downarrow} = \mathrm{Alg}(E^-)$ and $\mathsf{K}^{\downarrow}$ is an equational class of $\mathcal{L}$-algebras. ■

We close the exposition by using Proposition 42 to exhibit a close connection between the operator $\mathbb{V} : \mathcal{P}(\mathrm{Alg}(\mathcal{L})) \to \mathcal{P}(\mathrm{Alg}(\mathcal{L}))$, that maps a class of $\mathcal{L}$-algebras to the equational class it generates, and the operator $\mathbb{V}^{\mathsf{Syn}} : \mathcal{P}(\mathrm{AlSys}(\mathbf{F}^{\mathcal{L}})) \to \mathcal{P}(\mathrm{AlgSys}(\mathbf{F}^{\mathcal{L}}))$, mapping a class of $\mathbf{F}^{\mathcal{L}}$-algebraic systems to the syntactic variety it generates.

Corollary 43 *Let $\mathcal{L}$ be an algebraic signature, $\mathbf{F}^{\mathcal{L}}$ the corresponding algebraic system, $\mathsf{A} \subseteq \mathrm{Alg}(\mathcal{L})$ and $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F}^{\mathcal{L}})$. Then*

$$\mathbb{V}(\mathsf{A}) = \mathbb{V}^{\mathsf{Syn}}(\mathsf{A}^{\uparrow})^{\downarrow} \quad \textit{and} \quad \mathbb{V}^{\mathsf{Syn}}(\mathsf{K}) = \mathbb{V}(\mathsf{K}^{\downarrow})^{\uparrow}.$$

Proof: Suppose, first, that $\mathsf{A} \subseteq \mathrm{Alg}(\mathcal{L})$. Then, by definition, $\mathbb{V}^{\mathsf{Syn}}(\mathsf{A}^{\uparrow})$ is a syntactic variety. Thus, by Proposition 42, $\mathbb{V}^{\mathsf{Syn}}(\mathsf{A}^{\uparrow})^{\downarrow}$ is an equational class. But, obviously, $\mathsf{A} \subseteq \mathbb{V}^{\mathsf{Syn}}(\mathsf{A}^{\uparrow})^{\downarrow}$ and, therefore, by the definition of $\mathbb{V}$, we get $\mathbb{V}(\mathsf{A}) \subseteq \mathbb{V}^{\mathsf{Syn}}(\mathsf{A}^{\uparrow})^{\downarrow}$. On the other hand, $\mathbb{V}(\mathsf{A})$ is an equational class by definition. Therefore, by Proposition 42, $\mathbb{V}(\mathsf{A})^{\uparrow}$ is a syntactic variety. Moreover, clearly, $\mathsf{A}^{\uparrow} \subseteq \mathbb{V}(\mathsf{A})^{\uparrow}$. hence, by the definition of $\mathbb{V}^{\mathsf{Syn}}$, we get $\mathbb{V}^{\mathsf{Syn}}(\mathsf{A}^{\uparrow}) \subseteq \mathbb{V}(\mathsf{A})^{\uparrow}$. Now, applying ${}^{\downarrow}$ and taking into account Lemma 38, we get $\mathbb{V}^{\mathsf{Syn}}(\mathsf{A}^{\uparrow})^{\downarrow} \subseteq \mathbb{V}(\mathsf{A})$. This yields that $\mathbb{V}(\mathsf{A}) = \mathbb{V}^{\mathsf{Syn}}(\mathsf{A}^{\uparrow})^{\downarrow}$.

Suppose, next, that $\mathsf{K} \subseteq \mathrm{AlgSys}(\mathbf{F}^{\mathcal{L}})$. Then, by definition, $\mathbb{V}(\mathsf{K}^{\downarrow})$ is an equational class. Thus, by Proposition 42, $\mathbb{V}(\mathsf{K}^{\downarrow})^{\uparrow}$ is a syntactic variety. But, obviously, $\mathsf{K} \subseteq \mathbb{V}(\mathsf{K}^{\downarrow})^{\uparrow}$ and, therefore, by the definition of $\mathbb{V}^{\mathsf{Syn}}$, we get $\mathbb{V}^{\mathsf{Syn}}(\mathsf{K}) \subseteq \mathbb{V}(\mathsf{K}^{\downarrow})^{\uparrow}$. On the other hand, $\mathbb{V}^{\mathsf{Syn}}(\mathsf{K})$ is a syntactic variety by definition. Therefore, by Proposition 42, $\mathbb{V}^{\mathsf{Syn}}(\mathsf{K})^{\downarrow}$ is an equational class. Moreover, clearly, $\mathsf{K}^{\downarrow} \subseteq \mathbb{V}^{\mathsf{Syn}}(\mathsf{K})^{\downarrow}$. hence, by the definition of $\mathbb{V}$, we get $\mathbb{V}(\mathsf{K}^{\downarrow}) \subseteq \mathbb{V}^{\mathsf{Syn}}(\mathsf{K})^{\downarrow}$. Now, applying $^{\uparrow}$ and taking into account Lemmas 40 and 38, we get $\mathbb{V}(\mathsf{K}^{\downarrow})^{\uparrow} \subseteq \mathbb{V}^{\mathsf{Syn}}(\mathsf{K})$. This yields that $\mathbb{V}^{\mathsf{Syn}}(\mathsf{K}) = \mathbb{V}(\mathsf{K}^{\downarrow})^{\uparrow}$. ■

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