# Combinatorial Analysis of the State Space Structure of Finite Automata Networks*

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**Abstract**

Several questions concerning the structure of the state space of Finite Automata Networks (FANs) are considered and many of them are answered in the context of special classes of FANs, the most important of which, in our studies, is the class of Threshold Agent Networks (TANs). Namely, the number of strongly non-equivalent FANs and TANs on the complete digraph $C_n$ with $n$ vertices is computed, by using the techniques of generating functions and recurrence relations together with Pólya's theory of counting from elementary combinatorics [5] and combining them as in [6]. Then a complete description is given of the limit point structure of positive and negative TANs on $C_n$. These results use only simple combinatorial counting arguments exploiting the threshold structure available. Finally, a combinatorial algebra is introduced whose product, scalar product and sum provide in many cases an easy way to compute the description of the state space of products and other constructs of networks from the corresponding descriptions of their components.

## 1 Introduction

A **Finite Automata Network** (FAN) $N = \langle G, \{f_i\}_{i\in V}\rangle$ [3] consists of a digraph $G = \langle V, E\rangle$ together with a collection $\{f_i\}_{i\in V}$ of functions $f_i : \{0,1\}^V \to \{0,1\}, i \in V,$

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such that $f_i$ only depends on those $j$, such that $\langle j, i\rangle \in E$. The **global update function** $f : \{0,1\}^V \to \{0,1\}^V$ of the FAN $N$ is the function given by

$$f(x)_i = f_i(x), \quad \text{for all } x \in \{0,1\}^V, i \in V.$$

The **state space** $\mathrm{S}(N)$ of the FAN $N$ is the digraph with set of vertices $\{0,1\}^V$ and edges all pairs $\langle x, y\rangle \in (\{0,1\}^V)^2$, such that $y = f(x)$. A point $x$ is said to be a **fixed-point** if $x = f(x)$ and a sequence of points $x_1, \ldots, x_m$ is said to form a **limit cycle** of length $m$ if, for all $1 \leq i \leq m-1, x_{i+1} = f(x_i)$ and $x_1 = f(x_m)$. Thus fixed points are limit cycles of length 1. All points in limit cycles are collectively termed **limit points**.

Two state spaces $\mathrm{S}(N_1)$ and $\mathrm{S}(N_2)$ are said to be **isomorphic** if they are isomorphic as digraphs. They are said to be **strongly isomorphic** if they are isomorphic via an isomorphism that is induced by a bijection between the sets $V_1$ and $V_2$ of the vertices of the digraphs $G_1$ and $G_2$ of the FANs $N_1$ and $N_2$, respectively. Two FANs $N_1$ and $N_2$ are said to be **equivalent** if $\mathrm{S}(N_1)$ and $\mathrm{S}(N_2)$ are isomorphic and are said to be **strongly equivalent** if their state spaces are strongly isomorphic. In Section 2, a formula will be given to compute the number of strongly non-equivalent FANs over the complete digraph with $n$ vertices. This is accomplished by applying Pólya's theory of counting as done in [6], together with simple tools from the theory of generating functions and recurrence relations.

Next, the focus will be shifted to a special class of FANs. This class is a subclass of neural or threshold networks [3] and it was introduced in [6] as an alternative platform to sequential dynamical systems of [1, 2] for modelling and analytically studying properties of computer simulations.

A **Threshold Agent Network** (TAN) is a FAN $A = \langle G, \{f_i\}_{i\in V}\rangle$, whose functions $f_i$ are integer threshold functions, i.e., $f_i, i \in V$, is determined by an integer $t_i$, in the following way, for all $x \in \{0,1\}^V$,

$$f_i(x) = \begin{cases} 1, & \text{if } |\{j : \langle j, i\rangle \in E \text{ and } x_j = 1\}| \geq t_i \\ 0, & \text{otherwise} \end{cases}, \quad \text{if } t_i \geq 0,$$

and

$$f_i(x) = \begin{cases} 0, & \text{if } |\{j : \langle j, i\rangle \in E \text{ and } x_j = 1\}| \geq -t_i \\ 1, & \text{otherwise} \end{cases}, \quad \text{if } t_i < 0.$$

Since the $f_i$'s are completely determined by the thresholds $t_i$, the TAN $A$ is most often denoted by $A = \langle G, t\rangle$, where $t = \langle t_i : i \in V\rangle$ is the sequence of integer thresholds.

A TAN $A$ is said to be **positive** if, for all $i \in V, 0 \leq t_i \leq |V|$ and it is said to be **negative** if, for all $i \in V, -|V| \leq t_i \leq -1$.

In [6], Pólya's theory of counting is used to obtain a tight upper bound on the number of strongly non-equivalent TANs over a given digraph. This result will be applied in Section 2 to obtain a recursive formula for the number of strongly non-equivalent TANs over the complete digraph $C_n$ with $n$ vertices. In Section 3, the limit point structure of positive and of negative TANs over $C_n$ is completely determined. The problem for arbitrary TANs over $C_n$ and for TANs over other more complicated digraphs is open. The results presented here are also of computational significance since they simplify the complexity of computing the limit structure.

In [7, 8] a categorical treatment of FANs and TANs was presented and the notion of product of two FANs and TANs was introduced and studied. The same notion for sequential dynamical systems was studied in [4]. In Section 4, based on the combinatorial idea of generating functions and combinatorial algebras, a representation is given of the main features of the state space of a FAN in terms of sequences. The collection of all sequences is endowed with operations that make it a $\mathbf{Z}$-algebra, where $\mathbf{Z}$ is the ring of integers. Not every element of this algebra is a valid description of a state space of a FAN, but, when two elements are, their product in the algebra is the description of the state space of their product as given in [7]. The meaning of some of the other algebraic operations is also explored, but many questions on this very interesting correspondence between the structure of FANs themselves and the algebraic descriptions of their state spaces remain open.

Section 5 makes explicit some of the questions that we were unable to answer at this point. The hope is that more work on this topic will give satisfactory answers to many of these and other related issues.

## 2 The Number of FANs and TANs over $C_n$

It is straightforward to verify that the number of FANs over the complete digraph $C_n$ with $n$ vertices is $(2^{2^n})^n = 2^{n2^n}$. This is because, for each vertex, one may arbitrarily choose two output values for each of the $2^n$ possible inputs.

In a similar way, we may compute the number of different TANs over $C_n$. This number is obtained by noting that an arbitrary selection of a threshold may be made for each vertex in the TAN but that the thresholds must be chosen in the range $-n, \ldots, n+1$, since all values below $-n$ will determine exactly the same output function with the value 0 and all values above $n+1$ will determine exactly the same output function with the value $n+1$. Thus, the number of different TANs over $C_n$ is $(2n+2)^n$.

As an illustration consider the number of different FANs and TANs over the complete graphs with 1, 2, 3 and 4 vertices. The numbers are given in the following table

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| FANs | 4 | 256 | $2^{24}$ | $2^{64}$ |
| TANs | 4 | 36 | 512 | $10,000$ |

A more challenging question is to determine the number of non-equivalent and of strongly non-equivalent FANs and TANs over $C_n$. We do the latter for the complete digraphs on 1,2,3 and 4 vertices by exploring a result in [6]. Namely, we compute, using Pólya's theory of counting, all the possible non-isomorphic labelings of $C_n$, $n = 1, 2, 3, 4$, under the automorphism group of $C_n$, first with labels from the set of $2^{2^n}$ possible functions with $n$ variables, and then with labels from the $2n+2$ allowable thresholds. To do this, the cycle index $C_{C_n}(x_1, x_2, \ldots, x_n)$ of $C_n$ has to be computed first. To compute the cycle index, the number of permutations of the $n$ vertices and their cycle structures have to be computed. The number of strongly non-equivalent FANs over $C_n$ is then given by $C_{C_n}(2^{2^n}, \ldots, 2^{2^n})$. Similarly, the number of strongly non-equivalent TANs over $C_n$ is given by $C_{C_n}(2n + 2, \ldots, 2n + 2)$.

For $n = 1$, there is only one permutation on 1 vertex, namely the identity that fixes the vertex. So its cycle structure is $x_1$ and the cycle index of $C_1$ is $C_{C_1}(x_1) = \frac{1}{1!}x_1 = x_1$. The first column of the table below follows.

For $n = 2$, there is one permutation (the identity) fixing both vertices and one that interchanges them and therefore contains one cycle of length 2. So the corresponding cycle structures are $x_1^2$ and $x_2$ and the cycle index is $C_{C_2}(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2)$. Now the second column of the table below follows.

For $n = 3$, there is the identity that fixes all three vertices and has cycle structure $x_1^3$, three permutations that fix one vertex each and have structures $x_1x_2$ and two permutations that do not fix any vertex and have cycle structure $x_3$. Therefore the cycle index in this case is $C_{C_3}(x_1, x_2, x_3) = \frac{1}{6}(x_1^3 + 3x_1x_2 + 2x_3)$. The third column of the table follows.

Finally, for $n = 4$, there is the identity with cycle structure $x_1^4$, eight permutations that fix only one point each with cycle structures $x_1x_3$, six permutations that fix two points each and interchange the other two with cycle structure $x_1^2x_2$, three permutations that interchange two pairs of points with cycle structures $x_2^2$ and, finally, six more permutations that do not fix any point and have cycle structure representations $x_4$. Thus, the cycle index in this case is $C_{C_4}(x_1, x_2, x_3, x_4) = \frac{1}{24}(x_1^4 + 8x_1x_3 + 6x_1^2x_2 + 3x_2^2 + 6x_4)$. Thus the last column of the table follows.

| $n$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $2^{2^n}$ | 4 | 16 | 256 | $2^{16}$ |
| $2n+2$ | 4 | 6 | 8 | 10 |
| FANs | 4 | 136 | $2,829,056$ | $768,684,707,117,285,376$ |
| TANs | 4 | 10 | 120 | 715 |

After seeing the analysis for the special cases of $n = 1, 2, 3$ and 4, we give a combinatorial recursion formula for computing the cycle index $C_{C_n}$ of the full group of permutations on $n$ vertices, which is the symmetry group of the complete digraph on $n$ vertices. Denote by $C_n$ this group for simplicity and set $D_n = n!C_n$. $D_n$ is the same polynomial as $C_n$ in the $n$ variables $x_1, \ldots, x_n$ without the constant multiplicative term $\frac{1}{n!}$.

**Proposition 1** *The polynomial $D_n$ obeys the recursive formula*

$$D_n = \sum_{k=1}^{n} \binom{n-1}{k-1}(k-1)!x_k D_{n-k}.$$

*Thus, the cycle index of the full permutation group on $n$ vertices $C_n$ obeys the recursive formula*

$$C_n = \frac{1}{n!}\sum_{k=1}^{n}(n-1)!x_k C_{n-k}.$$

**Proof:**

An easy combinatorial proof follows from the observation that the permutation group on $n$ vertices may be constructed as the disjoint union of the permutations that contain a fixed $k$-cycle containing a specific element, say 1, for $k = 0, 1, \ldots, n$. Each such cycle may be selected in $\binom{n-1}{k-1}(k-1)!$ ways and the sum of the cycle structure representations of the permutations containing such a $k$-cycle is $x_k D_{n-k}$. ∎

We may now verify the results obtained above by computing, using these recursive formulas, the polynomials computed in an ad-hoc fashion above.

$$\begin{aligned}
D_0 &= 1. \\
D_1 &= \tbinom{0}{0}0!x_1 = x_1. \\
D_2 &= \tbinom{1}{0}0!x_1D_1 + \tbinom{1}{1}1!x_2D_0 = x_1^2 + x_2. \\
D_3 &= \tbinom{2}{0}0!x_1D_2 + \tbinom{2}{1}1!x_2D_1 + \tbinom{2}{2}2!x_3D_0 \\
&= x_1(x_1^2 + x_2) + 2x_1x_2 + 2x_3 \\
&= x_1^3 + 3x_1x_2 + 2x_3 \\
D_4 &= \tbinom{3}{0}0!x_1D_3 + \tbinom{3}{1}1!x_2D_2 + \tbinom{3}{2}2!x_3D_1 + \tbinom{3}{3}3!x_4D_0 \\
&= x_1(x_1^3 + 3x_1x_2 + 2x_3) + 3x_2(x_1^2 + x_2) + 6x_1x_3 + 6x_4 \\
&= x_1^4 + 6x_1^2x_2 + 8x_1x_3 + 3x_2^2 + 6x_4.
\end{aligned}$$

Having Proposition 1 at hand one may now derive from the results in [6] the following theorem

**Theorem 2** *The number of strongly non-equivalent FANs over $C_n$ is $C_n(2^{2^n}, \ldots, 2^{2^n})$ and the number of strongly non-equivalent TANs over $C_n$ is $C_n(2n+2, \ldots, 2n+2)$.*

# 3 The Limit Cycle Structure of TANs over $C_n$

In this section, a study is undertaken to determine the structure of the state space of a TAN over $C_n$. More precisely, we would like, given threshold values on the vertices of $C_n$, to be able to determine the structure of the state space of the corresponding TAN. In the ideal case, it would be desirable to have this ability for an arbitrary TAN. The study of TANs over the complete digraph forms only a simple first step and will, hopefully, provide some insights on how to tackle the general case. The present study will be furthermore restricted to the cases when all the thresholds are non-negative and when all the thresholds are negative. These threshold agent networks were called **positive** and **negative,** respectively, in [6].

## Positive TANs over $C_n$

First, an analysis is provided of the limit point structure of positive TANs over the complete digraph $C_n$ on $n$ vertices. These are TANs whose thresholds are in the range $0, \ldots, n$. Note that the local update functions $f_i : \{0,1\}^V \rightarrow \{0,1\}, 1 \leq i \leq n$ of a positive TAN are nondecreasing functions. Thus, a positive TAN cannot have a limit cycle of length greater than 1. In other words, all limit points of a positive TAN are fixed-points. So the limit point structure of a positive TAN is completely determined by the number of fixed-points in its state space. The following analysis completely determines the number of fixed-points of a positive TAN over $C_n$. Such a TAN is completely determined by the sequence $t = \langle t_i : i \in V \rangle$ of its thresholds. Let $T_k : \{0, \ldots, n\}^V \rightarrow \{0,1\}, 0 \leq k \leq n$ be the function (really a predicate on allowable sequences of thresholds)

$$T_k(t) = \begin{cases} 1, & \text{if } |\{i \in V : t_i \leq k\}| = k, \\ 0, & \text{otherwise} \end{cases} , 0 \leq k \leq n.$$

Then, the following holds

**Theorem 3 (Fixed-Points of a Positive TAN over $C_n$)** *The number* fp$(t)$ *of fixed-points of a positive TAN over $C_n$ with thresholds $t = \langle t_i : i \in V \rangle$ is given by the formula* $\text{fp}(t) = \sum_{i=0}^{n} T_i(t)$.

**Proof:**

The formula follows directly from the observation that for

$$\overbrace{11\ldots1}^{k}\overbrace{00\ldots0}^{n-k}$$

to be a fixed-point, the first $k$ vertices must have thresholds not exceeding $k$ and the last $n-k$ vertices thresholds at least $k+1$. ■

## Negative TANs over $C_n$

In this section, the analysis of the limit point structure of the positive TANs over $C_n$ is modified to present an analysis of the limit point structure of negative TANs over $C_n$. These are TANs whose thresholds take values in the range $-1, \ldots, -n$. The first observation that can be made is that all these TANs possess the two-state limit cycle

$$\overbrace{00\ldots0}^{n} \rightleftarrows \overbrace{11\ldots1}^{n}$$

This is true, because the thresholds are in the range $-1, \ldots, -n$.

Now, let $R_k : \{-1, \ldots, -n\}^V \to \{0,1\}, 0 \le k \le n$, be the function defined by

$$R_k(t) = \begin{cases} 1, & \text{if } |\{i \in V : t_i < -k\}| = k \\ 0, & \text{otherwise} \end{cases}, 0 \le k \le n.$$

Moreover, let $S_{l,k} : \{-1, \ldots, -n\}^V \to \{0,1\}, 0 \le l \le k \le n$, be the function defined, for all $0 \le l \le k \le n$, by

$$S_{l,k}(t) = \begin{cases} 1, & \text{if } |\{i \in V : t_i < -k\}| = l \text{ and } |\{i \in V : t_i \ge -l\}| = n-k \\ 0, & \text{otherwise} \end{cases}.$$

Note, that, for all $0 \le k \le n$, $S_{k,k} = R_k$, so that $R_k$ will be replaced by $S_{k,k}$ in the following discussion. The reason why they are both given is because $R_k$ will be used to address fixed-points and $S_{l,k}, l < k$, will be used to detect limit cycles of length 2.

The following theorem reveals the limit point structure of a negative TAN over $C_n$. Two lemmas will be stated first to break up the proof into smaller pieces.

**Lemma 4** *Let $s \to s'$ be a single transition in a limit cycle of a negative TAN over $C_n$. There do not exist $1 \le i, j \le n$, such that $s_i = 0, s_j = 1$ and $s'_i = 1, s'_j = 0$.*

**Proof:**

Suppose that this is not the case. Assume that $s'' \to s \to s'$ and that $s, s', s''$ contain $k, l, m$ 1's, respectively, and let $i$ and $j$ be such that $s_i = 0, s_j = 1$ and $s'_i = 1, s'_j = 0$. Then, $s \to s'$ gives $k < -t_i$ and $k \ge -t_j$. Therefore $t_i < t_j$. But, $s'' \to s$ gives $m \ge -t_i$ and $m < -t_j$. These two combined give $t_j < t_i$, which is a contradiction. ■

**Lemma 5** *The state space of a negative TAN over $C_n$ does not contain limit cycles of length greater than 2.*

**Proof:**

Suppose that a negative TAN over $C_n$ contained a limit cycle of length $l \geq 3$. Then the state with the most 1's, say $m$, would follow the state with the most 0's, containing say $l$ 1's, and the state with the most 0's would follow the state with the most 1's. This can only happen if there is at least one vertex with threshold $t$, such that $m < -t \leq m$, a contradiction. ■

**Theorem 6 (Limit Points of a Negative TAN over $C_n$)** *The number* $\mathrm{fp}(t)$ *of fixed-points of a negative TAN over $C_n$ with thresholds $t = \langle t_i : i \in V \rangle$ is given by the formula* $\mathrm{fp}(t) = \sum_{i=0}^{n} S_{i,i}(t)$. *The number of its limit cycles of length 2* $\mathrm{lc}_2(t)$ *is given by* $\mathrm{lc}_2(t) = \sum_{0 \leq l < k \leq n} S_{l,k}(t)$. *Thus, the number of its limit cycles is* $\mathrm{lc}(t) = \mathrm{fp}(t) + \mathrm{lc}_2(t)$ *and the number of its limit points is* $\mathrm{lp}(t) = \mathrm{fp}(t) + 2\mathrm{lc}_2(t)$.

**Proof:**

First, it is clear that the two states $00\ldots0$ and $11\ldots1$ form a limit cycle by themselves.

Suppose that

$$\overbrace{11\ldots1}^{k}\overbrace{00\ldots0}^{n-k}$$

is a fixed point. Then the first $k$ states must have thresholds less than $-k$, whereas the remaining $n-k$ states must have thresholds greater than or equal to $-k$. Hence points of that form are counted by the sum $\sum_{i=0}^{n} R_i(t) = \sum_{i=0}^{n} S_{i,i}(t)$.

Next, suppose that

$$\overbrace{11\ldots1}^{k}\overbrace{00\ldots0}^{n-k} \rightleftarrows \overbrace{11\ldots1}^{l}\overbrace{00\ldots0}^{n-l}$$

is a limit cycle of length 2 with $l < k$. Then, by Lemma 4, it is not difficult to see that exactly $l$ vertices must have thresholds less than $-k$ and exactly $n-k$ vertices thresholds greater than or equal to $l$. Thus, the limit cycles of length 2 are counted by $\sum_{0 \leq l < k \leq n} S_{l,k}(t)$.

Finally, by Lemma 5, a negative TAN on the complete digraph cannot have any limit cycles of length exceeding 2. Thus the statement follows. ■

Note that Theorems 3 and 6 are of great computational value because they provide means for computing the limit structure of positive and negative TANs over the complete digraph with $n$ vertices in time polynomial to the size of the input, whereas the obvious brute force approach would require exponential time.

# 4 On the Structure of Products

By a (state space) **description** of a FAN or TAN is meant an enumeration of the number of cycles of each length and of the number of paths of each length on the

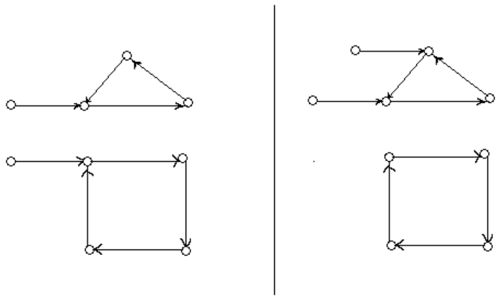

Figure 1: Two non-isomorphic state spaces with the same description.

transients in its state space. Note that each FAN or TAN has a unique description but, usually, a single description corresponds to many non-isomorphic state spaces. This is because, on the one hand, a description does not specify the limit cycles that each transient ends up into and, on the other, it does not specify how paths of different lengths combine with each-other to form transients entering the limit cycles. To exhibit the first point consider the following description: one cycle of length 3, one cycle of length 4 and two transients of length 1. One may formally denote such a description by two sequences $\langle \alpha_1, \alpha_2, \ldots \rangle$ and $\langle \beta_1, \beta_2, \ldots, \rangle$ indexed by the nonnegative integers with finitely many nonzero entries. $\alpha_i$ is the number of cycles of length $i$ and $\beta_j$ is the number of transient paths of length $j$. Thus, the above example corresponds to the description $\langle 0, 0, 1, 1, 0, \ldots \rangle$ and $\langle 2, 0, \ldots \rangle$. Two non-isomorphic state spaces with this description are depicted in Figure 1. The second point may be exhibited by considering the description with one cycle of length 2, two transient paths with length 1 and four transients of length 2, i.e., the description given by $\langle 0, 1, 0, \ldots \rangle$ and $\langle 2, 4, 0, \ldots \rangle$. Two non-isomorphic state spaces with this description are depicted in Figure 2.

In this section, a state space description is given of the product of two FANs in terms of the descriptions of the FANs themselves. Recall from [7] and [8] that the product of two FANs, respectively TANs, consists of the disjoint union of their graphs, where each vertex has the same local update function, respectively threshold, that it had in the original FAN, respectively TAN. On the state space level, the product of the two state spaces is their product when they are viewed as unary algebras. So a description of the product state space may be given by combining the description of the individual state spaces as follows:

A limit cycle of length $i$ in the first FAN and a limit cycle of length $j$ in the second combine to give $i \wedge j$ limit cycles of length $i \vee j$, where by $i \wedge j, i \vee j$ are denoted the

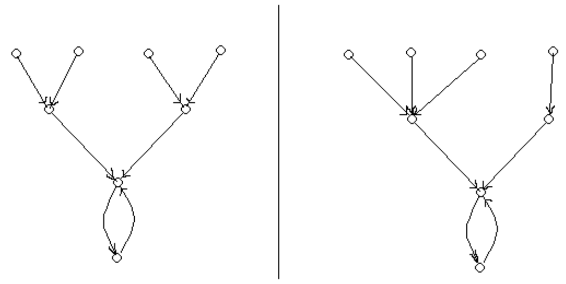

Figure 2: Another pair of non-isomorphic state spaces with the same description.

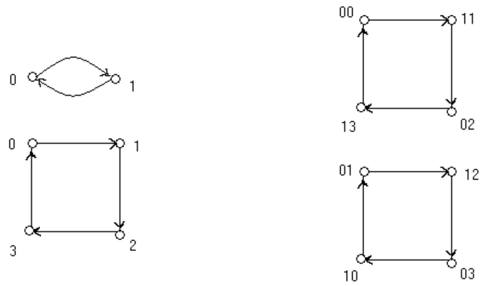


Figure 3: Product of two limit cycles.

greatest common divisor and the least common multiple of $i$ and $j$, respectively. As an example consider the two limit cycles of lengths 2 and 4 on the left of Figure 3. These give rise in the product space to the two limit cycles of lenth 4 on the right of Figure 3. A transient path of length $i$ combined in the product with a limit cycle of length $j$ will give $j$ transient paths of length $i$. This is depicted in Figure 4, where a transient of length 2 is combined in the product with a limit cycle of length 3 to give 3 transients of length 2. Finally, a transient path of length $i$ and a transient path of length $j$ combine in the product to give one transient path of length $\max(i, j)$. This is illustrated by Figure 5, where a transient of length 2 and a transient of length 3 are combined to a transient of length 3. Note that the paths of lengths 1 and 1 and 2, respectively, on these transients combine with each other and with the full transients to give the paths starting from the shaded nodes in Figure 5. Also note that the remaining transients appear as a result of combining transients with limit cycles.

All the observations made above on combining descriptions of FANs to provide a

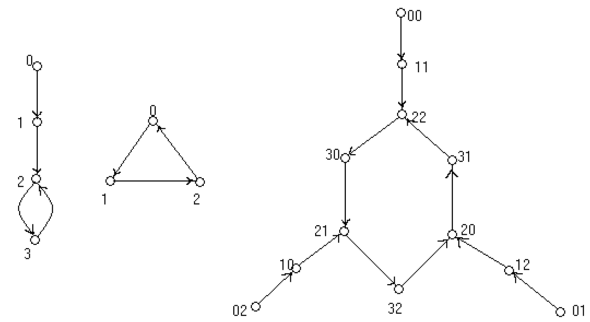


Figure 4: Product of two limit cycles.

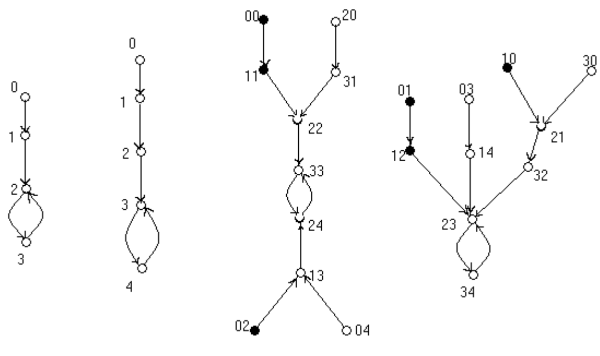


Figure 5: Product of two transients.

description of the product FAN may be nicely put together to provide a **calculus** or **algebra of descriptions** as shown below.

## A Combinatorial Calculus for Descriptions

We will be dealing with a $\mathbf{Z}$-algebra $\mathbf{A}$, where $\mathbf{Z}$ is the ring of integers. The elements of this algebra are pairs of sequences $\langle\langle\alpha_1, \alpha_2, \ldots\rangle, \langle\beta_1, \beta_2, \ldots\rangle\rangle$, with $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$ integers such that only finitely many of the $\alpha_i$'s and the $\beta_j$'s are nonzero. An alternative representation of the elements of $A$ is as formal sums

$$\sum_{i=1}^{\infty} \alpha_i c_i + \sum_{j=1}^{\infty} \beta_j d_j,$$

where $\alpha_i, \beta_j \in \mathbf{Z}, i, j \in \mathbb{N}^*$, with only finitely many of the $\alpha_i, \beta_j$'s being nonzero. Addition then is defined component-wise, i.e.,

$$(\sum_{i=1}^{\infty} \alpha_i c_i + \sum_{j=1}^{\infty} \beta_j d_j) + (\sum_{i=1}^{\infty} \gamma_i c_i + \sum_{j=1}^{\infty} \delta_j d_j) = \sum_{i=1}^{\infty} (\alpha_i + \gamma_i) c_i + \sum_{j=1}^{\infty} (\beta_j + \delta_j) d_j.$$

Multiplication is defined on the $c_i$'s and the $d_j$'s by the entries in table (1) and then extended by linearity to the whole algebra.

$$\begin{array}{c|cc} & c_k & d_l \\ \hline c_i & (i \wedge j) c_{i \vee j} & i d_l \\ d_j & k d_j & d_{\max(j,l)} \end{array} \tag{1}$$

Finally, scalar multiplication acts again componentwise

$$k(\sum_{i=1}^{\infty} \alpha_i c_i + \sum_{j=1}^{\infty} \beta_j d_j) = \sum_{i=1}^{\infty} (k\alpha_i) c_i + \sum_{j=1}^{\infty} (k\beta_j) d_j.$$

Not all elements of the $\mathbf{Z}$-algebra $\mathbf{A}$ have, of course, an interpretation as descriptions of FANs. However, every valid description is an element of this algebra and, furthermore, given two descriptions, the description of the product FAN is obtained by computing their product in this algebra. Moreover, given the description of a FAN, its multiple by a positive integer $2^n$ is the description of the network containing $n$ additional isolated vertices endowed with loops and having the identity functions (or thresholds 1), called its $n$-**increment**. As an illustration of the first statement consider the state spaces of two FANs on the left of Figure 6. The state space of the product FAN is shown on the right of Figure 6. The description of the first FAN is

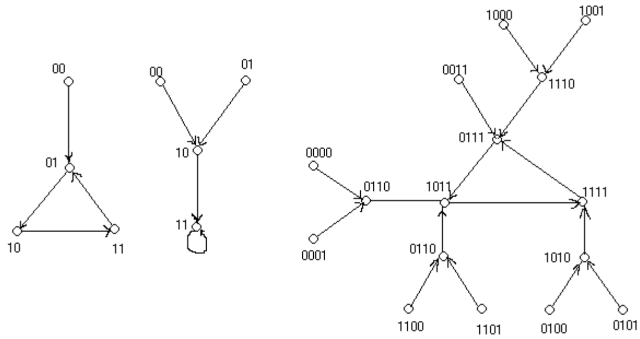


Figure 6: Product FAN.

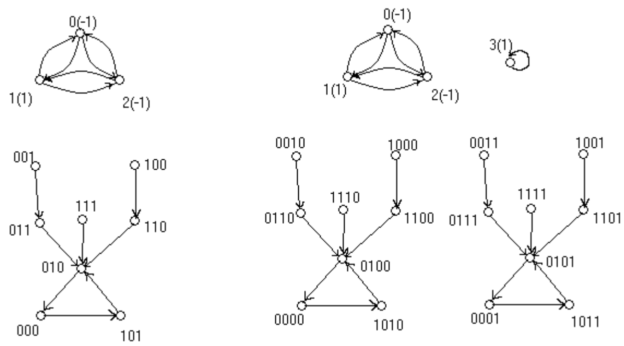


Figure 7: 1-increment of a TAN.

$c_3 + d_1$ and of the second FAN $c_1 + d_1 + 2d_2$. The description of the product FAN is then computed by

$$\begin{aligned}(c_3 + d_1)(c_1 + d_1 + 2d_2) &= c_3c_1 + d_1c_1 + c_3d_1 + d_1d_1 + 2c_3d_2 + 2d_1d_2 \\ &= c_3 + d_1 + 3d_1 + d_1 + 6d_2 + 2d_2 \\ &= c_3 + 5d_1 + 8d_2,\end{aligned}$$

which is in fact the description of the product. For the second statement, a TAN and its 1-increment together with their state spaces are given in Figure 7. The description of the first TAN is $c_3 + 3d_1 + 2d_2$ and of its 1-increment is $2c_3 + 6d_1 + 4d_2$.

Suppose, now, that $u', u'' \in A$ are the descriptions of the FANs $N' = \langle G', \{f'_i\}_{i \in V'} \rangle$, $N'' = \langle G'', \{f''_i\}_{i \in V''} \rangle$, respectively. Is there a FAN $N = \langle G, \{f_i\}_{i \in V} \rangle$ that has $u = u' + u''$ as its description? First, note that $u$ may not even be a valid description of a FAN at all. For instance, $c_3 + d_1$ and $c_2$ are two valid descriptions corresponding to the TANs of Figure 8 but $c_2 + c_3 + d_1$ is not a valid description, since a state space

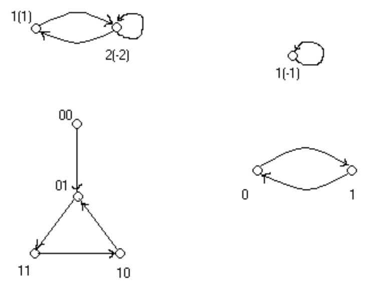


Figure 8: Two TANs with descriptions $c_3 + d_1$ and $c_2$.

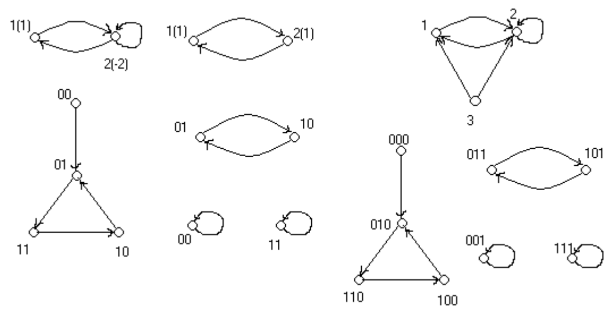


Figure 9: Two TANs and the FAN that has as its description the sum of their descriptions.

with one cycle of length 2, one cycle of length 3 and one transient of length 1 must necessarily have 6 points and 6 is not a power of 2. The only combination of points that guarantees that this cannot happen is $2^n$ and $2^n$ for some $n$. So, suppose that $|V'| = |V''| = n$. Then, given a bijection $h : V' \to V''$, a FAN with description $u$ may be constructed as follows: Its graph has set of nodes $V' \cup \{z\}, z \notin V'$, and set of edges $E' \cup h^{-1}(E'') \cup \{\langle z, v\rangle : v \in V'\}$. Its functions are determined, for all $i \in V$, by

$$f_i(x) = (\overline{x_z} \wedge f'_i(x \upharpoonright V')) \vee (x_z \wedge f''_{h(i)}(h^*(x \upharpoonright V'))) \quad \text{and} \quad f_z(x) = x_z,$$

where

$$h^*(y)_j = y_{h^{-1}(j)}, \text{ for all } y \in \{0,1\}^{V'}, j \in V''.$$

As an example consider the two TANs on the left of Figure 9. The first has functions

$$f'_1(x_1, x_2) = x_2, \quad \text{and} \quad f'_2(x_1, x_2) = \overline{x_1} \vee \overline{x_2}$$

and description $c_3 + d_1$. The second has functions

$$f_1''(x_1, x_2) = x_2 \quad \text{and} \quad f_2''(x_1, x_2) = x_1$$

and description $2c_1 + c_2$. Using the identity function from $V'$ to $V''$ as the bijection $h$, a FAN with description $2c_1 + c_2 + c_3 + d_1$ may be given as on the right in Figure 9. This FAN has functions

$$\begin{array}{rcl} f_1(x_1, x_2, x_3) & = & (x_2 \wedge \overline{x_3}) \vee (x_2 \wedge x_3) = x_2 \\ f_2(x_1, x_2, x_3) & = & ((\overline{x_1} \vee \overline{x_2}) \wedge \overline{x_3}) \vee (x_1 \wedge x_3) = (\overline{x_1} \wedge \overline{x_3}) \vee (\overline{x_2} \wedge \overline{x_3}) \vee (x_1 \wedge x_3) \\ f_3(x_1, x_2, x_3) & = & x_3 \end{array}$$

and state space also depicted on the right in Figure 9.

# 5 Open Questions

This paper addresses the problem of describing the structure of the state space of a TAN as a function of the adjacency relations of its underlying digraph and its thresholds without actually "running" the network. Some questions are answered but a lot more are left open. One obvious open question is to give a description of the limit point structure of an arbitrary TAN over the complete digraph. This was done in this paper only for positive and negative TANs of this kind. Another, more general, question would be to extent these or related results to TANs over different underlying digraphs. Further, related to the composition and decomposition of TANs, it would be very interesting to investigate whether an algebraic formula may be given for the description of the TAN that results by taking the product graph and product threshold structure of two TANs in terms of the description of these TANs. Finally, pertaining to finite control issues on TANs, a point of interest would be to investigate whether controlling some aspects of a TAN corresponds nicely to optimizing or, otherwise changing, in a specific way, quantities related to the description of the TAN. This would transform combinatorial control problems to algebraic, opening the possibility of applying powerful algebraic tools for their solution.

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