Chapter 4

Graded Classes of Models

4.1 Introduction

In work done during the last quarter of the twentieth century by, among others, Czelakowski [13, 14], Blok and Pigozzi [5, 6] and Font and Jansana [28] (see, also, the monograph [15], the survey [29] and the textbook [27]) a truly abstract framework was developed for the algebraization of arbitrary sentential logics. Depending on the strength of the ties this process establishes between a logic and its associated class of algebras, logics are classified in the steps of a hierarchy, known as the algebraic or Leibniz hierarchy.

In particular, in [6] (see, also, [8] and [7]) the authors pointed out the special role that first order logic without equality plays in formalizing a deductive system. Following this line of thought in earnest, two doctoral dissertations in Barcelona in the mid '90s went quite a long way in pursuing this point of view and in clarifying the scope of this approach. The first was Elgueta's Dissertation [22] and accompanying work [23, 24, 25] and [26] and the second was Dellunde's Dissertation [17] and accompanying work [18, 19] and [20].

Here, we are concerned especially with part of the work of Elgueta (which partially overlaps with other works in the preceding paragraph, e.g., [20]) presented in [23], culminating in the characterization of classes of models of first order structures defined without equality. In Section 5, titled "Main Theorems", Elgueta presents characterizations of classes of first order models without equality, including elementary classes (Subsection 5.1), universal classes (Subsection 5.2), universal Horn classes (Subsection 5.3) and universal atomic classes (Subsection 5.4). In each case several characterizations are provided based on operators on classes of models that preserve those classes. Moreover, Elgueta characterizes the corresponding reduced classes, i.e., those resulting by applying to models the operation of reduction modulo Leibniz equality (which is first order definable without equality and plays the role of "equality" in this equality free setting).

Following Elgueta [23], we make an attempt at developing the fundamentals of an extended framework in which interpretations of first order formulas (without equality) are multi-valued. However, we are faced with rather limited success. Many wished-for analogs of the more difficult results of [23] remain elusive. It is not clear yet whether this is due to inherent difficulties, or whether modifications in the framework are needed, or the failures are because different techniques are needed. So the work in this chapter cannot but be viewed as preliminary and exploratory in nature.

Let us give, nevertheless, a brief overview of what is accomplished and point out some key elements that are lacking and need to be modified and/or addressed differently.

In Section 4.2, basic notation and terminology are introduced and the values of a Boolean algebra (or, perhaps, more generally, a structure with the appropriate operations) are used to interpret formulas of a first order language without equality. Section 4.3 deals with *substructures*, *filter ex-*

tensions, *elementary substructures* and *elementarily equivalent structures*. Section 4.4 deals with morphisms of structures. We present morphisms, epimorphisms, embeddings and the corresponding strict versions, which are the ones that, roughly speaking, preserve and reflect the values of the interpreted symbols. We also define image structures and pre-image structures and what it means for a morphism to be elementary.

Section 4.5 is the one that introduces products.

In Section 4.6, we introduce graded congruences, or G-congruences, of structures. Graded congruences are G-congruences on the underlying G-algebra which are compatible with all relations of the structure. We use compatibility to define the Leibniz G-congruence of a structure. We prove some properties essential for applying the theory of Leibniz G-congruences. Key among those are the way they interact with morphisms and, in particular, the fact that they commute with inverse strict epimorphisms. Section 4.7 is dedicated into proving an analog of the well known result of Blok and Pigozzi characterizing Leibniz G-congruences.

Section 4.8 is dedicated entirely to studying quotients of structures and special morphisms relating them. Given a G-congruence Θ of a structure $\mathfrak{A}$, we define the quotient structure $\mathfrak{A}/\Theta$ of $\mathfrak{A}$ by Θ. The definition makes the canonical projection $\pi_\Theta : A \to A/\Theta$ a reductive morphism, i.e., a strict epimorphism, between the corresponding structures. Working along the lines of the classical universal algebraic results, we are able to obtain a sequence of analogs of the Homomorphism Theorems. More precisely, we prove analogs of the Homomorphism, the Second and Third Isomorphism and of the Correspondence Theorems.

A natural quotient to consider, as it is the one that reduces the structures "to the largest extent possible", is the quotient by the largest G-congruence of the structure, i.e., its Leibniz G-congruence. For this reason this quotient is termed the Leibniz quotient. It is considered in Section 4.9. There it is shown that every reductive morphism between two structures induces an isomorphism between the corresponding Leibniz quotients.

Section 4.10 quickly readjusts the notions of models and of semantic consequence to the current framework. A novelty here, already encompassed in [23], is that, apart from ordinary model classes, one may use reduced model classes, which are classes of models obtained by applying the Leibniz reduction operation to each model of a certain class.

Class operators on structures are introduced and studied in Section 4.11. Let us summarize these operators. We consider the operator $\mathbb{S}$ of taking substructures, $\mathbb{F}$ of taking filter extensions, $\mathbb{H}$ of taking morphic images, $\mathbb{R}$ of taking reductions and $\mathbb{E}$ of taking extensions (preimages under reductive morphisms), and $\mathbb{P}$ of taking products. The first result of the section asserts that all these operators are idempotent. Subsequent results examine the way any two different of these operators compose with one another.

Section 4.12 adapts the Diagram Lemma of Model Theory to the multi-valued framework. The result is an extension of the version established by Elgueta (Theorem 4.5 of [23]). Section 4.13 revisits and establishes an analog of the Reduction Operator Lemma (Theorem 4.7 of [23]). The success in extending those two key lemmas gave (false?) hope that many of the results of the main section, Section 5 on "The Main Theorems" of [23], would be adaptable to the multi-valued context. The main theorems include characterizations of (from narrower to wider) universal atomic classes (Subsection 5.4 of [23]), universal Horn classes (Subsection 5.3 ibid.), universal classes (Subsection 5.2 ibid.) and elementary classes (Subsection 5.1 ibid.). However, we were only able to characterize universal atomic classes in Section 4.14. It is not clear yet whether the problems encountered in characterizing the other classes in a similar way are inherent, whether they need some tweaking of the framework or whether they require new methodology. For now these characterizations are left open for future work. So this work (and this report) should be considered as preliminary and leaves many questions open for further investigation.

4.2 Notation and Terminology

Let $\mathbf{G} = \langle G, \leq \rangle$ be a fixed complete Boolean algebra, which may be assumed to have additional structure, as needed. Recall that, given a set X, the set of all functions G^X is ordered pointwise, i.e., by setting, for all $f, g : X \to G$,

$$f \leq g \quad \text{iff, for all } x \in X, \ f(x) \leq g(x).$$

We consider a first-order language $\mathcal{L} = \langle F, R, \rho \rangle$, where F is a set of function symbols, R is a set of relation symbols and ρ is the arity function from $F \cup R$ into the natural numbers. The formulas of $\mathcal{L}$ are the usual first-order formulas. That is, the set of **$\mathcal{L}$-terms** $\mathrm{Tm}_{\mathcal{L}}(V)$ is constructed using the function symbols in F starting from a countably infinite collection V of individual variables. Moreover, the set of **$\mathcal{L}$-formulas** $\mathrm{Fm}_{\mathcal{L}}(V)$ is constructed using the usual logical connectives (say, $\neg$, $\wedge$ and $\vee$ and the quantifiers $\forall$ and $\exists$) starting from atomic formulas of the form $r(t_1, \ldots, t_n)$, where $r \in R$, with $\rho(r) = n$, and $t_1, \ldots, t_n$ are arbitrary $\mathcal{L}$-terms.

To introduce the notion of $\mathcal{L}$-structure, let us revisit, first, the notions of G-equivalence and G-congruence and, also, of reduced G-congruence.

Let A be a set. A **G-equivalence Θ on A** is a function $\Theta : A^2 \to G$, such that:

(**Reflexivity**) $\Theta(a, a) = \top$, for all $a \in A$;

(**Symmetry**) $\Theta(b, a) = \Theta(a, b)$, for all $a, b \in A$;

(**Transitivity**) $\Theta(a, b) \wedge \Theta(b, c) \leq \Theta(a, c)$, for all $a, b, c \in A$.

Θ is said to be **reduced** if, for all $a, b \in A$,

$$\Theta(a,b) = \top \quad \text{iff} \quad a = b.$$

Let $\mathbf{A} = \langle A, F^{\mathbf{A}} \rangle$ be an F-algebra. An operation $f^{\mathbf{A}} \in F^{\mathbf{A}}$, with arity $\rho(f) = n$, is said to be **compatible with** Θ if, for all $a_1, b_1, \dots, a_n, b_n \in A$,

$$\bigwedge_{i=1}^{n} \Theta(a_i, b_i) \le \Theta(\lambda^{\mathbf{A}}(a_1, \dots, a_n), \lambda^{\mathbf{A}}(b_1, \dots, b_n)).$$

If all operations in $F^{\mathbf{A}}$ are compatible with Θ, then Θ is called a G-**congruence of A**.

Let R be a set of relation symbols with arity function $\rho : R \to \omega$. A G-**relation** $r^A \in R^A$, with $\rho(r) = n$, is a function $r^A : A^n \to G$. It is said to be **compatible with** Θ if, for all $a_1, b_1, \dots, a_n, b_n \in A$,

$$\bigwedge_{i=1}^{n} \Theta(a_i, b_i) \wedge r^A(a_1, \dots, a_n) \le r^A(b_1, \dots, b_n).$$

If all relations in $R^{\mathbf{A}}$ are compatible with Θ, then Θ is called a G-**congruence of** $\langle A, R^A \rangle$.

A **structure over** $\mathcal{L}$ or $\mathcal{L}$-**structure** $\mathfrak{A} = \langle A, E, F^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ consists of:

- A set A, the **universe** of $\mathfrak{A}$;
- A reduced G-equivalence E on A;
- The **interpretations** $F^{\mathfrak{A}}$ of the function symbols as functions $f^{\mathfrak{A}} : A^n \to A$, where $f \in F$, with $\rho(f) = n$, all of which are compatible with E, i.e., such that E is a reduced G-congruence on $\mathbf{A} = \langle A, F^{\mathfrak{A}} \rangle$;
- The **interpretations** $R^{\mathfrak{A}}$ of the relation symbols as G-relations $r^{\mathfrak{A}} : A^n \to G$, where $r \in R$, with $\rho(r) = n$, all of which are compatible with E, i.e., such that E is a reduced G-congruence of $\langle A, R^{\mathfrak{A}} \rangle$.

In general, capital Gothic letters $\mathfrak{A}, \mathfrak{B}, \dots$ represent structures over $\mathcal{L}$. The corresponding boldface letter $\mathbf{A}$ is used to denote the underlying algebra $\mathbf{A} = \langle A, F^{\mathfrak{A}} \rangle$ of $\mathfrak{A}$, and we sometimes write $f^{\mathbf{A}}$ instead of $f^{\mathfrak{A}}$. The corresponding calligraphic letter $\mathcal{A}$ is used to denote the underlying G-algebra $\mathcal{A} = \langle \mathbf{A}, E \rangle$ of $\mathfrak{A}$. Lowercase boldface letters $\mathbf{a}, \mathbf{b}, \dots$ are used to indicate members of the cartesian product of some family of sets. So, if $\mathfrak{A}$ is an $\mathcal{L}$-structure, $\mathbf{a} = \langle a_1, \dots, a_n \rangle$ belongs to A^n, $f \in F$, with $\rho(f) = n$, $r \in R$, with $\rho(r) = n$, and h is any mapping with domain A, then $f^{\mathbf{A}}(\mathbf{a})$, $r^{\mathfrak{A}}(\mathbf{a})$ and $h(\mathbf{a})$ are short-hand for $f^{\mathbf{A}}(a_1, \dots, a_n)$, $r^{\mathfrak{A}}(a_1, \dots, a_n)$ and $\langle h(a_1), \dots, h(a_n) \rangle$, respectively.

By an $\mathcal{L}$-**algebra** we mean the underlying algebra of any $\mathcal{L}$-structure. By a G-**algebra** we mean the underlying G-algebra of any $\mathcal{L}$-structure. If the set of function symbols is empty, an $\mathcal{L}$-algebra simply means an arbitrary set and a G-algebra means a set with a reduced G-equivalence on it. The absolutely free $\mathcal{L}$-algebra over the set V of variables is the algebra of all $\mathcal{L}$-terms over V. It is denoted by $\mathbf{Tm}_{\mathcal{L}}(V)$. We also write $\mathcal{T}m_{\mathcal{L}}(V) = \langle \mathbf{Tm}_{\mathcal{L}}(V), \Delta_{\mathrm{Tm}_{\mathcal{L}}(V)}\rangle$ for the G-algebra of terms, where the reduced G-congruence is simply the identity.

Formulas are represented by lowercase Greek letters $\varphi, \psi, \ldots$, and uppercase ones are used to denote sets of formulas. We write $\varphi(x_1, \ldots, x_n)$ to indicate that the free variables that occur in φ are among $x_1, \ldots, x_n$.

Now we define the truth value $\varphi^{\mathfrak{A}}[h]$ of an $\mathcal{L}$-formula φ in a structure $\mathfrak{A}$ under an assignment $h : \mathbf{Tm}_{\mathcal{L}}(V) \to \mathbf{A}$.

For atomic $\varphi = r(t_1, \ldots, t_n)$, we get the value

$$\varphi^{\mathfrak{A}}[h] \coloneqq r^{\mathfrak{A}}(h(t_1), \ldots, h(t_n)).$$

Assume, inductively, that the values $\varphi^{\mathfrak{A}}[h]$, $\varphi_1^{\mathfrak{A}}[g]$ and $\varphi_2^{\mathfrak{A}}[h]$ have been computed. Then

$$\begin{array}{rcl}(\neg\varphi)^{\mathfrak{A}}[h] & = & \neg\varphi^{\mathfrak{A}}[h];\\ (\varphi_1 \wedge \varphi_2)^{\mathfrak{A}}[h] & = & \varphi_1^{\mathfrak{A}}[h] \wedge \varphi_2^{\mathfrak{A}}[h];\\ (\varphi_1 \vee \varphi_2)^{\mathfrak{A}}[h] & = & \varphi_1^{\mathfrak{A}}[h] \vee \varphi_2^{\mathfrak{A}}[h].\end{array}$$

Finally, suppose, inductively, that the values $\varphi(x)^{\mathfrak{A}}[h]$, for all h, have been computed. Denote, as usual, by $h[a/x]$ or $h(a/x)$ the assignment that maps x to a and acts as h on the remaining variables. Then

$$\begin{array}{rcl}(\forall x\varphi)^{\mathfrak{A}}[h] & = & \bigwedge_{a\in A}\varphi^{\mathfrak{A}}[h(a/x)];\\ (\exists x\varphi)^{\mathfrak{A}}[h] & = & \bigvee_{a\in A}\varphi^{\mathfrak{A}}[h(a/x)].\end{array}$$

It is clear that, in this context, the De Morgan Laws hold, that is, for a structure $\mathfrak{A}$ under an assignment $h : \mathbf{Tm}_{\mathcal{L}}(V) \to \mathbf{A}$, we have

$$\begin{array}{rcl}[\neg(\varphi_1 \wedge \varphi_2)]^{\mathfrak{A}}[h] & = & (\neg\varphi_1 \vee \neg\varphi_2)^{\mathfrak{A}}[h],\\ [\neg(\varphi_1 \vee \varphi_2)]^{\mathfrak{A}}[h] & = & (\neg\varphi_1 \wedge \neg\varphi_2)^{\mathfrak{A}}[h],\\ (\neg\forall x\varphi)^{\mathfrak{A}}[h] & = & (\exists x\neg\varphi)^{\mathfrak{A}}[h],\\ (\neg\exists x\varphi)^{\mathfrak{A}}[h] & = & (\forall x\neg\varphi)^{\mathfrak{A}}[h].\end{array}$$

This implies that, in a context where a proof by structural induction on a formula is to be carried out, once the case of negation has been dealt with, then only one of the cases of conjunction or disjunction and of universal or existential quantifications have to be undertaken.

We show that the compatibility of the functions and of the G-relations of a structure with its reduced G-congruence relation extends to the compatibility of all its term functions and all its formula defined G-relations with the reduced G-congruence.

Proposition 122 *Let $\mathfrak{A} = \langle A, E, F^{\mathfrak{A}}, R^{\mathfrak{A}}\rangle$ be an $\mathcal{L}$-structure. Then, for all $\mathcal{L}$-terms $t(\bar{x})$ and all $\mathcal{L}$-formulas $\varphi(\bar{x})$, we have, for all $a_1, b_1, \ldots, a_n, b_n \in A$,* *(a)* $\bigwedge_{i=1}^n E(a_i,b_i) \leq E(t^{\mathbf{A}}(\mathbf{a}), t^{\mathbf{A}}(\mathbf{b}))$*;*

(b) $\bigwedge_{i=1}^n E(a_i,b_i) \wedge \varphi^{\mathfrak{A}}(\mathbf{a}) \leq \varphi^{\mathfrak{A}}(\mathbf{b})$*.*

Proof:

(a) We work by induction on the structure of an $\mathcal{L}$-term. Suppose, for the base case, that $t = x_j$ is a variable. Then, we get

$$\begin{array}{rcll} \bigwedge_{i=1}^n E(a_i,b_i) & \leq & E(a_j,b_j) & \text{(Property of } \wedge\text{)} \\ & = & E(x_j^{\mathbf{A}}(\mathbf{a}), x_j^{\mathbf{A}}(\mathbf{b})). & \text{(Definition of } x_j^{\mathbf{A}}\text{)} \end{array}$$

Assume, for the induction hypothesis, that, for the $\mathcal{L}$-terms $t_1(\bar{x}),\ldots, t_m(\bar{x})$, we have

$$\bigwedge_{i=1}^n E(a_i,b_i) \leq E(t_j^{\mathbf{A}}(\mathbf{a}), t_j^{\mathbf{A}}(\mathbf{b})).$$

Consider the term $t(\bar{x}) = f(t_1,\ldots,t_m)$, where $f \in F$, with $\rho(f) = m$. Then, we get

$$\begin{array}{rcl} \bigwedge_{i=1}^n E(a_i,b_i) & \leq & \bigwedge_{j=1}^m E(t_j^{\mathbf{A}}(\mathbf{a}), t_j^{\mathbf{A}}(\mathbf{b})) \\ & & \quad \text{(Induction Hypothesis and Property of } \wedge\text{)} \\ & \leq & E(f^{\mathbf{A}}(t_1^{\mathbf{A}}(\mathbf{a}),\ldots,t_m^{\mathbf{A}}(\mathbf{a})), f^{\mathbf{A}}(t_1^{\mathbf{A}}(\mathbf{b}),\ldots,t_m^{\mathbf{A}}(\mathbf{b}))) \\ & & \quad \text{(Compatibility of } f^{\mathbf{A}} \text{ with } E\text{)} \\ & = & E(t^{\mathbf{A}}(\mathbf{a}), t^{\mathbf{A}}(\mathbf{b})). \quad \text{(Definition of } t\text{)} \end{array}$$

So, for all terms $t(\bar{x})$, $\bigwedge_{i=1}^n E(a_i,b_i) \leq E(t^{\mathbf{A}}(\mathbf{a}), t^{\mathbf{A}}(\mathbf{b}))$.

(b) We work by induction on the structure of $\mathcal{L}$-formulas. For the base case, let $r(t_1,\ldots,t_m)$ be atomic, with $t_j = t_j(\bar{x})$, for $j = 1,\ldots,m$. Then, we get

$$\begin{array}{l} \bigwedge_{i=1}^n E(a_i,b_i) \wedge r^{\mathfrak{A}}(t_1^{\mathbf{A}}(\mathbf{a}),\ldots,t_m^{\mathbf{A}}(\mathbf{a})) \\ \quad \leq \bigwedge_{j=1}^m E(t_j^{\mathbf{A}}(\mathbf{a}), t_j^{\mathbf{A}}(\mathbf{b})) \wedge r^{\mathfrak{A}}(t_1^{\mathbf{A}}(\mathbf{a}),\ldots,t_m^{\mathbf{A}}(\mathbf{a})) \\ \quad\quad \text{(Part (a) and Property of } \wedge\text{)} \\ \quad \leq r^{\mathfrak{A}}(t_1^{\mathbf{A}}(\mathbf{b}),\ldots,t_m^{\mathbf{A}}(\mathbf{b})). \\ \quad\quad \text{(Compatibility of } r^{\mathfrak{A}} \text{ with } E\text{)} \end{array}$$

For negation, consider the formula $\varphi(\bar{x}) = \neg\psi(\bar{x})$. Assume inductively, that, for all $\mathbf{a} = \langle a_1,\ldots,a_n\rangle$ and $\mathbf{b} = \langle b_1,\ldots,b_n\rangle$,

$$\bigwedge_{i=1}^n E(a_i,b_i) \wedge \psi^{\mathfrak{A}}(\mathbf{a}) \leq \psi^{\mathfrak{A}}(\mathbf{b}).$$

This is equivalent to $\bigwedge_{i=1}^n E(a_i,b_i) \leq \psi^{\mathfrak{A}}(\mathbf{a}) \leftrightarrow \psi^{\mathfrak{A}}(\mathbf{b})$. Thus, we get

$$\bigwedge_{i=1}^n E(a_i,b_i) \leq \neg\psi^{\mathfrak{A}}(\mathbf{a}) \leftrightarrow \neg\psi^{\mathfrak{A}}(\mathbf{b}),$$

i.e., $\bigwedge_{i=1}^n E(a_i,b_i) \leq \varphi^{\mathfrak{A}}(\mathbf{a}) \leftrightarrow \varphi^{\mathfrak{A}}(\mathbf{b})$. In particular, we obtain

$$\bigwedge_{i=1}^n E(a_i,b_i) \wedge \varphi^{\mathfrak{A}}(\mathbf{a}) \leq \varphi^{\mathfrak{A}}(\mathbf{b}).$$

We deal next with conjunction. Suppose $\varphi(\bar{x}) = \varphi_1(\bar{x}) \wedge \varphi_2(\bar{x})$ and, inductively, that

$$\bigwedge_{i=1}^n E(a_i,b_i) \wedge \varphi_1^{\mathfrak{A}}(\mathbf{a}) \leq \varphi_1^{\mathfrak{A}}(\mathbf{b}) \quad \text{and} \quad \bigwedge_{i=1}^n E(a_i,b_i) \wedge \varphi_2^{\mathfrak{A}}(\mathbf{a}) \leq \varphi_2^{\mathfrak{A}}(\mathbf{b}).$$

Then we get

$$\begin{aligned}
&\textstyle\bigwedge_{i=1}^n E(a_i,b_i) \wedge \varphi^{\mathfrak{A}}(\mathbf{a}) \\
&\qquad = \textstyle\bigwedge_{i=1}^n E(a_i,b_i) \wedge \varphi_1^{\mathfrak{A}}(\mathbf{a}) \wedge \varphi_2^{\mathfrak{A}}(\mathbf{a}) \\
&\qquad\quad \text{(Definition of } \varphi) \\
&\qquad \leq \varphi_1^{\mathfrak{A}}(\mathbf{b}) \wedge \varphi_2^{\mathfrak{A}}(\mathbf{b}) \\
&\qquad\quad \text{(Induction Hypothesis and Property of } \wedge) \\
&\qquad \leq \varphi^{\mathfrak{A}}(\mathbf{b}). \quad \text{(Definition of } \varphi)
\end{aligned}$$

Finally, for universal quantification, assume $\varphi(\bar{x}) = \forall y \psi(\bar{x},y)$ and that, inductively,

$$\bigwedge_{i=1}^n E(a_i,b_i) \wedge E(a,b) \wedge \psi^{\mathfrak{A}}(\mathbf{a},a) \leq \psi^{\mathfrak{A}}(\mathbf{b},b).$$

Then we get

$$\begin{aligned}
\textstyle\bigwedge_{i=1}^n E(a_i,b_i) \wedge \varphi^{\mathfrak{A}}(\mathbf{a}) &= \textstyle\bigwedge_{i=1}^n E(a_i,b_i) \wedge \bigwedge_{a\in A} \psi^{\mathfrak{A}}(\mathbf{a},a) \\
&\qquad \text{(Definition of } \varphi^{\mathfrak{A}}(\mathbf{a})) \\
&= \textstyle\bigwedge_{a\in A}(\bigwedge_{i=1}^n E(a_i,b_i) \wedge \psi^{\mathfrak{A}}(\mathbf{a},a)) \\
&\qquad \text{(Property of } \wedge) \\
&\leq \textstyle\bigwedge_{a\in A} \psi^{\mathfrak{A}}(\mathbf{b},a) \\
&\qquad \text{(Induction Hypothesis)} \\
&= \varphi^{\mathfrak{A}}(\mathbf{b}). \quad \text{(Definition of } \varphi^{\mathfrak{A}}(\mathbf{b}))
\end{aligned}$$

So, for every $\mathcal{L}$-formula φ, $\bigwedge_{i=1}^n E(a_i,b_i) \wedge \varphi^{\mathfrak{A}}(\mathbf{a}) \leq \varphi^{\mathfrak{A}}(\mathbf{b})$.

■

Let $X \subseteq G$. Then we write $\mathfrak{A} \vDash_X \varphi[h]$ to signify that $\varphi^{\mathfrak{A}}[h] \in X$. Further, $\mathfrak{A} \vDash_X \varphi$ expresses $\mathfrak{A} \vDash_X \forall\varphi$, where $\forall\varphi$ denotes the universal closure of φ. When we write

$$\mathfrak{A} \vDash_X \varphi(x_1,\ldots,x_k)[a_1,\ldots,a_k],$$

we mean $\mathfrak{A} \vDash \varphi[h]$, with respect to any assignment h, such that $h(x_i) = a_i$, for all $1 \leq i \leq k$.

We finish the section with a remark concerning **languages with** and **languages without G-equality**. We say $\mathcal{L}$ is a **language with G-equality** if R contains a binary relation symbol e, such that, for all structures $\mathfrak{A}$ and all $a, b \in A$,

$$e^{\mathfrak{A}}(a,b) = E(a,b).$$

$\mathcal{L}$ is **without G-equality** if such a relation symbol (forced to be interpreted in such a way in all structures) is not present in R.

4.3 Substructures and Elementarity

Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ and $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ be two $\mathcal{L}$-structures. We say that $\mathfrak{A}$ is a **substructure** of $\mathfrak{B}$, written $\mathfrak{A} \subseteq \mathfrak{B}$, if $\mathcal{A} = \langle \mathbf{A}, E^{\mathfrak{A}} \rangle$ is a subalgebra of $\mathcal{B} = \langle \mathbf{B}, E^{\mathfrak{B}} \rangle$ ($\mathcal{A} \leq \mathcal{B}$) and, for all $r \in R$,

$$r^{\mathfrak{A}} = r^{\mathfrak{B}} \restriction_{A^{\rho(r)}}.$$

The $\mathcal{L}$-structure $\mathfrak{B}$ is called a **filter extension of** $\mathfrak{A}$, written $\mathfrak{A} \lesssim \mathfrak{B}$, if $\mathcal{A} = \mathcal{B}$ and, for all $r \in R$,

$$r^{\mathfrak{A}} \leq r^{\mathfrak{B}},$$

that is, for all $\mathbf{a} \in A^{\rho(r)}$, $r^{\mathfrak{A}}(\mathbf{a}) \leq r^{\mathfrak{B}}(\mathbf{a})$.

Suppose that $\mathfrak{A} = \langle \mathbf{A}, R^{\mathfrak{A}} \rangle$ is an $\mathcal{L}$-structure. Let X be a subset of A and denote by $[X]$ the universe of the subalgebra of $\mathbf{A}$ generated by X. Then the **substructure of $\mathfrak{A}$ generated by** X, denoted $\mathfrak{A}\restriction_X$, is the substructure of $\mathfrak{A}$, with universe $[X]$, i.e.,

$$\mathfrak{A}\restriction_X = \left\langle [X], E^{\mathfrak{A}} \restriction_{[X]}, \left\{ f^{\mathfrak{A}}\restriction_{[X]} : f \in F \right\}, \left\{ r^{\mathfrak{A}}\restriction_{[X]} : r \in R \right\} \right\rangle.$$

The $\mathcal{L}$-structure $\mathfrak{A}$ is an **elementary substructure** of $\mathfrak{B}$, denoted $\mathfrak{A} \subseteq_e \mathfrak{B}$, if $\mathfrak{A} \subseteq \mathfrak{B}$ and, in addition, for every formula φ and any assignment $h : \mathbf{Tm}_{\mathcal{L}}(V) \to \mathbf{A}$,

$$\varphi^{\mathfrak{A}}[h] = \varphi^{\mathfrak{B}}[h].$$

More generally, we call two $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ **elementarily equivalent** and write $\mathfrak{A} \equiv \mathfrak{B}$ if, for every sentence φ over $\mathcal{L}$,

$$\varphi^{\mathfrak{A}} = \varphi^{\mathfrak{B}}.$$

Obviously, if $\mathfrak{A} \subseteq_e \mathfrak{B}$, we also have $\mathfrak{A} \equiv \mathfrak{B}$.

4.4 Morphisms

Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ and $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ be two $\mathcal{L}$-structures. A mapping $h : A \to B$ is called a **morphism** from $\mathfrak{A}$ to $\mathfrak{B}$, written $h : \mathfrak{A} \to \mathfrak{B}$, if it is a

G-algebra homomorphism (G-morphism) $h : \mathcal{A} \to \mathcal{B}$ and, in addition, for all $r \in R$, with $\rho(r) = n$, and all $\mathbf{a} \in A^n$,

$$\begin{tikzcd}
A^n \arrow[rr, "h^n"] \arrow[dr, "r^{\mathfrak{B}} \circ h^n"'] & & B^n \arrow[dl, "r^{\mathfrak{B}}"] \\
& G &
\end{tikzcd}$$

$$r^{\mathfrak{A}}(\mathbf{a}) \leq r^{\mathfrak{B}}(h(\mathbf{a})).$$

A G-morphism $h : \langle \mathbf{A}, E^{\mathfrak{A}} \rangle \to \langle \mathbf{B}, E^{\mathfrak{B}} \rangle$ is a G-**embedding**, written $h : \mathcal{A} \rightarrowtail \mathcal{B}$, if, for all $a_1, a_2 \in A$, $E^{\mathfrak{A}}(a_1, a_2) = E^{\mathfrak{B}}(h(a), h(b))$. Note that, since both $E^{\mathfrak{A}}$ and $E^{\mathfrak{B}}$ are reduced, this implies that $h : \mathbf{A} \rightarrowtail \mathbf{B}$ is an $\mathcal{L}$-algebra embedding. A morphism $h : \mathfrak{A} \to \mathfrak{B}$ is an **embedding** (of $\mathcal{L}$-structures) if it is a G-embedding. In this case we write $h : \mathfrak{A} \rightarrowtail \mathfrak{B}$. It is an **epimorphism** if it is an epimorphism between the corresponding $\mathcal{L}$-algebras. This is written $h : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$ and we say that $\mathfrak{B}$ is a **morphic image** of $\mathfrak{A}$. A morphism $h : \mathfrak{A} \to \mathfrak{B}$ is an **isomorphism**, written $h : \mathfrak{A} \cong \mathfrak{B}$, if it is bijective and $h^{-1} : \mathfrak{B} \to \mathfrak{A}$ is also a morphism.

A mapping $h : A \to B$ is called a **strong** or **strict morphism** from $\mathfrak{A}$ to $\mathfrak{B}$, written $h : \mathfrak{A} \to_s \mathfrak{B}$, if $h : \mathfrak{A} \to \mathfrak{B}$ and, moreover, the reverse inequality of the one displayed above, for all $r \in R$, with $\rho(r) = n$, and all $\mathbf{a} \in A^n$, also holds, that is, we have

$$r^{\mathfrak{A}}(\mathbf{a}) = r^{\mathfrak{B}}(h(\mathbf{a})).$$

A **strict embedding**, written $h : \mathfrak{A} \rightarrowtail_s \mathfrak{B}$ is a strict morphism that is also an embedding. A **strict epimorphism**, more often called a **reductive morphism**, written $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$, is a strict morphism that is an epimorphism. In case there exists a reductive morphism $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$, we say that $\mathfrak{B}$ is a **reduction** of $\mathfrak{A}$ and $\mathfrak{A}$ is an **expansion** of $\mathfrak{B}$.

Based on the following lemma, one can show that a reductive morphism, which is also an embedding, is an isomorphism and, moreover, if the language has G-equality, then a reductive morphism is an isomorphism.

Lemma 123 *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$, $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ *be* $\mathcal{L}$*-structures and* $h : \mathcal{A} \to \mathcal{B}$ *a* G*-morphism.*

(i) $h : \mathfrak{A} \to \mathfrak{B}$ *iff, for all* $r \in R$, *and all* $\mathbf{b} \in B^{\rho(r)}$,

$$\bigvee_{\mathbf{a} \in h^{-1}(\mathbf{b})} r^{\mathfrak{A}}(\mathbf{a}) \leq r^{\mathfrak{B}}(\mathbf{b}).$$

(ii) $h : \mathfrak{A} \to_s \mathfrak{B}$ *iff, for all* $r \in R$, *and all* $\mathbf{b} \in h(A)^{\rho(r)}$,

$$r^{\mathfrak{A}}(\mathbf{a}) = r^{\mathfrak{B}}(\mathbf{b}), \text{ for all } \mathbf{a} \in h^{-1}(\mathbf{b}).$$

(iii) $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$ *implies, for all* $r \in R$,

  - $r^{\mathfrak{A}}(\mathbf{a}) = r^{\mathfrak{B}}(h(\mathbf{a}))$, *for all* $\mathbf{a} \in A^{\rho(r)}$*;*
  - $r^{\mathfrak{B}}(\mathbf{b}) = r^{\mathfrak{A}}(\mathbf{a})$, *for all* $\mathbf{b} \in B^{\rho(r)}$ *and* $\mathbf{a} \in h^{-1}(\mathbf{b})$.

Proof:

(i) For the implication left to right, consider $\mathbf{b} \in B^{\rho(r)}$. Let $\mathbf{a} \in h^{-1}(\mathbf{b})$. By definition, $h(\mathbf{a}) = \mathbf{b}$. By hypothesis, $r^{\mathfrak{A}}(\mathbf{a}) \le r^{\mathfrak{B}}(h(\mathbf{a})) = r^{\mathfrak{B}}(\mathbf{b})$. Hence, since $\mathbf{a} \in h^{-1}(\mathbf{b})$ was arbitrary, $\bigvee_{\mathbf{a} \in h^{-1}(\mathbf{b})} r^{\mathfrak{A}}(\mathbf{a}) \le r^{\mathfrak{B}}(\mathbf{b})$.

For the converse, assume $\mathbf{a} \in A^{\rho(r)}$ and let $\mathbf{b} = h(\mathbf{a})$. Then we have, using the hypothesis,

$$r^{\mathfrak{A}}(\mathbf{a}) \le \bigvee_{\mathbf{a} \in h^{-1}(\mathbf{b})} r^{\mathfrak{A}}(\mathbf{a}) \le r^{\mathfrak{B}}(\mathbf{b}) = r^{\mathfrak{B}}(h(\mathbf{a})).$$

(ii) We have, for all $\mathbf{a} \in A^{\rho(r)}$ and all $\mathbf{b} \in h(A)^{\rho(r)}$,

$$\mathbf{b} = h(\mathbf{a}) \quad \text{iff} \quad \mathbf{a} \in h^{-1}(\mathbf{b}).$$

Thus, the condition $r^{\mathfrak{A}}(\mathbf{a}) = r^{\mathfrak{B}}(h(\mathbf{a}))$, for all $\mathbf{a} \in A^{\rho(r)}$, is equivalent to the condition $r^{\mathfrak{A}}(\mathbf{a}) = r^{\mathfrak{B}}(\mathbf{b})$, for all $\mathbf{b} \in h(A)^{\rho(r)}$ and all $\mathbf{a} \in h^{-1}(\mathbf{b})$.

(iii) By Part (ii), taking into account the fact that $h(A) = B$.

■

Corollary 124 *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$, $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ *be* $\mathcal{L}$*-structures and* $h : \mathcal{A} \to \mathcal{B}$.

(i) If $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$ *is an embedding, then* $h : \mathfrak{A} \cong \mathfrak{B}$.

(ii) If the language has G-equality and $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$, *then* $h : \mathfrak{A} \cong \mathfrak{B}$.

Proof: Part (i) follows from the definition of an embedding and from Part (iii) of Lemma 123. Part (ii) is a consequence of Part (i) and the fact that the presence of G-equality, together with strictness, implies that h is an embedding. ■

Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$, $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ be $\mathcal{L}$-structures and $h : \mathfrak{A} \to \mathfrak{B}$. Let $\mathfrak{A}' \subseteq \mathfrak{A}$. The **image of $\mathfrak{A}'$ under** h is the substructure of $\mathfrak{B}$ generated by $h(A')$,

$$h(\mathfrak{A}') = \langle h(A'), E^{\mathfrak{B}}\restriction_{h(A')}, F^{\mathfrak{B}}\restriction_{h(A')}, R^{\mathfrak{B}}\restriction_{h(A')} \rangle.$$

On the other hand, consider a substructure $\mathfrak{B}' \subseteq \mathfrak{B}$. The **preimage of $\mathfrak{B}'$ under** h is the structure

$$h^{-1}(\mathfrak{B}') = \langle h^{-1}(B'), E^{\mathfrak{A}}\restriction_{h^{-1}(B')}, F^{\mathfrak{A}}\restriction_{h^{-1}(B')}, R^{h^{-1}(\mathfrak{B}')} \rangle,$$

where $R^{h^{-1}(\mathfrak{B}')} = \{r^{h^{-1}(\mathfrak{B}')} : r \in R\}$ and, for all $r \in R$ and all $\mathbf{a} \in h^{-1}(B')^{\rho(r)}$,

$$\begin{tikzcd}
h^{-1}(B')^n \arrow[rr, "h"] \arrow[dr, "r^{\mathfrak{B}'} \circ h"'] & & B'^n \arrow[dl, "r^{\mathfrak{B}'}"] \\
& G &
\end{tikzcd}$$

$$r^{h^{-1}(\mathfrak{B}')}(\mathbf{a}) = r^{\mathfrak{B}'}(h(\mathbf{a})).$$

Both $h(\mathfrak{A}')$ and $h^{-1}(\mathfrak{B}')$ are $\mathcal{L}$-structures. For $h(\mathfrak{A}')$ this holds by definition. For $h^{-1}(\mathfrak{B}')$, on the other hand, one must show that all operations in $F^{\mathfrak{A}}{\restriction}_{h^{-1}(B')}$ and all G-relations in $R^{h^{-1}(\mathfrak{B}')}$ are compatible with $E^{\mathfrak{A}}{\restriction}_{h^{-1}(B')}$. The proof for operations is a direct consequence of the compatibility of $F^{\mathfrak{A}}$ with $E^{\mathfrak{A}}$. For relations, let us assume that $r \in R$, with $\rho(r) = n$, and $a_1, b_1, \dots, a_n, b_n \in h^{-1}(B')$. Then we have

$$\begin{aligned}
&\textstyle\bigwedge_{i=1}^n E^{\mathfrak{A}}{\restriction}_{h^{-1}(B')}(a_i, b_i) \wedge r^{h^{-1}(\mathfrak{B}')}(\mathbf{a})\\
&\qquad = \textstyle\bigwedge_{i=1}^n E^{\mathfrak{A}}(a_i, b_i) \wedge r^{\mathfrak{B}'}(h(\mathbf{a}))\\
&\qquad\quad \text{(Definitions of } E^{\mathfrak{A}}{\restriction}_{h^{-1}(B')} \text{ and of } r^{h^{-1}(\mathfrak{B}')}\text{)}\\
&\qquad \leq \textstyle\bigwedge_{i=1}^n E^{\mathfrak{B}}(h(a_i), h(b_i)) \wedge r^{\mathfrak{B}'}(h(\mathbf{a}))\\
&\qquad\quad (h : \mathfrak{A} \to \mathfrak{B})\\
&\qquad = \textstyle\bigwedge_{i=1}^n E^{\mathfrak{B}'}(h(a_i), h(b_i)) \wedge r^{\mathfrak{B}'}(h(\mathbf{a}))\\
&\qquad\quad (a_1, b_1, \dots, a_n, b_n \in h^{-1}(B'))\\
&\qquad \leq r^{\mathfrak{B}'}(h(\mathbf{b}))\\
&\qquad\quad (r^{\mathfrak{B}'} \text{ compatible with } E^{\mathfrak{B}'})\\
&\qquad = r^{h^{-1}(\mathfrak{B}')}(\mathbf{b}). \quad \text{(Definition of } r^{h^{-1}(\mathfrak{B}')}(\mathbf{b})\text{)}
\end{aligned}$$

Furthermore, by definition, $h(\mathfrak{A}') \subseteq \mathfrak{B}$. However, in general, $h^{-1}(\mathfrak{B}')$ may not be a substructure of $\mathfrak{A}$. This is the case, however, if $h : \mathfrak{A} \to_s \mathfrak{B}$ is a strict morphism.

Lemma 125 *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$, $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ *be* $\mathcal{L}$*-structures and* $h : \mathfrak{A} \to_s \mathfrak{B}$*. If* $\mathfrak{B}' \subseteq \mathfrak{B}$*, then* $h^{-1}(\mathfrak{B}') \subseteq \mathfrak{A}$*.*

Proof: Let $\mathbf{a} \in h^{-1}(B')$. Then

$$\begin{array}{rcl}
r^{h^{-1}(\mathfrak{B}')}(\mathbf{a}) & = & r^{\mathfrak{B}'}(h(\mathbf{a})) \quad \text{(Definition of } r^{h^{-1}(\mathfrak{B}')}(\mathbf{a})\text{)}\\
& = & r^{\mathfrak{B}}(h(\mathbf{a})) \quad (\mathfrak{B}' \subseteq \mathfrak{B} \text{ and } h(\mathbf{a}) \in B')\\
& = & r^{\mathfrak{A}}(\mathbf{a}). \quad (h : \mathfrak{A} \to_s \mathfrak{B})
\end{array}$$

So $h^{-1}(\mathfrak{B}') \subseteq \mathfrak{A}$. ■

We detail, next, how any given surjective morphism from a structure $\mathfrak{A}$ onto a structure $\mathfrak{B}$ can be canonically decomposed through a reductive morphism. Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}},, R^{\mathfrak{A}} \rangle$, $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ be $\mathcal{L}$-structures and $h : \mathfrak{A} \to \mathfrak{B}$. The map $h : A \to B$ gives rise to a reductive morphism $\hat{h}$:

$h^{-1}(\mathfrak{B}) \twoheadrightarrow_s h(\mathfrak{A})$. Denoting by $i : \mathfrak{A} \rightarrowtail h^{-1}(\mathfrak{B})$ and by $j : h(\mathfrak{A}) \rightarrowtail \mathfrak{B}$ the identity mappings, we get the commutative diagram

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, "h"] \arrow[d, tail, "i"'] & \mathfrak{B} \\
h^{-1}(\mathfrak{B}) \arrow[r, two heads, "s" description, "\hat{h}"'] & h(\mathfrak{A}) \arrow[u, tail, "j"']
\end{tikzcd}$$

Thus, in case $h : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$, we obtain the decomposition

$$\begin{tikzcd}
\mathfrak{A} \arrow[rr, two heads, "h"] \arrow[dr, tail, "i"'] & & \mathfrak{B} \\
& h^{-1}(\mathfrak{B}) \arrow[ur, two heads, "s" description, "\hat{h}"'] &
\end{tikzcd}$$

This is quite important for what follows because, as Elgueta points out [23], strict morphisms are the appropriate ones by which to replace homomorphisms when attempting to lift universal algebraic results to languages that may contain relation symbols in addition to function symbols.

Let, again, $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ and $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ be $\mathcal{L}$-structures and $h : \mathfrak{A} \to \mathfrak{B}$ a morphism. h is called **elementary**, written $h : \mathfrak{A} \to_e \mathfrak{B}$, if, for all $\mathcal{L}$-formulas φ and all assignments $g : \mathbf{Tm}_{\mathcal{L}}(V) \to \mathbf{A}$,

$$\varphi^{\mathfrak{A}}[g] = \varphi^{\mathfrak{B}}[h \circ g].$$

We close the section by showing that every reductive morphism is, in fact, elementary. A weak converse will be proven later on.

Proposition 126 *Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ and $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ be $\mathcal{L}$-structures. If $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$, then $h : \mathfrak{A} \to_e \mathfrak{B}$.*

Proof: Suppose $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$. We use structural induction on φ to show that, for every φ and every $g : \mathbf{Tm}_{\mathcal{L}}(V) \to \mathbf{A}$,

$$\varphi^{\mathfrak{A}}[g] = \varphi^{\mathfrak{B}}[h \circ g].$$

For $\varphi = r(t_1, \ldots, t_n)$ atomic, we have

$$\begin{array}{rcll}
\varphi^{\mathfrak{A}}[g] & = & r^{\mathfrak{A}}(g(t_1), \ldots, g(t_n)) & \text{(Definition)} \\
& = & r^{\mathfrak{B}}(h(g(t_1)), \ldots, h(g(t_n))) & (h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}) \\
& = & \varphi^{\mathfrak{B}}[h \circ g]. & \text{(Definition)}
\end{array}$$

For negation, we have

$$\begin{array}{rcll}
(\neg\varphi)^{\mathfrak{A}}[g] & = & \neg\varphi^{\mathfrak{A}}[g] & \text{(Definition)} \\
& = & \neg\varphi^{\mathfrak{B}}[h \circ g] & \text{(Induction Hypothesis)} \\
& = & (\neg\varphi)^{\mathfrak{B}}[h \circ g]. & \text{(Definition)}
\end{array}$$

The cases of conjunction and disjunction can be handled similarly.

For existential quantification, we obtain

$$\begin{array}{rcl}(\exists x\varphi)^{\mathfrak{A}}[g] & = & \bigvee_{a\in A}\varphi^{\mathfrak{A}}[g(a/x)] \quad \text{(Definition)}\\ & = & \bigvee_{a\in A}\varphi^{\mathfrak{B}}[h\circ g(a/x)] \quad \text{(Induction Hypothesis)}\\ & = & \bigvee_{a\in A}\varphi^{\mathfrak{B}}[(h\circ g)(h(a)/x)]\\ & = & \bigvee_{b\in B}\varphi^{\mathfrak{B}}[(h\circ g)(b/x)] \quad (h\text{ Surjective})\\ & = & (\exists x\varphi)^{\mathfrak{B}}[h\circ g]. \quad \text{(Definition)}\end{array}$$

Universal quantification can be handled analogously. ■

4.5 Products

Let $\mathfrak{A}_i = \langle \mathbf{A}_i, E^{\mathfrak{A}_i}, R^{\mathfrak{A}_i}\rangle$, $i\in I$, be a family of $\mathcal{L}$-structures. The **direct product** of the $\mathfrak{A}_i$, $i\in I$, is the $\mathcal{L}$-structure defined by

$$\prod_{i\in I}\mathfrak{A}_i = \left\langle \prod_{i\in I}\mathbf{A}_i, E^{\prod\mathfrak{A}_i}, R^{\prod\mathfrak{A}_i}\right\rangle,$$

where:

- $\prod_{i\in I}\mathbf{A}_i$ is the ordinary direct product of the $\mathcal{L}$-algebras $\mathbf{A}_i$, $i\in I$;
- $E^{\prod\mathfrak{A}_i}$ is defined, for all $\mathbf{a},\mathbf{b}\in\prod_{i\in I}A_i$,
$$E^{\prod_i\mathfrak{A}_i}(\mathbf{a},\mathbf{b}) = \bigwedge_{i\in I}E^{\mathfrak{A}_i}(a_i,b_i);$$
- $R^{\prod\mathfrak{A}_i}$ is defined by $R^{\prod\mathfrak{A}_i} = \{r^{\prod\mathfrak{A}_i} : r\in R\}$, where, for each $r\in R$, with $\rho(r)=n$, and all $\mathbf{a}_1,\dots,\mathbf{a}_n\in\prod_{i\in I}A_i$,
$$r^{\prod_i\mathfrak{A}_i}(\mathbf{a}_1,\dots,\mathbf{a}_n) = \bigwedge_{i\in I}r^{\mathfrak{A}_i}(a_{1i},\dots,a_{ni}).$$

In this construction, I is allowed to be empty. In this case, we assume that $\prod_{i\in\varnothing}\mathbf{A}_i$ is the trivial one-element $\mathcal{L}$-algebra and all relations (including the reduced G-congruence) interpret the unique tuple in the domain as $\top$.

We show that this construction does indeed give a bona fide $\mathcal{L}$-structure. To verify this, we must show that all operations in $F^{\prod\mathfrak{A}_i}$ and all G-relations in $R^{\prod\mathfrak{A}_i}$ are compatible with $E^{\prod\mathfrak{A}_i}$. Let $f\in F$, with $\rho(f)=n$, and $\mathbf{a}_1,\mathbf{b}_1,\dots,\mathbf{a}_n,\mathbf{b}_n\in\prod_{i\in I}A_i$. Then, by definition, for all $i\in I$,

$$\bigwedge_{j=1}^n E^{\mathfrak{A}_i}(a_{ji},b_{ji}) \le E^{\mathfrak{A}_i}(f^{\mathfrak{A}_i}(a_{1i},\dots,a_{ni}), f^{\mathfrak{A}_i}(b_{1i},\dots,b_{ni})).$$

Thus,

$$\bigwedge_{i\in I}\bigwedge_{j=1}^n E^{\mathfrak{A}_i}(a_{ji},b_{ji}) \le \bigwedge_{i\in I}E^{\mathfrak{A}_i}(f^{\mathfrak{A}_i}(a_{1i},\dots,a_{ni}), f^{\mathfrak{A}_i}(b_{1i},\dots,b_{ni})).$$

This yields

$$\bigwedge_{j=1}^{n}\bigwedge_{i\in I}E^{\mathfrak{A}_i}(a_{ji},b_{ji}) \leq \bigwedge_{i\in I}E^{\mathfrak{A}_i}(f^{\mathfrak{A}_i}(a_{1i},\dots,a_{ni}),f^{\mathfrak{A}_i}(b_{1i},\dots,b_{ni})),$$

i.e.,

$$\bigwedge_{j=1}^{n}E^{\prod\mathfrak{A}_i}(\mathbf{a}_j,\mathbf{b}_j) \leq E^{\prod\mathfrak{A}_i}(f^{\prod\mathfrak{A}_i}(\mathbf{a}_1,\dots,\mathbf{a}_n),f^{\prod\mathfrak{A}_i}(\mathbf{b}_1,\dots,\mathbf{b}_n)).$$

This proves that $F^{\prod\mathfrak{A}_i}$ is compatible with $E^{\prod\mathfrak{A}_i}$. Suppose, next, that $r \in R$, with $\rho(r) = n$, and $\mathbf{a}_1,\mathbf{b}_1,\dots,\mathbf{a}_n,\mathbf{b}_n \in \prod_{i\in I}A_i$. We get

$$\begin{array}{l}\bigwedge_{j=1}^{n}E^{\prod\mathfrak{A}_i}(\mathbf{a}_j,\mathbf{b}_j) \wedge r^{\prod\mathfrak{A}_i}(\mathbf{a}_1,\dots,\mathbf{a}_n)\\ \quad = \bigwedge_{j=1}^{n}\bigwedge_{i\in I}E^{\mathfrak{A}_i}(a_{ji},b_{ji}) \wedge \bigwedge_{i\in I}r^{\mathfrak{A}_i}(a_{1i},\dots,a_{ni})\\ \quad = \bigwedge_{i\in I}\bigwedge_{j=1}^{n}E^{\mathfrak{A}_i}(a_{ji},b_{ji}) \wedge \bigwedge_{i\in I}r^{\mathfrak{A}_i}(a_{1i},\dots,a_{ni})\\ \quad = \bigwedge_{i\in I}\left[\bigwedge_{j=1}^{n}E^{\mathfrak{A}_i}(a_{ji},b_{ji}) \wedge r^{\mathfrak{A}_i}(a_{1i},\dots,a_{ni})\right]\\ \quad \leq \bigwedge_{i\in I}r^{\mathfrak{A}_i}(b_{1i},\dots,b_{ni})\\ \quad = r^{\prod\mathfrak{A}_i}(\mathbf{b}_1,\dots,\mathbf{b}_n).\end{array}$$

This proves that $R^{\prod\mathfrak{A}_i}$ is also compatible with $E^{\prod\mathfrak{A}_i}$. Thus $\prod_{i\in I}\mathfrak{A}_i$ is an $\mathcal{L}$-structure.

4.6 G-Congruences

Consider a set A. Recall that a mapping $\Theta : A^2 \to G$ is a G-equivalence on A if, for all $a,b,c \in A$,

(**Reflexivity**) $\Theta(a,a) = \top$;

(**Symmetry**) $\Theta(a,b) = \Theta(b,a)$;

(**Transitivity**) $\Theta(a,b) \wedge \Theta(b,c) \leq \Theta(a,c)$.

Suppose that $\mathbf{A} = \langle A, F^{\mathbf{A}}\rangle$ is an F-algebra. Recall that $\Theta : A^2 \to G$ is G-congruence on $\mathbf{A}$ if Θ is a G-equivalence on A and, in addition, it satisfies, for all f in F of arity n and all $a_1,\dots,a_n$ and $b_1,\dots,b_n$ in A,

$$\bigwedge_{i=1}^{n}\Theta(a_i,b_i) \leq \Theta(f^{\mathbf{A}}(a_1,\dots,a_n),f^{\mathbf{A}}(b_1,\dots,b_n)).$$

Let $\mathrm{Gon}(\mathbf{A})$ denote the collection of all G-congruences on $\mathbf{A}$. This set is naturally ordered by

$$\Theta \leq \Theta' \quad \text{iff} \quad \Theta(a,b) \leq \Theta'(a,b), \text{ for all } a,b \in A.$$

Lemma 127 *Let* $\mathbf{A} = \langle A, F^{\mathbf{A}}\rangle$ *be an* F*-algebra. Then* $\mathbf{Gon(A)} = \langle \mathrm{Gon}(\mathbf{A}), \leq\rangle$ *is a complete lattice.*

Proof: Clearly, the function $\top : A^2 \to G$, with $\top(a,b) = \top$, for all $a,b \in A$, is a G-congruence on $\mathbf{A}$. Next, let Θ_i, $i \in I$, be a nonempty collection of G-congruences on $\mathbf{A}$. Then, for all $a,b,c \in A$,

- $\bigwedge_i \Theta_i(a,a) = \bigwedge_i \top = \top$;
- $\bigwedge_i \Theta_i(a,b) = \bigwedge_i \Theta_i(b,a)$;
- $\bigwedge_i \Theta_i(a,b) \wedge \bigwedge_i \Theta_i(b,c) = \bigwedge_i(\Theta_i(a,b) \wedge \Theta_i(b,c)) \leq \bigwedge_i \Theta_i(a,c)$.

Moreover, for all n-ary $f \in F$ and all $a_1,b_1,\dots,a_n,b_n \in A$,

$$\begin{array}{rcl}\bigwedge_{j=1}^n \bigwedge_i \Theta_i(a_j,b_j) & = & \bigwedge_i \bigwedge_{j=1}^n \Theta_i(a_j,b_j) \\ & \leq & \bigwedge_i \Theta_i(f^{\mathbf{A}}(a_1,\dots,a_n), f^{\mathbf{A}}(b_1,\dots,b_n)).\end{array}$$

Thus, $\mathbf{Gon(A)}$ forms a complete lattice under $\leq$. ■

Let $\mathcal{A} = \langle \mathbf{A}, E^{\mathcal{A}}\rangle$ be a G-algebra. A G-congruence $\Theta \in \mathrm{Gon}(\mathbf{A})$ is a **G-congruence on $\mathcal{A}$** if

$$E^{\mathcal{A}} \leq \Theta.$$

We denote by $\mathrm{Gon}(\mathcal{A})$ the collection of all G-congruences on $\mathcal{A}$. Clearly, $\mathrm{Gon}(\mathcal{A}) \subseteq \mathrm{Gon}(\mathbf{A})$ and, hence, $\mathrm{Gon}(\mathcal{A})$ inherits the order $\leq$ from $\mathbf{Gon(A)}$.

Lemma 128 *Let $\mathcal{A} = \langle \mathbf{A}, E^{\mathcal{A}}\rangle$ be a G-algebra. Then $\mathbf{Gon}(\mathcal{A}) = \langle \mathrm{Gon}(\mathcal{A}), \leq\rangle$ is a principal filter of $\mathbf{Gon(A)}$, and, hence, a complete lattice.*

Proof: This is clear from definition, since $\Theta \in \mathrm{Gon}(\mathcal{A})$ if and only if $\Theta \in \mathrm{Gon}(\mathbf{A})$ and $E^{\mathcal{A}} \leq \Theta$. ■

Now, given a set A, a mapping $\Theta : A^2 \to G$ and a G-relation $r : A^n \to G$, we say that Θ is **compatible with** r if, for all $a_1,b_1,\dots,a_n,b_n \in A$, we have

$$\bigwedge_{i=1}^n \Theta(a_i,b_i) \leq r(a_1,\dots,a_n) \leftrightarrow r(b_1,\dots,b_n).$$

Note that, if Θ is symmetric, this is equivalent to saying that the G-relation r is compatible with Θ, according to previously introduced terminology.

Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}}\rangle$ be an $\mathcal{L}$-structure. We say that $\Theta : A^2 \to G$ is a **G-congruence on $\mathfrak{A}$** if:

- Θ is a G-congruence on $\mathcal{A} = \langle \mathbf{A}, E^{\mathfrak{A}}\rangle$;
- Θ is compatible with $r^{\mathfrak{A}}$, for all $r \in R$.

Denote by $\mathrm{Gon}(\mathfrak{A})$ the collection of all G-congruences on $\mathfrak{A}$. Clearly, we have $\mathrm{Gon}(\mathfrak{A}) \subseteq \mathrm{Gon}(\mathcal{A})$ and, furthermore, $\mathrm{Gon}(\mathfrak{A})$ inherits from $\mathbf{Gon}(\mathcal{A})$ the ordering $\leq$.

Since, for a given structure $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}}\rangle$, $E^{\mathfrak{A}} \in \mathrm{Gon}(\mathfrak{A})$ by definition, we get that, for every structure $\mathfrak{A}$, the largest G-congruence on $\mathfrak{A}$ exists (under sufficient conditions on the complete lattice $\mathbf{G}$, of course). It will be called the **Leibniz G-congruence** of $\mathfrak{A}$ and denoted by $\Omega(\mathfrak{A})$. So we have

$$\mathrm{Gon}(\mathfrak{A}) = \{\Theta \in \mathrm{Gon}(\mathcal{A}) : \Theta \leq \Omega(\mathfrak{A})\}.$$

We have the following observations that follow directly from the definitions.

Lemma 129 *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ *be an* $\mathcal{L}$*-structure. Suppose there exists an* n*-ary relation symbol* $r \in R$*, such that, for some* $a_1, b_1, \dots, a_n, b_n \in A$*,*

$$r^{\mathfrak{A}}(a_1, \dots, a_n) \nleftrightarrow r^{\mathfrak{A}}(b_1, \dots, b_n).$$

Then $\Omega(\mathfrak{A}) \neq \top$*.*

Proof: Suppose, towards a contradiction, that $\Omega(\mathfrak{A}) = \top$ and let $r \in R$ and $a_1, b_1, \dots, a_n, b_n \in A$. Then, we have

$$\top = \bigwedge_{i=1}^{n} \Omega(\mathfrak{A})(a_i, b_i) \leq r^{\mathfrak{A}}(a_1, \dots, a_n) \leftrightarrow r^{\mathfrak{A}}(b_1, \dots, b_n).$$

This contradicts the hypothesis. ■

If the signature $\mathcal{L}$ includes a binary relation symbol r whose interpretation $r^{\mathfrak{A}}$ in a structure $\mathfrak{A}$ happens to be a G-congruence on $\mathfrak{A}$, then $r^{\mathfrak{A}}$ is necessarily the Leibniz G-congruence on $\mathfrak{A}$.

Lemma 130 *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ *be an* $\mathcal{L}$*-structure. If* $\mathcal{L}$ *has a binary relation symbol* r*, such that* $r^{\mathfrak{A}} \in \mathrm{Gon}(\mathfrak{A})$*, then* $r^{\mathfrak{A}} = \Omega(\mathfrak{A})$*.*

Proof: By hypothesis and the maximality property of $\Omega(\mathfrak{A})$, we have $r^{\mathfrak{A}} \leq \Omega(\mathfrak{A})$. On the other hand, for all $a, b \in A$,

$$\begin{array}{rcl}
\Omega(\mathfrak{A})(a,b) & = & \Omega(\mathfrak{A})(a,b) \wedge \Omega(\mathfrak{A})(b,b) \quad (\Omega(\mathfrak{A})(b,b) = \top) \\
& \leq & r^{\mathfrak{A}}(a,b) \leftrightarrow r^{\mathfrak{A}}(b,b) \quad (\Omega(\mathfrak{A}) \in \mathrm{Gon}(\mathfrak{A})) \\
& = & r^{\mathfrak{A}}(a,b). \quad (r^{\mathfrak{A}}(b,b) = \top)
\end{array}$$

Since a, b were arbitrary, $\Omega(\mathfrak{A}) \leq r^{\mathfrak{A}}$. Combining, we get the conclusion. ■

We provide, next, a series of lemmas relating G-congruences with strong homomorphisms, substructures and strong homomorphic images.

Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$, $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ be $\mathcal{L}$-structures and $h : \mathbf{A} \to \mathbf{B}$. Define the **$G$-kernel** $\mathrm{Ker}(h) : A^2 \to G$ **of** h by

$$\mathrm{Ker}(h) = E^{\mathfrak{B}} \circ h,$$

$$\begin{tikzcd}
A^2 \arrow[rr, "h^2"] \arrow[dr, "\mathrm{Ker}(h)"'] & & B^2 \arrow[dl, "E^{\mathfrak{B}}"] \\
& G &
\end{tikzcd}$$

that is, for all $a, b \in A$,

$$\mathrm{Ker}(h)(a,b) = E^{\mathfrak{B}}(h(a), h(b)).$$

Lemma 131 *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$, $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ *be* $\mathcal{L}$*-structures and* $h : \mathbf{A} \to \mathbf{B}$*. If* $h : \mathfrak{A} \to_s \mathfrak{B}$*, then* $\mathrm{Ker}(h) \in \mathrm{Gon}(\mathfrak{A})$*.*

Proof: First, we show that $\mathrm{Ker}(h)$ is a G-equivalence on A. Let $a, b, c \in A$.

- For Reflexivity,

$$\begin{array}{rcl} \mathrm{Ker}(h)(a,a) & = & E^{\mathfrak{B}}(h(a), h(a)) \quad (\text{Definition of } \mathrm{Ker}(h)) \\ & = & \top; \quad (E^{\mathfrak{B}} \ G\text{-Congruence}) \end{array}$$

- For Symmetry,

$$\begin{array}{rcl} \mathrm{Ker}(h)(b,a) & = & E^{\mathfrak{B}}(h(b), h(a)) \quad (\text{Definition of } \mathrm{Ker}(h)) \\ & = & E^{\mathfrak{B}}(h(a), h(b)) \quad (E^{\mathfrak{B}} \ G\text{-Congruence}) \\ & = & \mathrm{Ker}(h)(a,b); \quad (\text{Definition of } \mathrm{Ker}(h)) \end{array}$$

- For Transitivity,

$$\begin{array}{rcl} \mathrm{Ker}(h)(a,b) \wedge \mathrm{Ker}(h)(b,c) & = & E^{\mathfrak{B}}(h(a), h(b)) \wedge E^{\mathfrak{B}}(h(b), h(c)) \\ & & \quad (\text{Definition of } \mathrm{Ker}(h)) \\ & \leq & E^{\mathfrak{B}}(h(a), h(c)) \\ & & \quad (E^{\mathfrak{B}} \ G\text{-Congruence}) \\ & = & \mathrm{Ker}(h)(a,c). \\ & & \quad (\text{Definition of } \mathrm{Ker}(h)) \end{array}$$

Next, we show that $\mathrm{Ker}(h)$ is compatible with all operations in $F^{\mathfrak{A}}$. Let $f \in F$, with $\rho(f) = n$, and $a_1, b_1, \ldots, a_n, b_n \in A$.

$$\begin{array}{rcl} \bigwedge_{i=1}^n \mathrm{Ker}(h)(a_i, b_i) & = & \bigwedge_{i=1}^n E^{\mathfrak{B}}(h(a_i), h(b_i)) \\ & & \quad (\text{Definition of } \mathrm{Ker}(h)) \\ & \leq & E^{\mathfrak{B}}(f^{\mathfrak{B}}(h(\mathbf{a})), f^{\mathfrak{B}}(h(\mathbf{b}))) \\ & & \quad (E^{\mathfrak{B}} \ G\text{-Congruence}) \\ & = & E^{\mathfrak{B}}(h(f^{\mathfrak{A}}(\mathbf{a})), h(f^{\mathfrak{A}}(\mathbf{b}))) \\ & & \quad (h : \mathbf{A} \to \mathbf{B}) \\ & = & \mathrm{Ker}(h)(f^{\mathfrak{A}}(\mathbf{a}), f^{\mathfrak{A}}(\mathbf{b})). \\ & & \quad (\text{Definition of } \mathrm{Ker}(h)) \end{array}$$

By definition of a morphism of structures, $E^{\mathfrak{A}} \leq E^{\mathfrak{B}} \circ h = \mathrm{Ker}(h)$.

Finally, we show that $\mathrm{Ker}(h)$ is compatible with all G-relations in $R^{\mathfrak{A}}$. Note that this is the only part where strictness is needed. We have, for all $r \in R$, with $\rho(r) = n$, and all $a_1, b_1, \ldots, a_n, b_n \in A$,

$$\begin{array}{rcl}\bigwedge_{i=1}^n \mathrm{Ker}(h)(a_i, b_i) \wedge r^{\mathfrak{A}}(\mathbf{a}) & = & \bigwedge_{i=1}^n E^{\mathfrak{B}}(h(a_i), h(b_i)) \wedge r^{\mathfrak{A}}(\mathbf{a}) \\ & & \quad (\text{Definition of } \mathrm{Ker}(h)) \\ & = & \bigwedge_{i=1}^n E^{\mathfrak{B}}(h(a_i), h(b_i)) \wedge r^{\mathfrak{B}}(h(\mathbf{a})) \\ & & \quad (h : \mathfrak{A} \to_s \mathfrak{B}) \\ & \leq & r^{\mathfrak{B}}(h(\mathbf{b})) \quad (E^{\mathfrak{B}}\ G\text{-Congruence}) \\ & = & r^{\mathfrak{A}}(\mathbf{b}). \quad (h : \mathfrak{A} \to_s \mathfrak{B})\end{array}$$

We conclude that $\mathrm{Ker}(h) \in \mathrm{Gon}(\mathfrak{A})$. ■

Lemma 132 *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$, $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ *be* $\mathcal{L}$*-structures, such that* $\mathfrak{B} \subseteq \mathfrak{A}$. *Given* $\Theta : A^2 \to G$, *define* $\Theta_B = \Theta\restriction_{B^2}$. *If* $\Theta \in \mathrm{Gon}(\mathfrak{A})$, *then* $\Theta_B \in \mathrm{Gon}(\mathfrak{B})$.

Proof: It follows almost immediately from the definition of Θ_B and the fact that Θ is a G-congruence on $\mathbf{A}$ that Θ_B is a G-congruence on $\mathbf{B}$. Further, for all $b_1, b_2 \in B$, we have

$$\begin{array}{rcl}E^{\mathfrak{B}}(b_1, b_2) & = & E^{\mathfrak{A}}(b_1, b_2) \quad (\mathfrak{B} \subseteq \mathfrak{A}) \\ & \leq & \Theta(b_1, b_2) \quad (\Theta \in \mathrm{Gon}(\mathfrak{A})) \\ & = & \Theta_B(b_1, b_2). \quad (\text{Definition of } \Theta_B)\end{array}$$

Finally, for all $r \in R$, with $\rho(r) = n$, and all $b_1, b_1', \ldots, b_n, b_n' \in B$, we have

$$\begin{array}{rcl}\bigwedge_{i=1}^n \Theta_B(b_i, b_i') \wedge r^{\mathfrak{B}}(\mathbf{b}) & = & \bigwedge_{i=1}^n \Theta(b_i, b_i') \wedge r^{\mathfrak{B}}(\mathbf{b}) \quad (\Theta_B = \Theta\restriction_{B^2}) \\ & = & \bigwedge_{i=1}^n \Theta(b_i, b_i') \wedge r^{\mathfrak{A}}(\mathbf{b}) \quad (\mathfrak{B} \subseteq \mathfrak{A}) \\ & \leq & r^{\mathfrak{A}}(\mathbf{b}') \quad (\Theta \in \mathrm{Gon}(\mathfrak{A})) \\ & = & r^{\mathfrak{B}}(\mathbf{b}'). \quad (\mathfrak{B} \subseteq \mathfrak{A})\end{array}$$

Thus, $\Theta_B \in \mathrm{Gon}(\mathfrak{B})$. ■

Lemma 133 *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$, $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ *be* $\mathcal{L}$*-structures and* $h : \mathfrak{A} \to_s \mathfrak{B}$.

(a) *If* $\Theta \in \mathrm{Gon}(\mathfrak{B})$, *then* $\Theta \circ h \in \mathrm{Gon}(\mathfrak{A})$.

(b) *If* $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$ *and* $\Theta \in \mathrm{Gon}(\mathfrak{A})$, *such that* $\mathrm{Ker}(h)$ *is compatible with* Θ, *then, there exists* $\Theta' \in \mathrm{Gon}(\mathfrak{B})$, *such that* $\Theta = \Theta' \circ h$.

Proof:

(a) First, we must show that, if Θ is a G-congruence on $\mathbf{B}$, then $\Theta \circ h$ is a G-congruence on $\mathbf{A}$. Let us show Transitivity and Congruence. Suppose $a_1, a_2, a_3 \in A$. We have

$$\begin{aligned}
&(\Theta \circ h)(a_1, a_2) \wedge (\Theta \circ h)(a_2, a_3)\\
&\qquad = \Theta(h(a_1), h(a_2)) \wedge \Theta(h(a_2), h(a_3)) \quad \text{(Composition)}\\
&\qquad \leq \Theta(h(a_1), h(a_3)) \quad (\Theta \in \mathrm{Gon}(\mathfrak{B}))\\
&\qquad = (\Theta \circ h)(a_1, a_3). \quad \text{(Composition)}
\end{aligned}$$

For Congruence, suppose $f \in F$, with $\rho(f) = n$, and let $a_1, \dots, a_n$, $a'_1, \dots, a'_n \in A$. We have

$$\begin{aligned}
&\textstyle\bigwedge_{i=1}^n (\Theta \circ h)(a_i, a'_i)\\
&\qquad = \textstyle\bigwedge_{i=1}^n \Theta(h(a_i), h(a'_i)) \quad \text{(Composition)}\\
&\qquad \leq \Theta(f^{\mathbf{B}}(h(\mathbf{a})), f^{\mathbf{B}}(h(\mathbf{a}'))) \quad (\Theta \in \mathrm{Gon}(\mathfrak{B}))\\
&\qquad = \Theta(h(f^{\mathbf{A}}(\mathbf{a})), h(f^{\mathbf{A}}(\mathbf{a}'))) \quad (h : \mathbf{A} \to \mathbf{B})\\
&\qquad = (\Theta \circ h)(f^{\mathbf{A}}(\mathbf{a}), f^{\mathbf{A}}(\mathbf{a}')). \quad \text{(Composition)}
\end{aligned}$$

Further, for all $a_1, a_2 \in A$,

$$\begin{array}{rcll}
E^{\mathfrak{A}}(a_1, a_2) & \leq & E^{\mathfrak{B}}(h(a_1), h(a_2)) & (h : \mathfrak{A} \to \mathfrak{B})\\
& \leq & \Theta(h(a_1), h(a_2)) & (\Theta \in \mathrm{Gon}(\mathfrak{B}))\\
& = & (\Theta \circ h)(a_1, a_2). & \text{(Composition)}
\end{array}$$

It remains to show that, if Θ is compatible with $R^{\mathfrak{B}}$, then $\Theta \circ h$ is compatible with $R^{\mathfrak{A}}$. Suppose $r \in R$, with $\rho(r) = n$, and let $a_1, a'_1, \dots, a_n$, $a'_n \in A$. We have

$$\begin{aligned}
&\textstyle\bigwedge_{i=1}^n (\Theta \circ h)(a_i, a'_i) \wedge r^{\mathfrak{A}}(\mathbf{a})\\
&\qquad = \textstyle\bigwedge_{i=1}^n \Theta(h(a_i), h(a'_i)) \wedge r^{\mathfrak{A}}(\mathbf{a}) \quad \text{(Composition)}\\
&\qquad = \textstyle\bigwedge_{i=1}^n \Theta(h(a_i), h(a'_i)) \wedge r^{\mathfrak{B}}(h(\mathbf{a})) \quad (h : \mathfrak{A} \to_s \mathfrak{B})\\
&\qquad \leq r^{\mathfrak{B}}(h(\mathbf{a}')) \quad (\Theta \in \mathrm{Gon}(\mathfrak{B}))\\
&\qquad = r^{\mathfrak{A}}(\mathbf{a}') \quad (h : \mathfrak{A} \to_s \mathfrak{B})
\end{aligned}$$

Thus, if $\Theta \in \mathrm{Gon}(\mathfrak{B})$, then $\Theta \circ h \in \mathrm{Gon}(\mathfrak{A})$.

(b) The condition that Θ' needs to satisfy compels the definition of Θ'. Suppose $b_1, b_2 \in B$. Since h is reductive, there exist $a_1, a_2 \in A$, such that $b_1 = h(a_1)$ and $b_2 = h(a_2)$. We define

$$\Theta'(b_1, b_2) = \Theta(a_1, a_2).$$

Of course, one has to show that Θ' is well defined. So suppose, a_1, a'_1, $a_2, a'_2 \in A$, such that $h(a_1) = h(a'_1) = b_1$ and $h(a_2) = h(a'_2) = b_2$. Then $E^{\mathfrak{B}}(h(a_1), h(a'_1)) = \top$ and $E^{\mathfrak{B}}(h(a_2), h(a'_2)) = \top$. By the definition of $\mathrm{Ker}(h)$, $\mathrm{Ker}(h)(a_1, a'_1) = \top$ and $\mathrm{Ker}(h)(a_2, a'_2) = \top$. Thus, since, by hypothesis, $\mathrm{Ker}(h) \le \Theta$, $\Theta(a_1, a_1') = \Theta(a_2, a_2') = \top$. Thus, using Transitivity,

$$\begin{aligned}\Theta(a_1,a_2) &= \top \wedge \Theta(a_1,a_2) \wedge \top\\ &= \Theta(a_1',a_1) \wedge \Theta(a_1,a_2) \wedge \Theta(a_2,a_2')\\ &\le \Theta(a_1',a_2')\end{aligned}$$

and, by symmetry, $\Theta(a_1, a_2) = \Theta(a_1', a_2')$. Hence, Θ' is well-defined.

Since the definition yields $\Theta = \Theta' \circ h$, it suffices to show that Θ' is a congruence on $\mathfrak{B}$. We show Transitivity, Congruence, Inclusion of $E^{\mathfrak{B}}$ and Compatibility. Reflexivity and Symmetry can be proven similarly.

Let $b_1, b_2, b_3 \in B$. Then there exist $a_1, a_2, a_3 \in A$, such that $b_i = h(a_i)$, $i = 1, 2, 3$. So we have

$$\begin{aligned}\Theta'(b_1,b_2) \wedge \Theta'(b_2,b_3) &= \Theta(a_1,a_2) \wedge \Theta(a_2,a_3) \quad (\text{Definition of } \Theta')\\ &\le \Theta(a_1,a_3) \quad (\Theta \in \mathrm{Gon}(\mathbf{A}))\\ &= \Theta'(b_1,b_3). \quad (\text{Definition of } \Theta')\end{aligned}$$

Next, suppose $f \in F$, with $\rho(f) = n$, and let $b_1, b_1', \ldots, b_n, b_n' \in B$. Consider $a_1, a_1', \ldots, a_n, a_n' \in A$, such that $\mathbf{b} = h(\mathbf{a})$ and $\mathbf{b}' = h(\mathbf{a}')$. Note that $f^{\mathbf{B}}(\mathbf{b}) = f^{\mathbf{B}}(h(\mathbf{a})) = h(f^{\mathbf{A}}(\mathbf{a}))$ and, similarly, $f^{\mathbf{B}}(\mathbf{b}') = h(f^{\mathbf{A}}(\mathbf{a}'))$. Then we have

$$\begin{aligned}\textstyle\bigwedge_{i=1}^n \Theta'(b_i,b_i') &= \textstyle\bigwedge_{i=1}^n \Theta(a_i,a_i') \quad (\text{Definition of } \Theta')\\ &\le \Theta(f^{\mathbf{A}}(\mathbf{a}), f^{\mathbf{A}}(\mathbf{a}')) \quad (\Theta \in \mathrm{Gon}(\mathbf{A}))\\ &= \Theta'(f^{\mathbf{B}}(\mathbf{b}), f^{\mathbf{B}}(\mathbf{b}')). \quad (\text{Definition of } \Theta')\end{aligned}$$

Next, for $b_1, b_2 \in B$, such that $b_1 = h(a_1)$ and $b_2 = h(a_2)$, for some $a_1, a_2 \in A$, we get

$$\begin{aligned}E^{\mathfrak{B}}(b_1,b_2) &= E^{\mathfrak{B}}(h(a_1),h(a_2)) \quad (b_i = h(a_i))\\ &\le \Theta(a_1,a_2) \quad (\mathrm{Ker}(h) \le \Theta)\\ &= \Theta'(b_1,b_2). \quad (\text{Definition of } \Theta')\end{aligned}$$

Finally, suppose $r \in R$, with $\rho(r) = n$, and let $b_1, b_1', \ldots, b_n, b_n' \in B$. Consider $a_1, a_1', \ldots, a_n, a_n' \in A$, such that $\mathbf{b} = h(\mathbf{a})$ and $\mathbf{b}' = h(\mathbf{a}')$. Note that, since $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$, $r^{\mathfrak{B}}(\mathbf{b}) = r^{\mathfrak{B}}(h(\mathbf{a})) = r^{\mathfrak{A}}(\mathbf{a})$ and, similarly, $r^{\mathfrak{B}}(\mathbf{b}') = r^{\mathfrak{A}}(\mathbf{a}')$. Thus, we have

$$\begin{aligned}\textstyle\bigwedge_{i=1}^n \Theta'(b_i,b_i') \wedge r^{\mathfrak{B}}(\mathbf{b}) &= \textstyle\bigwedge_{i=1}^n \Theta(a_i,a_i') \wedge r^{\mathfrak{A}}(\mathbf{a})\\ &\qquad (\text{Definition of } \Theta' \text{ and } h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B})\\ &\le r^{\mathfrak{A}}(\mathbf{a}') \quad (\Theta \in \mathrm{Gon}(\mathfrak{A}))\\ &= r^{\mathfrak{B}}(\mathbf{b}'). \quad (h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B})\end{aligned}$$

Hence, $\Theta' \in \mathrm{Gon}(\mathfrak{B})$.

■

We can also show that Leibniz G-congruences commute with inverse reductive morphisms. This property abstracts a similar property in the setting of ordinary first order structures that was presented in Theorem 2.5 of [23].

Corollary 134 *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ *and* $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ *be* $\mathcal{L}$*-structures and* $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$*. Then*

$$\Omega(\mathfrak{A}) = \Omega(\mathfrak{B}) \circ h.$$

Proof: Let us set $\Theta := \Omega(\mathfrak{B}) \circ h$. By Part (a) of Lemma 133, $\Theta \in \mathrm{Gon}(\mathfrak{A})$. Hence, by the maximality property of $\Omega(\mathfrak{A})$, we get $\Omega(\mathfrak{B}) \circ h = \Theta \le \Omega(\mathfrak{A})$.

Assume, conversely, taking into account Lemma 131, that $\Theta' \in \mathrm{Gon}(\mathfrak{B})$ is the G-congruence obtained by applying Part (b) of Lemma 133 to the G-congruence $\Omega(\mathfrak{A}) \in \mathrm{Gon}(\mathfrak{A})$. Then, by the lemma and the maximality of $\Omega(\mathfrak{B})$, we have

$$\Theta' \circ h = \Omega(\mathfrak{A}) \quad \text{and} \quad \Theta' \le \Omega(\mathfrak{B}).$$

Hence, $\Omega(\mathfrak{A}) = \Theta' \circ h \le \Omega(\mathfrak{B}) \circ h$. We conclude that $\Omega(\mathfrak{B}) \circ h = \Omega(\mathfrak{A})$. ■

4.7 Leibniz Equality

A **Leibniz formula over** $\mathcal{L}$ is a formula $\psi(x,y)$, with two free variables, that has the form

$$\psi(x,y) := \forall \bar{z}(\varphi(x,\bar{z}) \leftrightarrow \varphi(y,\bar{z})),$$

for some atomic $\mathcal{L}$-formula $\varphi(x,\bar{z})$, with at least one free variable x.

Using Leibniz formulas, we may obtain characterizations of the Leibniz G-congruence on a given $\mathcal{L}$-structure.

Theorem 135 *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ *be an* $\mathcal{L}$*-structure. Then, for all* $a, b \in A$*, we have*

$$\Omega(\mathfrak{A})(a,b) = \bigwedge\{\psi^{\mathfrak{A}}(a,b) : \psi \text{ Leibniz}\}.$$

Proof: Let us define

$$\Theta := \bigwedge\{\psi^{\mathfrak{A}}(a,b) : \psi \text{ Leibniz}\}.$$

Suppose, first, that $a, b \in A$ and let $\varphi(x,\bar{z})$ be an atomic formula and $\mathbf{c}$ in A. Then, using compatibility of $\Omega(\mathfrak{A})$ with $R^{\mathfrak{A}}$, we get

$$\Omega(\mathfrak{A})(a,b) \le \varphi^{\mathfrak{A}}(a,\mathbf{c}) \leftrightarrow \varphi^{\mathfrak{A}}(b,\mathbf{c}).$$

Thus, for every Leibniz formula $\psi(x,y)$,

$$\Omega(\mathfrak{A})(a,b) \le \psi^{\mathfrak{A}}(a,b).$$

This yields

$$\Omega(\mathfrak{A})(a,b) \le \bigwedge\{\psi^{\mathfrak{A}}(a,b) : \psi \text{ Leibniz}\},$$

i.e., $\Omega(\mathfrak{A}) \le \Theta$.

For the reverse inequality, by the maximality property of $\Omega(\mathfrak{A})$, it suffices to show that Θ is a G-congruence on $\mathfrak{A}$. As done previously, we show Transitivity, Congruence, Inclusion of $E^{\mathfrak{A}}$ and Compatibility.

Let $a,b,c \in A$. Then

$$\begin{array}{rcl}\Theta(a,b) \wedge \Theta(b,c) & = & \bigwedge_\psi \psi^{\mathfrak{A}}(a,b) \wedge \bigwedge_\psi \psi^{\mathfrak{A}}(b,c) \quad \text{(Definition of } \Theta) \\ & = & \bigwedge_\psi (\psi^{\mathfrak{A}}(a,b) \wedge \psi^{\mathfrak{A}}(b,c)) \quad \text{(Property of meet)} \\ & \le & \bigwedge_\psi \psi^{\mathfrak{A}}(a,c) \quad \text{(Property of } \leftrightarrow) \\ & = & \Theta(a,c). \quad \text{(Definition of } \Theta)\end{array}$$

Next, suppose $f \in F$, with $\rho(f) = n$, and $a_1,b_1,\ldots,a_n,b_n \in A$. Then we get

$$\begin{array}{rcl}\bigwedge_{i=1}^n \Theta(a_i,b_i) & = & \bigwedge_{i=1}^n \bigwedge_\psi \psi^{\mathfrak{A}}(a_i,b_i) \quad \text{(Definition of } \Theta) \\ & \le & \bigwedge_\psi \bigwedge_{i=1}^n \psi^{\mathfrak{A}}(f^{\mathbf{A}}(b_1,\ldots,b_{i-1},a_i,a_{i+1},\ldots,a_n), \\ & & \qquad f^{\mathbf{A}}(b_1,\ldots,b_{i-1},b_i,a_{i+1},\ldots,a_n)) \\ & & \text{(Meets over special formulas)} \\ & \le & \bigwedge_\psi \psi^{\mathfrak{A}}(f^{\mathbf{A}}(\mathbf{a}),f^{\mathbf{A}}(\mathbf{b})) \quad \text{(Property of } \leftrightarrow) \\ & = & \Theta(f^{\mathbf{A}}(\mathbf{a}),f^{\mathbf{A}}(\mathbf{b})). \quad \text{(Definition of } \Theta)\end{array}$$

Next, recall that all relations of $\mathfrak{A}$ are compatible with the reduced G-congruence $E^{\mathfrak{A}}$. Thus, for every atomic $\varphi(x,\bar{z})$ and all $a,b,\mathbf{c} \in A$,

$$E^{\mathfrak{A}}(a,b) \le \varphi^{\mathfrak{A}}(a,\mathbf{c}) \leftrightarrow \varphi^{\mathfrak{A}}(b,\mathbf{c}).$$

This yields, that, for every Leibniz $\mathcal{L}$-formula ψ, $E^{\mathfrak{A}}(a,b) \le \psi^{\mathfrak{A}}(a,b)$ and, therefore, $E^{\mathfrak{A}} \le \Theta$.

Finally, suppose $r \in R$, with $\rho(r) = n$, and $a_1,b_1,\ldots,a_n,b_n \in A$. Then we get

$$\begin{array}{rcl}\bigwedge_{i=1}^n \Theta(a_i,b_i) & = & \bigwedge_{i=1}^n \bigwedge_\psi \psi^{\mathfrak{A}}(a_i,b_i) \quad \text{(Definition of } \Theta) \\ & \le & \bigwedge_{i=1}^n r^{\mathfrak{A}}(b_1,\ldots,b_{i-1},a_i,a_{i+1},\ldots,a_n) \\ & & \qquad \leftrightarrow r^{\mathfrak{A}}(b_1,\ldots,b_{i-1},b_i,a_{i+1},\ldots,a_n) \\ & & \text{(Meets over special formulas)} \\ & \le & r^{\mathfrak{A}}(\mathbf{a}) \leftrightarrow r^{\mathfrak{A}}(\mathbf{b}). \quad \text{(Property of } \leftrightarrow)\end{array}$$

We conclude that $\Theta \le \Omega(\mathfrak{A})$. Now the characterization of the statement follows. ■

Applying induction on the structure of formulas, we obtain the following consequence.

Corollary 136 *Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}}\rangle$ be an $\mathcal{L}$-structure. Then, for all $a,b \in A$, we have*

$$\Omega(\mathfrak{A})(a,b) = \bigwedge\{\varphi^{\mathfrak{A}}(a,\mathbf{c}) \leftrightarrow \varphi^{\mathfrak{A}}(b,\mathbf{c}) : \varphi(x,\bar{z}) \textit{ an } \mathcal{L}\textit{-formula, } \mathbf{c} \textit{ in } A\}.$$

4.8 Quotient Structures and Morphisms

Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ be an $\mathcal{L}$-structure and suppose $\Theta \in \mathrm{Gon}(\mathfrak{A})$. Define $\hat{\Theta} = \{\hat{\Theta}_g\}_{g \in G}$, where, for all $g \in G$, $\hat{\Theta}_g \subseteq A^2$ is given by

$$\hat{\Theta}_g = \{\langle a, b \rangle \in A : \Theta(a, b) \geq g\}.$$

Lemma 137 *Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ be an $\mathcal{L}$-structure and $\Theta \in \mathrm{Gon}(\mathfrak{A})$. Then, for all $g \in G$, $\hat{\Theta}_g$ is a congruence of the algebra $\mathbf{A}$.*

Proof: Since, for all $a \in A$, $\Theta(a, a) = \top \geq g$, we get $\langle a, a \rangle \in \hat{\Theta}_g$, whence $\hat{\Theta}_g$ is reflexive.

If $\langle a, b \rangle \in \hat{\Theta}_g$, then $\Theta(b, a) = \Theta(a, b) \geq g$. Hence, $\langle b, a \rangle \in \hat{\Theta}_g$ and $\hat{\Theta}_g$ is symmetric.

If $\langle a, b \rangle, \langle b, c \rangle \in \hat{\Theta}_g$, then

$$\Theta(a, c) \geq \Theta(a, b) \wedge \Theta(b, c) \geq g,$$

whence $\langle a, c \rangle \in \hat{\Theta}_g$ and $\hat{\Theta}_g$ is also transitive.

Finally, if $f \in F$, with $\rho(f) = n$, and $a_1, b_1, \ldots, a_n, b_n \in A$, such that $\langle a_1, b_1 \rangle, \ldots, \langle a_n, b_n \rangle \in \hat{\Theta}_g$, then

$$\Theta(f^{\mathbf{A}}(\mathbf{a}), f^{\mathbf{A}}(\mathbf{b})) \geq \bigwedge_{i=1}^{n} \Theta(a_i, b_i) \geq g.$$

Hence, $\langle f^{\mathbf{A}}(\mathbf{a}), f^{\mathbf{a}}(\mathbf{b}) \rangle \in \hat{\Theta}_g$ and $\hat{\Theta}_g$ is a congruence on $\mathbf{A}$. ∎

$\hat{\Theta}$ is called the **stratified congruence associated with** the G-congruence Θ. More generally, a **stratified congruence on A** is any G-indexed family of congruences $\{\theta_g\}_{g \in G}$, such that, for all $g, g' \in G$,

$$g \leq g' \quad \text{implies} \quad \theta_{g'} \subseteq \theta_g.$$

We call θ_g the g-**stratum** of the stratified congruence. By slightly overloading notation, we shall write $\hat{\Theta}$ to denote $\hat{\Theta}_\top$ as well.

Consider the quotient algebra $\mathbf{A}/\hat{\Theta}$. Define the tuple

$$\mathfrak{A}/\Theta = \langle \mathbf{A}/\hat{\Theta}, \bar{\Theta}, R^{\mathfrak{A}/\Theta} \rangle$$

by setting:

- For all $a, b \in A$,
$$\bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) = \Theta(a, b);$$
- $R^{\mathfrak{A}/\Theta} = \{r^{\mathfrak{A}/\Theta} : r \in R\}$, where, for all $r \in R$, with $\rho(r) = n$, and all $a_1, \ldots, a_n \in A$,

$$r^{\mathfrak{A}/\Theta}(a_1/\hat{\Theta}, \ldots, a_n/\hat{\Theta}) = r^{\mathfrak{A}}(a_1, \ldots, a_n).$$

These definitions do not depend on the chosen representatives. Assume, first, that $a, a', b, b' \in A$, such that $\langle a, a'\rangle, \langle b, b'\rangle \in \hat{\Theta}$. Then

$$\begin{array}{rcl}\bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) & = & \Theta(a,b) \quad (\text{Definition of } \bar{\Theta}) \\ & = & \Theta(a',a) \wedge \Theta(a,b) \wedge \Theta(b,b') \quad (\Theta(a',a) = \Theta(b,b') = \top) \\ & \leq & \Theta(a',b') \quad (\Theta \in \text{Gon}(\mathfrak{A})) \\ & = & \bar{\Theta}(a'/\hat{\Theta}, b'/\hat{\Theta}). \quad (\text{Definition of } \bar{\Theta})\end{array}$$

Thus, $\bar{\Theta}$ is well defined. Further, if $r \in R$, with $\rho(r) = n$, and $a_1, b_1, \ldots, a_n, b_n \in A$, such that $\langle a_i, b_i\rangle \in \hat{\Theta}$, for all $i = 1, \ldots, n$, then

$$\begin{array}{rcl}r^{\mathfrak{A}/\Theta}(a_1/\hat{\Theta}, \ldots, a_n/\hat{\Theta}) & = & r^{\mathfrak{A}}(a_1, \ldots, a_n) \quad (\text{Definition of } r^{\mathfrak{A}/\Theta}) \\ & = & \bigwedge_{i=1}^n \Theta(a_i, b_i) \wedge r^{\mathfrak{A}}(a_1, \ldots, a_n) \quad (\Theta(a_i,b_i) = \top) \\ & \leq & r^{\mathfrak{A}}(b_1, \ldots, b_n) \quad (\Theta \in \text{Gon}(\mathfrak{A})) \\ & = & r^{\mathfrak{A}/\Theta}(b_1/\hat{\Theta}, \ldots, b_n/\hat{\Theta}). \quad (\text{Definition of } r^{\mathfrak{A}/\Theta})\end{array}$$

Hence $R^{\mathfrak{A}/\Theta}$ is also well defined.

Lemma 138 *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}}\rangle$ *be an* $\mathcal{L}$*-structure and* $\Theta \in \text{Gon}(\mathfrak{A})$*. Then* $\mathfrak{A}/\Theta = \langle \mathbf{A}/\hat{\Theta}, \bar{\Theta}, R^{\mathfrak{A}/\Theta}\rangle$ *is an* $\mathcal{L}$*-structure.*

Proof: We must show that $\bar{\Theta}$ is a reduced G-congruence and that $\bar{\Theta}$ is compatible with all operations in $F^{\mathbf{A}/\Theta}$ and all G-relations in $R^{\mathfrak{A}/\Theta}$.

Let $a, a \in A$. Then

$$\begin{array}{rcl}\bar{\Theta}(a/\hat{\Theta}, a/\hat{\Theta}) & = & \Theta(a,a) \quad (\text{Definition of } \bar{\Theta}) \\ & = & \top. \quad (\Theta \in \text{Gon}(\mathfrak{A}))\end{array}$$

Let $a, b \in A$. Then

$$\begin{array}{rcl}\bar{\Theta}(b/\hat{\Theta}, a/\hat{\Theta}) & = & \Theta(b,a) \quad (\text{Definition of } \bar{\Theta}) \\ & = & \Theta(a,b) \quad (\Theta \in \text{Gon}(\mathfrak{A})) \\ & = & \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}). \quad (\text{Definition of } \bar{\Theta})\end{array}$$

Let $a, b, c \in A$. Then

$$\begin{array}{rcl}\bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) \wedge \bar{\Theta}(b/\hat{\Theta}, c/\hat{\Theta}) & = & \Theta(a,b) \wedge \Theta(b,c) \quad (\text{Definition of } \bar{\Theta}) \\ & \leq & \Theta(a,c) \quad (\Theta \in \text{Gon}(\mathfrak{A})) \\ & = & \bar{\Theta}(a/\hat{\Theta}, c/\hat{\Theta}). \quad (\text{Definition of } \bar{\Theta})\end{array}$$

Let $f \in F$, with $\rho(f) = n$, and $a_1, b_1, \ldots, a_n, b_n \in A$. Then

$$\begin{array}{rcl}\bigwedge_{i=1}^n \bar{\Theta}(a_i/\hat{\Theta}, b_i/\hat{\Theta}) & = & \bigwedge_{i=1}^n \Theta(a_i, b_i) \quad (\text{Definition of } \bar{\Theta}) \\ & \leq & \Theta(f^{\mathbf{A}}(\mathbf{a}), f^{\mathbf{A}}(\mathbf{b})) \quad (\Theta \in \text{Gon}(\mathfrak{A})) \\ & = & \bar{\Theta}(f^{\mathbf{A}}(\mathbf{a})/\hat{\Theta}, f^{\mathbf{A}}(\mathbf{b}/\hat{\Theta})) \quad (\text{Definition of } \bar{\Theta}) \\ & = & \bar{\Theta}(f^{\mathbf{A}/\Theta}(\mathbf{a}/\hat{\Theta}), f^{\mathbf{A}/\Theta}(\mathbf{b}/\hat{\Theta})). \\ & & \quad (\text{Definition of } \mathbf{A}/\Theta)\end{array}$$

To see that $\bar{\Theta}$ is reduced, suppose $a, b \in A$, such that $\bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) = \top$. Then $\Theta(a, b) = \top$. But this gives $\langle a, b\rangle \in \hat{\Theta}$. Thus, $a/\hat{\Theta} = b/\hat{\Theta}$. Therefore, $\bar{\Theta}$ is a reduced G-congruence on $\mathbf{A}/\Theta$. To show compatibility with the G-relations, suppose $r \in R$, with $\rho(r) = n$, and let $a_1, b_1, \ldots, a_n, b_n \in A$. Then

$$\begin{array}{rcl}\bigwedge_{i=1}^n \bar{\Theta}(\mathbf{a}/\hat{\Theta}) \wedge r^{\mathfrak{A}/\Theta}(\mathbf{a}/\hat{\Theta}) & = & \bigwedge_{i=1}^n \Theta(\mathbf{a}) \wedge r^{\mathfrak{A}}(\mathbf{a}) \\ & & \quad \text{(Definitions of } \bar{\Theta} \text{ and } r^{\mathfrak{A}/\Theta}) \\ & \leq & r^{\mathfrak{A}}(\mathbf{b}) \quad (\Theta \in \mathrm{Gon}(\mathfrak{A})) \\ & = & r^{\mathfrak{A}/\Theta}(\mathbf{b}/\hat{\Theta}). \quad \text{(Definition of } r^{\mathfrak{A}/\Theta})\end{array}$$

■

We call $\mathfrak{A}/\Theta$ the **quotient structure of $\mathfrak{A}$ by Θ**.

We define the **natural** or **canonical projection morphism**, or **quotient morphism**, $\pi_\Theta : A \to A/\hat{\Theta}$ by setting, for all $a \in A$,

$$\pi_\Theta(a) = a/\hat{\Theta}.$$

Then the following proposition applies, which forms a kind of converse to Lemma 131.

Proposition 139 *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}}\rangle$ *be an* $\mathcal{L}$*-structure and* $\Theta \in \mathrm{Gon}(\mathfrak{A})$*. Then* $\pi_\Theta : \mathfrak{A} \twoheadrightarrow_s \mathfrak{A}/\Theta$ *is a reductive morphism, with* $\mathrm{Ker}(\pi_\Theta) = \Theta$*.*

Proof: First, let us show that π_Θ is an algebra homomorphism. Let $f \in F$, with $\rho(f) = n$, and $a_1, \ldots, a_n \in A$. We have

$$\begin{array}{rcll}\pi_\Theta(f^{\mathbf{A}}(\mathbf{a})) & = & f^{\mathbf{A}}(\mathbf{a})/\hat{\Theta} & \text{(Definition of } \pi_\Theta) \\ & = & f^{\mathbf{A}/\Theta}(\mathbf{a}/\hat{\Theta}) & \text{(Definition of } f^{\mathbf{A}/\Theta}) \\ & = & f^{\mathbf{A}/\Theta}(\pi_\Theta(\mathbf{a})). & \text{(Definition of } \pi_\Theta)\end{array}$$

Next, we show that it is, in fact, a G-algebra morphism. Let $a, b \in A$. We have

$$\begin{array}{rcll}E^{\mathfrak{A}}(a, b) & \leq & \Theta(a, b) & (\Theta \in \mathrm{Gon}(\mathfrak{A})) \\ & = & \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) & \text{(Definition of } \bar{\Theta}) \\ & = & \bar{\Theta}(\pi_\Theta(a), \pi_\Theta(b)). & \text{(Definition of } \pi_\Theta)\end{array}$$

To see that it is a reductive morphism of $\mathcal{L}$-structures, suppose $r \in R$, with $\rho(r) = n$, and $a_1, \ldots, a_n \in A$. Then

$$\begin{array}{rcll}r^{\mathfrak{A}}(\mathbf{a}) & = & r^{\mathfrak{A}/\Theta}(\mathbf{a}/\hat{\Theta}) & \text{(Definition of } r^{\mathfrak{A}/\Theta}) \\ & = & r^{\mathfrak{A}/\Theta}(\pi_\Theta(\mathbf{a})). & \text{(Definition of } \pi_\Theta)\end{array}$$

Finally, for the last claim in the statement, we have, for all $a, b \in A$,

$$\begin{array}{rcll}\mathrm{Ker}(\pi_\Theta)(a, b) & = & \bar{\Theta}(\pi_\Theta(a), \pi_\Theta(b)) & \text{(Definition of } \mathrm{Ker}(\pi_\Theta)) \\ & = & \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) & \text{(Definition of } \pi_\Theta) \\ & = & \Theta(a, b). & \text{(Definition of } \bar{\Theta})\end{array}$$

So $\mathrm{Ker}(\pi_\Theta) = \Theta$ and all claims in the statement are proven. ■

Our next goal is to prove a series of Homomorphism Theorems paralleling the ones in Universal Algebra. On our way we establish a lemma.

Lemma 140 *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$, $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ *and* $\mathfrak{C} = \langle \mathbf{C}, E^{\mathfrak{C}}, R^{\mathfrak{C}} \rangle$ *be* $\mathcal{L}$*-structures and* $h : \mathfrak{A} \to \mathfrak{B}$ *and* $g : \mathfrak{A} \twoheadrightarrow_s \mathfrak{C}$, *such that* $\mathrm{Ker}(g) \leq \mathrm{Ker}(h)$. *Then, there exists unique* $k : \mathfrak{C} \to \mathfrak{B}$, *such that* $h = k \circ g$.

$$\begin{tikzcd}
\mathfrak{A} \arrow[rr, "h"] \arrow[dr, "g"'] & & \mathfrak{B} \\
& \mathfrak{C} \arrow[ur, dotted, "k"'] &
\end{tikzcd}$$

Moreover, the morphism h *is strict if and only if* k *is strict.*

Proof: To define k, let $c \in C$. Then, since g is surjective, there exist $a \in A$, such that $g(a) = c$. We let

$$k(c) = h(a).$$

To see this is well defined, note that, if $a_1, a_2 \in A$ are such that $g(a_1) = g(a_2) = c$, then, since $\mathrm{Ker}(g) \leq \mathrm{Ker}(h)$,

$$\top = E^{\mathfrak{C}}(g(a_1), g(a_2)) \leq E^{\mathfrak{B}}(h(a_1), h(a_2)).$$

Hence, $E^{\mathfrak{B}}(h(a_1), h(a_2)) = \top$ and, since $E^{\mathfrak{B}}$ is reduced, $h(a_1) = h(a_2)$. So k is well defined.

Next, we show that k is an algebra homomorphism $k : \mathbf{C} \to \mathbf{B}$. Indeed, if $f \in F$, with $\rho(f) = n$, and $c_1, \ldots, c_n \in C$, such that $c_i = g(a_i)$, $i = 1, \ldots, n$, for some $a_1, \ldots, a_n \in A$, then we have $g(f^{\mathbf{A}}(\mathbf{a})) = f^{\mathbf{C}}(g(\mathbf{a})) = f^{\mathbf{C}}(\mathbf{c})$, whence

$$\begin{array}{rcll}
f^{\mathbf{B}}(k(\mathbf{c})) & = & f^{\mathbf{B}}(h(\mathbf{a})) & (\text{Definition of } k \text{ and } \mathbf{c} = g(\mathbf{a})) \\
& = & h(f^{\mathbf{A}}(\mathbf{a})) & (h : \mathbf{A} \to \mathbf{B}) \\
& = & k(f^{\mathbf{C}}(\mathbf{c})). & (\text{Definition of } k \text{ and } f^{\mathbf{C}}(\mathbf{c}) = g(f^{\mathbf{A}}(\mathbf{a})))
\end{array}$$

Further, for all $c_1, c_2 \in C$, such that $g(a_1) = c_1$ and $g(a_2) = c_2$, for some $a_1, a_2 \in A$,

$$\begin{array}{rcll}
E^{\mathfrak{C}}(c_1, c_2) & = & E^{\mathfrak{C}}(g(a_1), g(a_2)) & (\text{Hypothesis}) \\
& \leq & E^{\mathfrak{B}}(h(a_1), h(a_2)) & (\mathrm{Ker}(g) \leq \mathrm{Ker}(h)) \\
& = & E^{\mathfrak{B}}(k(c_1), k(c_2)). & (\text{Definition of } k)
\end{array}$$

Hence k is a G-algebra morphism.

Next, we show that $k : \mathfrak{C} \to \mathfrak{A}$. So suppose $r \in R$, with $\rho(r) = n$, and let $c_1, \ldots, c_n \in C$, such that $c_i = g(a_i)$, $i = 1, \ldots, n$, for some $a_1 \ldots, a_n \in A$. Then we have

$$\begin{array}{rcll}
r^{\mathfrak{C}}(\mathbf{c}) & = & r^{\mathfrak{C}}(g(\mathbf{a})) & (\mathbf{c} = g(\mathbf{a})) \\
& = & r^{\mathfrak{A}}(\mathbf{a}) & (g : \mathfrak{A} \twoheadrightarrow_s \mathfrak{C}) \\
& \leq & r^{\mathfrak{B}}(h(\mathbf{a})) & (h : \mathfrak{A} \to \mathfrak{B}) \\
& = & r^{\mathfrak{B}}(k(\mathbf{c})). & (\text{Definition of } k)
\end{array}$$

So $k : \mathfrak{C} \to \mathfrak{A}$. From this sequence, we can deduce that, if h is strict, then so is k. And the converse of this is straightforward by the fact that $h = k \circ g$. So h is strict if and only if k is strict. ■

We are now ready for the sequence of Homomorphism Theorems.

Theorem 141 (Homomorphism) *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ *and* $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ *be* $\mathcal{L}$*-structures and* $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$*. Then* $\mathfrak{A}/\mathrm{Ker}(h) \cong \mathfrak{B}$*.*

Proof: By Lemma 131, $\mathrm{Ker}(h) \in \mathrm{Gon}(\mathfrak{A})$. By Lemma 138, $\mathfrak{A}/\mathrm{Ker}(h)$ is an $\mathcal{L}$-structure and, by Lemma 139, $\pi_{\mathrm{Ker}(h)} : \mathfrak{A} \twoheadrightarrow_s \mathfrak{A}/\mathrm{Ker}(h)$ is a reductive morphism. Consider now the following diagram, where $k : \mathfrak{A}/\mathrm{Ker}(h) \to_s \mathfrak{B}$ is the strict morphism guaranteed by Lemma 140.

$$\begin{tikzcd}
\mathfrak{A} \arrow[rr, two heads, "h"] \arrow[dr, "\pi_{\mathrm{Ker}(h)}"'] & & \mathfrak{B} \\
& \mathfrak{A}/\mathrm{Ker}(h) \arrow[ur, dotted, "k"'] &
\end{tikzcd}$$

It now suffices to show that k is a bijection, and that it preserves the reduced congruences. That k is surjective follows from the fact that h is surjective. For injectivity, suppose $a, a' \in A$, such that $k(a/\widehat{\mathrm{Ker}(h)}) = k(a'/\widehat{\mathrm{Ker}(h)})$. Then, by definition, $h(a) = h(a')$. Hence, $E^{\mathfrak{B}}(h(a), h(a')) = \top$. By definition of $\mathrm{Ker}(h)$, $\mathrm{Ker}(h)(a, a') = \top$. Therefore, $a/\widehat{\mathrm{Ker}(h)} = a'/\widehat{\mathrm{Ker}(h)}$. So k is indeed one-to-one.

Finally, for $a, a' \in A$, we have

$$\begin{aligned}
E^{\mathfrak{A}/\mathrm{Ker}(h)}(a/\widehat{\mathrm{Ker}(h)}, a'/\widehat{\mathrm{Ker}(h)}) &= \overline{\mathrm{Ker}(h)}(a/\widehat{\mathrm{Ker}(h)}, a'/\widehat{\mathrm{Ker}(h)}) \\
&\qquad \text{(Definition of } E^{\mathfrak{A}/\mathrm{Ker}(h)}\text{)} \\
&= \mathrm{Ker}(h)(a, a') \\
&\qquad \text{(Definition of } \overline{\mathrm{Ker}(h)}\text{)} \\
&= E^{\mathfrak{B}}(h(a), h(a')) \\
&\qquad \text{(Definition of } \mathrm{Ker}(h)\text{)} \\
&= E^{\mathfrak{B}}(k(a/\widehat{\mathrm{Ker}(h)}), k(a'/\widehat{\mathrm{Ker}(h)})). \\
&\qquad \text{(Definition of } k\text{)}
\end{aligned}$$

Therefore, $k : \mathfrak{A}/\mathrm{Ker}(h) \cong \mathfrak{B}$. ■

The next theorem in the series is the analog of the well known Second Isomorphism Theorem of Universal Algebra.

Theorem 142 (Second Isomorphism) *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ *be an* $\mathcal{L}$*-structure and* $\Theta, \Theta' \in \mathrm{Gon}(\mathfrak{A})$*, such that* $\Theta \leq \Theta'$*. Then the function*

$$\pi : (A/\hat{\Theta})/(\hat{\Theta}'/\hat{\Theta}) \to A/\hat{\Theta}',$$

defined, for all $a \in A$, by

$$\pi((a/\hat{\Theta})/(\hat{\Theta}'/\hat{\Theta})) = a/\hat{\Theta}',$$

establishes an isomorphism $\pi : (\mathfrak{A}/\Theta)/(\Theta'/\Theta) \cong \mathfrak{A}/\Theta'$.

Proof: Since $\Theta \in \mathrm{Gon}(\mathfrak{A})$, by Lemma 138, $\mathfrak{A}/\Theta$ is an $\mathcal{L}$-structure. On $A/\hat{\Theta}$, define the G-relation $\Theta'/\Theta : (A/\hat{\Theta})^2 \to G$ by setting, for all $a, b \in A$,

$$(\Theta'/\Theta)(a/\hat{\Theta}, b/\hat{\Theta}) = \Theta'(a, b).$$

To see that Θ'/Θ is well defined, let $a, a', b, b' \in A$, such that $\langle a, a'\rangle \in \hat{\Theta}$ and $\langle b, b'\rangle \in \hat{\Theta}$. This means that $\Theta(a', a) = \top$ and $\Theta(b, b') = \top$. Thus, we get

$$\begin{aligned}
\Theta'(a,b) &= \Theta(a',a) \wedge \Theta'(a,b) \wedge \Theta(b,b') \\
&\qquad (\Theta(a',a) = \top \text{ and } \Theta(b,b') = \top) \\
&\leq \Theta'(a',a) \wedge \Theta'(a,b) \wedge \Theta'(b,b') \quad (\Theta \leq \Theta') \\
&\leq \Theta'(a',b'). \quad (\Theta' \in \mathrm{Gon}(\mathfrak{A}))
\end{aligned}$$

By symmetry, $\Theta'(a,b) = \Theta'(a',b')$, whence Θ'/Θ is well defined. Since $\Theta' \in \mathrm{Gon}(\mathfrak{A})$, it is not difficult to see that $\Theta'/\Theta \in \mathrm{Gon}(\mathfrak{A}/\Theta)$. Let us show Congruence, Inclusion of $E^{\mathfrak{A}/\Theta}$ and Compatibility.

Suppose $f \in F$, with $\rho(f) = n$, and $a_1, b_1, \ldots, a_n, b_n \in A$. Then

$$\begin{aligned}
\textstyle\bigwedge_{i=1}^n (\Theta'/\Theta)(a_i/\hat{\Theta}, b_i/\hat{\Theta}) &= \textstyle\bigwedge_{i=1}^n \Theta'(a_i, b_i) \quad (\text{Definition of } \Theta'/\Theta) \\
&\leq \Theta'(f^{\mathbf{A}}(\mathbf{a}), f^{\mathbf{A}}(\mathbf{b})) \quad (\Theta' \in \mathrm{Gon}(\mathfrak{A})) \\
&= (\Theta'/\Theta)(f^{\mathbf{A}}(\mathbf{a})/\hat{\Theta}, f^{\mathbf{A}}(\mathbf{b})/\hat{\Theta}) \\
&\qquad (\text{Definition of } \Theta'/\Theta) \\
&= (\Theta'/\Theta)(f^{\mathbf{A}/\Theta}(\mathbf{a}/\hat{\Theta}), f^{\mathbf{A}/\Theta}(\mathbf{b}/\hat{\Theta})). \\
&\qquad (\text{Definition of } f^{\mathbf{A}/\Theta})
\end{aligned}$$

Next, let $a, b \in A$. We obtain

$$\begin{aligned}
E^{\mathfrak{A}/\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) &= \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) \quad (\text{Definition of } E^{\mathfrak{A}/\Theta}) \\
&= \Theta(a,b) \quad (\text{Definition of } \bar{\Theta}) \\
&\leq \Theta'(a,b) \quad (\Theta \leq \Theta') \\
&= (\Theta'/\Theta)(a/\hat{\Theta}, b/\hat{\Theta}). \quad (\text{Definition of } \Theta'/\Theta)
\end{aligned}$$

Finally, consider $r \in R$, with $\rho(r) = n$, and $a_1, b_1, \ldots, a_n, b_n \in A$. Then

$$\begin{aligned}
\textstyle\bigwedge_{i=1}^n (\Theta'/\Theta)(a_i/\hat{\Theta}, b_i/\hat{\Theta}) \wedge r^{\mathfrak{A}/\Theta}(\mathbf{a}/\hat{\Theta}) &= \textstyle\bigwedge_{i=1}^n \Theta'(a_i, b_i) \wedge r^{\mathfrak{A}}(\mathbf{a}) \\
&\qquad (\text{Definitions of } \Theta'/\Theta \text{ and } r^{\mathfrak{A}/\Theta}) \\
&\leq r^{\mathfrak{A}}(\mathbf{b}) \quad (\Theta' \in \mathrm{Gon}(\mathfrak{A})) \\
&= r^{\mathfrak{A}/\Theta}(\mathbf{b}/\hat{\Theta}). \quad (\text{Definition of } r^{\mathfrak{A}/\Theta})
\end{aligned}$$

Now, by Proposition 139, we get that $\pi_{\Theta'/\Theta} : \mathfrak{A}/\Theta \twoheadrightarrow_s (\mathfrak{A}/\Theta)/(\Theta'/\Theta)$ is a reductive morphism.

Define, next, $h : A/\hat{\Theta} \to A/\hat{\Theta}'$ by setting, for all $a \in A$,

$$h(a/\hat{\Theta}) = a/\hat{\Theta}'.$$

We show that this is a reductive morphism $h : \mathfrak{A}/\Theta \twoheadrightarrow_s \mathfrak{A}/\Theta'$. It is definitely well defined, since $\langle a, b\rangle \in \hat{\Theta}$ implies $\top = \Theta(a,b) \leq \Theta'(a,b)$ and, hence, $\langle a, b\rangle \in \hat{\Theta}'$. By Universal Algebra, it is an algebra homomorphism. Further, for all $a, b \in A$,

$$\begin{aligned}
E^{\mathfrak{A}/\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) &= \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) \quad \text{(Definition of } E^{\mathfrak{A}/\Theta}\text{)} \\
&= \Theta(a,b) \quad \text{(Definition of } \bar{\Theta}\text{)} \\
&\leq \Theta'(a,b) \quad (\Theta \leq \Theta') \\
&= \bar{\Theta}'(a/\hat{\Theta}', b/\hat{\Theta}') \quad \text{(Definition of } \bar{\Theta}'\text{)} \\
&= E^{\mathfrak{A}/\Theta'}(h(a/\hat{\Theta}), h(b/\hat{\Theta})). \quad \text{(Definition of } E^{\mathfrak{A}/\Theta'}\text{)}
\end{aligned}$$

Hence, h is a G-algebra morphism. Finally, for all $r \in R$, with $\rho(r) = n$, and $a_1, \ldots, a_n \in A$,

$$\begin{aligned}
r^{\mathfrak{A}/\Theta}(\mathbf{a}/\hat{\Theta}) &= r^{\mathfrak{A}}(\mathbf{a}) \quad \text{(Definition of } r^{\mathfrak{A}/\Theta}\text{)} \\
&= r^{\mathfrak{A}/\Theta'}(\mathbf{a}/\hat{\Theta}') \quad \text{(Definition of } r^{\mathfrak{A}/\Theta'}\text{)} \\
&= r^{\mathfrak{A}/\Theta'}(h(\mathbf{a}/\hat{\Theta})). \quad \text{(Definition of } h\text{)}
\end{aligned}$$

Since h is surjective and preserves the value of the G-relations, it is indeed a reductive morphism $h : \mathfrak{A}/\Theta \twoheadrightarrow_s \mathfrak{A}/\Theta'$.

We have now the following diagram with given the two solid reductive morphisms.

$$\begin{tikzcd}
\mathfrak{A}/\theta \arrow[rr, two heads, "h"] \arrow[dr, "\pi_{\theta'/\theta}"'] & & \mathfrak{A}/\theta' \\
& (\mathfrak{A}/\theta)/(\theta'/\theta) \arrow[ur, dotted, "k"'] &
\end{tikzcd}$$

By Theorem 141, we get a unique isomorphism $k : (\mathfrak{A}/\Theta)/(\Theta'/\Theta) \cong \mathfrak{A}/\Theta'$, such that $k \circ \pi_{\Theta'/\Theta} = h$, i.e., such that, for all $a \in A$,

$$k((a/\hat{\Theta})/(\hat{\Theta}'/\hat{\Theta})) = a/\hat{\Theta}'.$$

This concludes the proof. ■

In order to formalize the Third Isomorphism Theorem, one has to introduce a couple of additional notions, which we have partially encountered previously. We start with an $\mathcal{L}$-structure $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}}\rangle$ and consider $B \subseteq A$ and a G-congruence Θ on $\mathfrak{A}$. We augment B so as to be "closed under $\hat{\Theta}$-classes", obtaining

$$B^{\Theta} = \{a \in A : B \cap a/\hat{\Theta} \neq \varnothing\}.$$

Then take $\mathfrak{B}^{\Theta}$ to be the substructure of $\mathfrak{A}$ generated by B^{Θ}. As before, write $\Theta_B = \Theta{\restriction}_B$.

Lemma 143 *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}}\rangle$ *and* $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}}\rangle$ *be* $\mathcal{L}$*-structures, such that* $\mathfrak{B} \subseteq \mathfrak{A}$ *and* $\Theta \in \mathrm{Gon}(\mathfrak{A})$*. The universe of* $\mathfrak{B}^\Theta$ *is* B^Θ*.*

Proof: Suppose $f \in F$, with $\rho(f) = n$, and $a_1, \ldots, a_n \in B^\Theta$. By definition, there exist $b_1, \ldots, b_n \in A$, such that $b_i \in B \cap a_i/\hat{\Theta}$, for all $i = 1, \ldots, n$. Now we have

$$\begin{array}{rcll}
B \cap f^{\mathbf{A}}(\mathbf{a})/\hat{\Theta} & = & B \cap f^{\mathbf{A}/\Theta}(\mathbf{a}/\hat{\Theta}) & (\text{Definition of } f^{\mathbf{A}/\Theta}) \\
& = & B \cap f^{\mathbf{A}/\Theta}(\mathbf{b}/\hat{\Theta}) & (\mathbf{a}/\hat{\Theta} = \mathbf{b}/\hat{\Theta}) \\
& = & B \cap f^{\mathbf{A}}(\mathbf{b})/\hat{\Theta} & (\text{Definition of } f^{\mathbf{A}/\Theta}) \\
& \neq & \varnothing. & (\mathfrak{B} \subseteq \mathfrak{A})
\end{array}$$

Thus, $f^{\mathbf{A}/\Theta}(\mathbf{a}) \in B^\Theta$ and, hence, B^Θ is a subuniverse of $\mathfrak{A}$. ■

Now we have available the machinery and notation needed to formulate the analog of the Third Isomorphism Theorem.

Theorem 144 (Third Isomorphism) *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}}\rangle$ *and* $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}}\rangle$ *be* $\mathcal{L}$*-structures, such that* $\mathfrak{B} \subseteq \mathfrak{A}$*, and* $\Theta \in \mathrm{Gon}(\mathfrak{A})$*. Then*

$$\mathfrak{B}/\Theta_B \cong \mathfrak{B}^\Theta/\Theta_{B^\Theta}.$$

Proof: We define a mapping $\pi : B/\hat{\Theta}_B \to B^\Theta/\hat{\Theta}_{B^\Theta}$ by setting, for all $b \in B$,

$$\pi(b/\hat{\Theta}_B) = b/\hat{\Theta}_{B^\Theta}.$$

We must show that $\pi : \mathfrak{B}/\Theta_B \cong \mathfrak{B}^\Theta/\Theta_{B^\Theta}$. We know that π gives an isomorphism of the underlying algebras. Moreover, it is an isomorphism of G-algebras, since, for all $b_1, b_2 \in B$,

$$\begin{array}{rcll}
E^{\mathfrak{B}/\Theta_B}(b_1/\hat{\Theta}_B, b_2/\hat{\Theta}_B) & = & \bar{\Theta}_B(b_1/\hat{\Theta}_B, b_2/\hat{\Theta}_B) & (\text{Definition of } E^{\mathfrak{B}/\Theta_B}) \\
& = & \Theta_B(b_1, b_2) & (\text{Definition of } \bar{\Theta}_B) \\
& = & \Theta(b_1, b_2) & (\text{Definition of } \Theta_B) \\
& = & \Theta_{B^\Theta}(b_1, b_2) & (\text{Definition of } \Theta_{B^\Theta}) \\
& = & \bar{\Theta}_{B^\Theta}(b_1/\hat{\Theta}_{B^\Theta}, b_2/\hat{\Theta}_{B^\Theta}) & (\text{Definition of } \bar{\Theta}_{B^\Theta}) \\
& = & E^{\mathfrak{B}^\Theta/\Theta_{B^\Theta}}(\pi(b_1/\hat{\Theta}_B), \pi(b_2/\hat{\Theta}_B)). & \\
& & \quad (\text{Definition of } E^{\mathfrak{B}^\Theta/\Theta_{B^\Theta}}) &
\end{array}$$

To see that it is a strict homomorphism of $\mathcal{L}$-structures, note that, for all $r \in R$, with $\rho(r) = n$, and all $b_1, \ldots, b_n \in B$,

$$\begin{array}{rcll}
r^{\mathfrak{B}/\Theta_B}(\mathbf{b}/\Theta_B) & = & r^{\mathfrak{B}}(\mathbf{b}) & (\text{Definition of } r^{\mathfrak{B}/\Theta_B}) \\
& = & r^{\mathfrak{A}}(\mathbf{b}) & (\mathfrak{B} \subseteq \mathfrak{A}) \\
& = & r^{\mathfrak{B}^\Theta}(\mathbf{b}) & (\mathfrak{B}^\Theta \subseteq \mathfrak{A}) \\
& = & r^{\mathfrak{B}^\Theta/\Theta_{B^\Theta}}(\mathbf{b}/\Theta_{B^\Theta}). & (\text{Definition of } r^{\mathfrak{B}^\Theta/\Theta_{B^\Theta}})
\end{array}$$

Thus, π is a strict homomorphism. ■

For the Correspondence Theorem, we need the interval notation for lattices. For $[a, b]$ a closed interval of a lattice $\mathbf{L}$, where $a \leq b$, we define $[\mathbf{a}, \mathbf{b}]$ to be the corresponding sublattice of $\mathbf{L}$.

Theorem 145 (Correspondence) *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ *be an* $\mathcal{L}$*-structure and* $\Theta \in \mathrm{Gon}(\mathfrak{A})$*. Then the mapping* α *defined on the interval* $[\Theta, \Omega(\mathfrak{A})]$ *of* $\mathbf{Gon}(\mathfrak{A})$ *by*

$$\alpha(\Theta') = \Theta'/\Theta,$$

is a lattice isomorphism from $[\Theta, \Omega(\mathfrak{A})]$ *to* $\mathbf{Gon}(\mathfrak{A}/\Theta)$*, where* $[\Theta, \Omega(\mathfrak{A})]$ *is the interval sublattice of* $\mathbf{Gon}(\mathfrak{A})$*.*

Proof: By Theorem 142, the mapping α is well defined. If $\Theta'' \in \mathrm{Gon}(\mathfrak{A}/\Theta)$, then we define $\beta(\Theta'') : A^2 \to G$ by setting, for all $a, b \in A$,

$$\beta(\Theta'')(a,b) = \Theta''(a/\hat{\Theta}, b/\hat{\Theta}).$$

We show that $\beta(\Theta'') \in \mathrm{Gon}(\mathfrak{A})$ and that $\beta(\Theta'')/\Theta = \Theta''$. We skip Reflexivity, Symmetry and Transitivity and show Congruence, Inclusion of $E^{\mathfrak{A}}$ and Compatibility with $R^{\mathfrak{A}}$. Let $f \in F$, with $\rho(f) = n$, and $a_1, b_1, \ldots, a_n, b_n \in A$. Then

$$\begin{array}{rcll}
\bigwedge_{i=1}^n \beta(\Theta'')(a_i, b_i) & = & \bigwedge_{i=1}^n \Theta''(a_i/\hat{\Theta}, b_i/\hat{\Theta}) & (\text{Definition of } \beta(\Theta'')) \\
& \leq & \Theta''(f^{\mathfrak{A}/\Theta}(\mathbf{a}/\hat{\Theta}), f^{\mathfrak{A}/\Theta}(\mathbf{b}/\hat{\Theta})) & (\Theta'' \in \mathrm{Gon}(\mathfrak{A}/\Theta)) \\
& = & \Theta''(f^{\mathfrak{A}}(\mathbf{a})/\hat{\Theta}, f^{\mathfrak{A}}(\mathbf{b})/\hat{\Theta}) & (\text{Definition of } f^{\mathfrak{A}/\Theta}) \\
& = & \beta(\Theta'')(f^{\mathfrak{A}}(\mathbf{a}), f^{\mathfrak{A}}(\mathbf{b})). & (\text{Definition of } \beta(\Theta''))
\end{array}$$

Next, let $a, b \in A$. We have

$$\begin{array}{rcll}
E^{\mathfrak{A}}(a,b) & \leq & \Theta(a,b) & (\Theta \in \mathrm{Gon}(\mathfrak{A})) \\
& = & \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) & (\text{Definition of } \bar{\Theta}) \\
& = & E^{\mathfrak{A}/\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) & (\text{Definition of } E^{\mathfrak{A}/\Theta}) \\
& \leq & \Theta''(a/\hat{\Theta}, b/\hat{\Theta}). & (\Theta'' \in \mathrm{Gon}(\mathfrak{A}/\Theta))
\end{array}$$

Now consider $r \in R$, with $\rho(r) = n$, and $a_1, b_1, \ldots, a_n, b_n \in A$. Then

$$\begin{array}{rcll}
\bigwedge_{i=1}^n \beta(\Theta'')(a_i, b_i) \wedge r^{\mathfrak{A}}(\mathbf{a}) & = & \bigwedge_{i=1}^n \Theta''(a_i/\hat{\Theta}, b_i/\hat{\Theta}) \wedge r^{\mathfrak{A}/\Theta}(\mathbf{a}/\hat{\Theta}) & \\
& & \quad (\text{Definitions of } \beta(\Theta'') \text{ and } r^{\mathfrak{A}/\Theta}) & \\
& \leq & r^{\mathfrak{A}/\Theta}(\mathbf{b}/\hat{\Theta}) \quad (\Theta'' \in \mathrm{Gon}(\mathfrak{A}/\Theta)) & \\
& = & r^{\mathfrak{A}}(\mathbf{b}). \quad (\text{Definition of } r^{\mathfrak{A}/\Theta}) &
\end{array}$$

Finally, for all $a, b \in A$,

$$\begin{array}{rcll}
(\beta(\Theta'')/\Theta)(a/\hat{\Theta}, b/\hat{\Theta}) & = & \beta(\Theta'')(a,b) & (\text{Definition of } \beta(\Theta'')/\Theta) \\
& = & \Theta''(a/\hat{\Theta}, b/\hat{\Theta}). & (\text{Definition of } \beta(\Theta''))
\end{array}$$

We conclude that α and β are inverse maps.

Finally, notice that, on the one hand, $\Theta/\Theta = \bar{\Theta} = E^{\mathfrak{A}/\Theta}$. Indeed, for all $a, b \in A$,

$$\begin{array}{rcll}
(\Theta/\Theta)(a/\hat{\Theta}, b/\hat{\Theta}) & = & \Theta(a,b) & (\text{Definition of } \Theta/\Theta) \\
& = & \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) & (\text{Definition of } \bar{\Theta}) \\
& = & E^{\mathfrak{A}/\Theta}(a/\hat{\Theta}, b/\hat{\Theta}). & (\text{Definition of } E^{\mathfrak{A}/\Theta})
\end{array}$$

On the other hand, by Corollary 134, taking $\pi_\Theta : \mathfrak{A} \twoheadrightarrow_s \mathfrak{A}/\Theta$, $\Omega(\mathfrak{A}/\Theta) = \Omega(\mathfrak{A})/\Theta$. ■

4.9 Leibniz Quotient

Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ be an $\mathcal{L}$-structure. Its **Leibniz quotient** is the $\mathcal{L}$-structure

$$\mathfrak{A}^* = \mathfrak{A}/\Omega(\mathfrak{A}) := \langle \mathbf{A}/\widehat{\Omega(\mathfrak{A})}, \overline{\Omega(\mathfrak{A})}, R^{\mathfrak{A}/\Omega(\mathfrak{A})} \rangle.$$

We write $\mathbf{A}^*$ for the underlying algebra $\mathbf{A}/\Omega(\mathfrak{A}) := \mathbf{A}/\widehat{\Omega(\mathfrak{A})}$ of $\mathfrak{A}^*$, $\mathcal{A}^*$ for the underlying G-algebra $\mathcal{A}/\Omega(\mathfrak{A})$ of $\mathfrak{A}^*$, and we use a^* to denote an element $a/\widehat{\Omega(\mathfrak{A})}$ of the quotient, for $a \in A$.

Given $\mathcal{L}$-structures $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ and $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ and a mapping $h : A \to B$, we write $h^* : A^* \to B^*$ for the correspondence

$$a^* \mapsto (h(a))^*$$

induced by h on the quotients. However, this is not, in general, a well defined mapping.

By Proposition 139, $\mathfrak{A}^*$ is a reduction of $\mathfrak{A}$. By the Correspondence Theorem, $\mathfrak{A}^*$ is a reduced structure. So we have $\mathfrak{A}^{**} \cong \mathfrak{A}^*$. We can show that $\mathfrak{A}^*$ is minimal in the sense that it is a reduction of any other reduction of the $\mathcal{L}$-structure $\mathfrak{A}$.

Theorem 146 *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ *and* $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ *be* $\mathcal{L}$*-structures and let* $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$*. Then* $h^* : \mathfrak{A}^* \cong \mathfrak{B}^*$*. More generally, if* $h : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$*, then* $h^* : h^{-1}(\mathfrak{B})^* \cong \mathfrak{B}^*$*.*

Proof: Let $a, a' \in A$. Then, we have

$$\begin{array}{lll} a^* = a'^* & \text{iff} & \langle a, a' \rangle \in \widehat{\Omega(\mathfrak{A})} \quad (\text{Definition of }{}^*) \\ & \text{iff} & \langle h(a), h(a') \rangle \in \widehat{\Omega(\mathfrak{B})} \quad (\text{Corollary 134}) \\ & \text{iff} & h(a)^* = h(a')^*. \quad (\text{Definition of }{}^*) \end{array}$$

So h^* is both well defined and one-to-one. It is clearly onto, since h is onto. To see that it is an algebra homomorphism, let $f \in F$, with $\rho(f) = n$, and $a_1, \ldots, a_n \in A$. Then

$$\begin{array}{lll} h^*(f^{\mathbf{A}^*}(\mathbf{a}^*)) & = & h^*(f^{\mathbf{A}}(\mathbf{a})^*) \quad (\text{Definition of } \mathbf{A}^*) \\ & = & h(f^{\mathbf{A}}(\mathbf{a}))^* \quad (\text{Definition of } h^*) \\ & = & f^{\mathbf{B}}(h(\mathbf{a}))^* \quad (h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}) \\ & = & f^{\mathbf{B}^*}(h(\mathbf{a})^*) \quad (\text{Definition of } \mathbf{B}^*) \\ & = & f^{\mathbf{B}^*}(h^*(\mathbf{a}^*)). \quad (\text{Definition of } h^*) \end{array}$$

In fact, it is a G-algebra morphism, since, for all $a_1, a_2 \in A$,

$$\begin{array}{lll} E^{\mathfrak{A}^*}(a_1^*, a_2^*) & = & \overline{\Omega(\mathfrak{A})}(a_1^*, a_2^*) \quad (\text{Definition of } E^{\mathfrak{A}^*}) \\ & = & \Omega(\mathfrak{A})(a_1, a_2) \quad (\text{Definition of } \overline{\Omega(\mathfrak{A})}) \\ & = & \Omega(\mathfrak{B})(h(a_1), h(a_2)) \quad (\text{Corollary 134}) \\ & = & \overline{\Omega(\mathfrak{B})}(h(a_1)^*, h(a_2)^*) \quad (\text{Definition of } \overline{\Omega(\mathfrak{B})}) \\ & = & E^{\mathfrak{B}^*}(h^*(a_1^*), h^*(a_2^*)). \quad (\text{Definitions of } E^{\mathfrak{B}^*} \text{ and } h^*) \end{array}$$

Finally, to see that it is a strict homomorphism of structures, consider $r \in R$, with $\rho(r) = n$, and $a_1, \ldots, a_n \in A$. We have

$$\begin{array}{rcll} r^{\mathfrak{A}^*}(\mathbf{a}^*) & = & r^{\mathfrak{A}}(\mathbf{a}) & \text{(Definition of } r^{\mathfrak{A}^*}) \\ & = & r^{\mathfrak{B}}(h(\mathbf{a})) & (h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}) \\ & = & r^{\mathfrak{B}^*}(h(\mathbf{a})^*) & \text{(Definition of } r^{\mathfrak{B}^*}) \\ & = & r^{\mathfrak{B}^*}(h^*(\mathbf{a}^*)). & \text{(Definition of } h^*) \end{array}$$

So $h^* : \mathfrak{A}^* \cong \mathfrak{B}^*$. For the more general statement, assume that $h : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$. Consider the diagram following Lemma 125.

$$\begin{tikzcd}
\mathfrak{A} \arrow[rr, two heads, "h"] \arrow[dr, "i"'] & & \mathfrak{B} \\
& h^{-1}(\mathfrak{B}) \arrow[ur, two heads, "s", "\hat{h}"'] &
\end{tikzcd}$$

The conclusion follows by using this together with the case just proven. ■

Corollary 147 *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ *and* $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ *be* $\mathcal{L}$*-structures, such that* $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$*. Then* $\mathfrak{A}^*$ *is a reduction of* $\mathfrak{B}$*. Moreover, if* $\mathfrak{A} \cong \mathfrak{B}$*, then* $\mathfrak{A}^* \cong \mathfrak{B}^*$*.*

Proof: Suppose $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$. Then, by Theorem 146, $\mathfrak{A}^* \cong \mathfrak{B}^*$. Thus, since $\mathfrak{B}^*$ is a reduction of $\mathfrak{B}$, $\mathfrak{A}^*$ is a reduction of $\mathfrak{B}$. If $\mathfrak{A} \cong \mathfrak{B}$, then, in particular, $\mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$. Thus, by Theorem 146, $\mathfrak{A}^* \cong \mathfrak{B}^*$. ■

4.10 Models and Semantic Consequence

Consider a **graded collection**

$$\Phi = \{\Phi_g : g \in G\}$$

of pairwise disjoint sets of $\mathcal{L}$-formulas. Define,

$$\mathsf{Mod}(\Phi) = \{\mathfrak{A} : \varphi^{\mathfrak{A}} = g, \text{ for all } \varphi \in \Phi_g, g \in G\}$$

and

$$\mathsf{Mod}^*(\Phi) = \{\mathfrak{A} \in \mathsf{Mod}(\Phi) : \mathfrak{A} \text{ reduced}\}.$$

$\mathsf{Mod}(\Phi)$ and $\mathsf{Mod}^*(\Phi)$ are called, respectively, the **full model class** and the **reduced model class** of Φ. The two classes $\mathsf{Mod}(\Phi)$ and $\mathsf{Mod}^*(\Phi)$ are closely related. Let K be a class of $\mathcal{L}$-structures. Define

$$\mathbb{L}(\mathsf{K}) = \{\mathfrak{A} : \mathfrak{A} \cong \mathfrak{B}^*, \text{ for some } \mathfrak{B} \in \mathsf{K}\}.$$

Sometimes we write K^* in lieu of $\mathbb{L}(\mathsf{K})$. By Proposition 126, $\mathfrak{A}^*$ is elementarily equivalent to $\mathfrak{A}$. Thus, we have

$$\mathsf{Mod}^*(\Phi) = \mathbb{L}(\mathsf{Mod}(\Phi)).$$

The operator $\mathbb{L}$ is called the **reduction operator**. If K is an arbitrary class of $\mathcal{L}$-structures, we say K is a **full class** whenever it is closed under expansions and reductions. We say that K is a **reduced class** if it is obtained by applying the reduction operator to some class. In particular, the whole class of reduced $\mathcal{L}$-structures is called **reduced semantics** to differentiate it from the class of all $\mathcal{L}$-structures, named **full semantics**.

Observe that, if, in every member $\mathfrak{A}$ of a class K, $r^{\mathfrak{A}}$ is constant, for all $r \in R$, then K^* consists of one-element algebras in which all relations are constant (the same constant value as in the parent structure in K).

Let $\Gamma = \{\Gamma_g : g \in G\}$ and $\Phi = \{\Phi_g : g \in G\}$ be graded collections of $\mathcal{L}$-formulas. We write $\Gamma \vDash \Phi$ to signify that, for all $\mathcal{L}$-structures $\mathfrak{A}$ and all assignments h,

$$\mathfrak{A} \vDash \Gamma[h] \quad \text{implies} \quad \mathfrak{A} \vDash \Phi[h].$$

Additionally, we write $\Gamma \vDash^* \Phi$ to signify that, for all reduced $\mathcal{L}$-structures $\mathfrak{A}$ and all assignments h,

$$\mathfrak{A} \vDash \Gamma[h] \quad \text{implies} \quad \mathfrak{A} \vDash \Phi[h].$$

4.11 Class Operators and Properties

We introduced operators on classes of $\mathcal{L}$-structures corresponding to the constructions introduced thus far in the text. Then we formulate a few technical lemmas that investigate some of the properties of those operators.

$$\begin{array}{rcl}
\mathbb{S}(\mathsf{K}) & = & \{\mathfrak{A} : \mathfrak{A} \cong \mathfrak{C} \text{ and } \mathfrak{C} \subseteq \mathfrak{B}, \text{ for some } \mathfrak{B} \in \mathsf{K}\}; \\
\mathbb{F}(\mathsf{K}) & = & \{\mathfrak{A} : \mathfrak{A} \cong \mathfrak{C} \text{ and } \mathfrak{B} \lesssim \mathfrak{C}, \text{ for some } \mathfrak{B} \in \mathsf{K}\}; \\
\mathbb{H}(\mathsf{K}) & = & \{\mathfrak{A} : \mathfrak{A} \cong \mathfrak{C} \text{ and } h : \mathfrak{B} \twoheadrightarrow \mathfrak{C}, \text{ for some } \mathfrak{B} \in \mathsf{K} \text{ and some } h\}; \\
\mathbb{R}(\mathsf{K}) & = & \{\mathfrak{A} : \mathfrak{A} \cong \mathfrak{C} \text{ and } h : \mathfrak{B} \twoheadrightarrow_s \mathfrak{C}, \text{ for some } \mathfrak{B} \in \mathsf{K} \text{ and some } h\}; \\
\mathbb{E}(\mathsf{K}) & = & \{\mathfrak{A} : \mathfrak{A} \cong \mathfrak{C} \text{ and } h : \mathfrak{C} \twoheadrightarrow_s \mathfrak{B}, \text{ for some } \mathfrak{B} \in \mathsf{K} \text{ and some } h\}; \\
\mathbb{P}(\mathsf{K}) & = & \{\mathfrak{A} : \mathfrak{A} \cong \prod_{i \in I} \mathfrak{A}_i \text{ and } \mathfrak{A}_i \in \mathsf{K}, \text{ for all } \in I\}.
\end{array}$$

Suppose $\mathbb{O}$ and $\mathbb{O}'$ are two of these operators. Then $\mathbb{O}\mathbb{O}'$ denotes their composition and $\mathbb{O} \leq \mathbb{O}'$ means that, for every class K of $\mathcal{L}$-structures, $\mathbb{O}(\mathsf{K}) \subseteq \mathbb{O}'(\mathsf{K})$. Moreover, $\mathbb{O}^*$ denotes the operator $\mathbb{L}\mathbb{O}$. By definition, the operator $\mathbb{P}$ applied to any class (even the empty class) yields a nonempty class of structures, since, for an empty index set, one gets the trivial structure, with the trivial underlying algebra, in which the reduced G-congruence and all G-relations are interpreted as $\top$. We denote by $\overline{\mathbb{P}}$ the operator $\mathbb{P}$ applied only to constructions with nonempty index sets.

Lemma 148 *For $\mathbb{O}$ any of the operators defined above, $\mathbb{O}^2 = \mathbb{O}$.*

Proof: Let $\mathbb{O} = \mathbb{S}$, K be a class of $\mathcal{L}$-structures and $\mathfrak{A} \in \mathbb{S}^2(\mathsf{K})$. Assume, for simplicity, that there exists $\mathfrak{B} \in \mathbb{S}(\mathsf{K})$, such that $\mathfrak{A} \subseteq \mathfrak{B}$. Hence, there exists $\mathfrak{C} \in \mathsf{K}$, such that $\mathfrak{B} \subseteq \mathfrak{C}$. Thus, $\mathfrak{A} \subseteq \mathfrak{C} \in \mathsf{K}$, yielding $\mathfrak{A} \in \mathbb{S}(\mathsf{K})$.

Let $\mathbb{O} = \mathbb{F}$, K be a class of $\mathcal{L}$-structures and $\mathfrak{A} \in \mathbb{F}^2(\mathsf{K})$. Assume, for simplicity, that there exists $\mathfrak{B} \in \mathbb{F}(\mathsf{K})$, such that $\mathfrak{B} \lesssim \mathfrak{A}$. Hence, there exists $\mathfrak{C} \in \mathsf{K}$, such that $\mathfrak{C} \lesssim \mathfrak{B}$. Thus, we get $\mathsf{K} \ni \mathfrak{C} \lesssim \mathfrak{A}$, yielding $\mathfrak{A} \in \mathbb{F}(\mathsf{K})$.

Let $\mathbb{O} = \mathbb{H}$, K be a class of $\mathcal{L}$-structures and $\mathfrak{A} \in \mathbb{H}^2(\mathsf{K})$. Assume, for simplicity, that there exists $\mathfrak{B} \in \mathbb{H}(\mathsf{K})$, such that $\mathfrak{B} \twoheadrightarrow \mathfrak{A}$. Hence, there exists $\mathfrak{C} \in \mathsf{K}$, such that $\mathfrak{C} \twoheadrightarrow \mathfrak{B}$. Thus, we get $\mathsf{K} \ni \mathfrak{C} \twoheadrightarrow \mathfrak{A}$, yielding $\mathfrak{A} \in \mathbb{H}(\mathsf{K})$.

The same reasoning applies for $\mathbb{O} = \mathbb{R}$ and for $\mathbb{O} = \mathbb{E}$, since the composite of two reductive homomorphisms is a reductive homomorphism.

Let $\mathbb{O} = \mathbb{P}$, K be a class of $\mathcal{L}$-structures and $\mathfrak{A} \in \mathbb{P}^2(\mathsf{K})$. Assume, for simplicity, that there exist $\mathfrak{A}_i \in \mathbb{P}(\mathsf{K})$, such that $\mathfrak{A} = \prod_{i\in I} \mathfrak{A}_i$. Hence, there exist $\mathfrak{A}_{ij} \in \mathsf{K}$, $j \in J_i$, such that $\mathfrak{A}_i = \prod_{j\in J_i} \mathfrak{A}_{ij}$. Thus, we get $\mathfrak{A} = \prod_{i\in I}\prod_{j\in J_i} \mathfrak{A}_{ij}$, yielding $\mathfrak{A} \in \mathbb{P}(\mathsf{K})$.

■

Next we look at how the reduction and expansion operators interact with some of the remaining operators.

Lemma 149 *(a) $\mathbb{SE} \le \mathbb{ES}$ and $\mathbb{PE} \le \mathbb{EP}$;*

(b) $\mathbb{SR} \le \mathbb{RS}$ and $\mathbb{PR} \le \mathbb{RP}$.

Proof:

(a) Suppose $\mathfrak{A} \in \mathbb{SE}(\mathsf{K})$. Then $\mathfrak{A} \subseteq \mathfrak{C}$ and there exists $h : \mathfrak{C} \twoheadrightarrow_s \mathfrak{B}$, for some $\mathfrak{B} \in \mathsf{K}$. We consider the structure $h(\mathfrak{A})$, with $h(\mathfrak{A}) \subseteq \mathfrak{B}$. Moreover, denoting by $h\restriction_{\mathfrak{A}} : \mathfrak{A} \to h(\mathfrak{A})$, the restriction of h on A, we get that $h\restriction_{\mathfrak{A}} : \mathfrak{A} \twoheadrightarrow_s h(\mathfrak{A})$, since, for every $r \in R$, with $\rho(r) = n$, and all $a_1, \dots, a_n \in A$,

$$\begin{array}{rcll} r^{\mathfrak{A}}(\mathbf{a}) & = & r^{\mathfrak{C}}(\mathbf{a}) & (\mathfrak{A} \subseteq \mathfrak{C}) \\ & = & r^{\mathfrak{B}}(h(\mathbf{a})) & (h : \mathfrak{C} \twoheadrightarrow_s \mathfrak{B}) \\ & = & r^{h(\mathfrak{A})}(h(\mathbf{a})). & (h(\mathfrak{A}) \subseteq \mathfrak{B}) \end{array}$$

This shows that $\mathfrak{A} \in \mathbb{ES}(\mathsf{K})$.

Suppose $\mathfrak{A} \in \mathbb{PE}(\mathsf{K})$. Then $\mathfrak{A} \cong \prod_{i\in I} \mathfrak{A}_i$, such that, there exist $h_i : \mathfrak{A}_i \twoheadrightarrow_s \mathfrak{B}_i$, for some $\mathfrak{B}_i \in \mathsf{K}$, for all $i \in I$. Consider the mapping $h : \prod_{i\in I} \mathfrak{A}_i \to \prod_{i\in I} \mathfrak{B}_i$, defined, for all $\mathbf{a} = \langle a_i : i \in I\rangle \in \prod_{i\in I} A_i$, by

$$h(\mathbf{a}) = \langle h_i(a_i) : i \in I\rangle.$$

This is an $\mathcal{L}$-algebra homomorphism. Further, it is a G-algebra morphism, since, for all $\mathbf{a}, \mathbf{b} \in \prod_{i\in I} A_i$,

$$\begin{array}{rcll} E^{\prod \mathfrak{A}_i}(\mathbf{a}, \mathbf{b}) & = & \bigwedge_{i\in I} E^{\mathfrak{A}_i}(a_i, b_i) & (\text{Definition of } E^{\prod \mathfrak{A}_i}) \\ & \le & \bigwedge_{i\in I} E^{\mathfrak{B}_i}(h_i(a_i), h_i(b_i)) & (h_i : \mathfrak{A}_i \twoheadrightarrow_s \mathfrak{B}_i) \\ & = & E^{\prod \mathfrak{B}_i}(h(\mathbf{a}), h(\mathbf{b})). & (\text{Definition of } E^{\prod \mathfrak{B}_i}) \end{array}$$

Moreover, for all $r \in R$, with $\rho(r) = n$, and all $\mathbf{a}^1, \dots, \mathbf{a}^n \in (\prod_{i\in I} A_i)^n$,

$$\begin{array}{rcl} r^{\prod \mathfrak{B}_i}(h(\mathbf{a}_1), \dots, h(\mathbf{a}_n)) & = & \bigwedge_{i\in I} r^{\mathfrak{B}_i}(h_i(a_{1i}), \dots, h_i(a_{n_i})) \\ & & \quad \text{(Definition of } r^{\prod \mathfrak{B}_i}) \\ & = & \bigwedge_{i\in I} r^{\mathfrak{A}_i}(a_{1i}, \dots, a_{ni}) \\ & & \quad (h_i : \mathfrak{A}_i \twoheadrightarrow_s \mathfrak{B}_i) \\ & = & r^{\prod \mathfrak{A}_i}(\mathbf{a}_1, \dots, \mathbf{a}_n). \\ & & \quad \text{(Definition of } r^{\prod \mathfrak{A}_i}) \end{array}$$

So $h : \prod_{i\in I} \mathfrak{A}_i \twoheadrightarrow_s \prod_{i\in I} \mathfrak{B}_i$, which shows that $\mathfrak{A} \in \mathbb{EP}(\mathsf{K})$.

(b) Suppose $\mathfrak{A} \in \mathbb{SR}(\mathsf{K})$. Then $\mathfrak{A} \subseteq \mathfrak{C}$ and there exists $h : \mathfrak{B} \twoheadrightarrow_s \mathfrak{C}$, for some $\mathfrak{B} \in \mathsf{K}$. By Lemma 125, $\mathfrak{A}' = h^{-1}(\mathfrak{A}) \subseteq \mathfrak{B}$. Moreover, $h{\restriction}_{\mathfrak{A}'} : \mathfrak{A}' \twoheadrightarrow_s \mathfrak{A}$ is a reductive morphism. Thus, $\mathfrak{A} \in \mathbb{RS}(\mathsf{K})$.

Suppose $\mathfrak{A} = \mathbb{PR}(\mathsf{K})$. Then $\mathfrak{A} \cong \prod_{i\in I} \mathfrak{A}_i$ and, for all $i \in I$, there exists $h_i : \mathfrak{B}_i \twoheadrightarrow_s \mathfrak{A}_i$, for some $\mathfrak{B}_i \in \mathsf{K}$. Then, the mapping $h : \prod_{i\in I} B_i \to \prod_{i\in I} A_i$ given, for all $\mathbf{b} \in \prod_{i\in I} B_i$, by

$$h(\mathbf{b}) = \langle h_i(b_i) : i \in I \rangle$$

is such that $h : \prod_{i\in I} \mathfrak{B}_i \twoheadrightarrow_s \prod_{i\in I} \mathfrak{A}_i$ (see Part (a)). Hence, we obtain $\mathfrak{A} \in \mathbb{RP}(\mathsf{K})$.

■

Next we turn to properties involving filter extensions.

Lemma 150 *(a)* $\mathbb{EF} \leq \mathbb{FE}$*;*

(b) $\mathbb{FR} \leq \mathbb{RF} = \mathbb{H}$*;*

(c) $\mathbb{FS} \leq \mathbb{SF}$*.*

Proof:

(a) Suppose $\mathfrak{A} \in \mathbb{EF}(\mathsf{K})$. Let $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{C}$, with $\mathfrak{B} \lesssim \mathfrak{C}$, for some $\mathfrak{B} \in \mathsf{K}$. Consider the structure $h^{-1}(\mathfrak{B})$. For every $r \in R$, with $\rho(r) = n$, and all $a_1, \dots, a_n \in A$,

$$\begin{array}{rcl} r^{h^{-1}(\mathfrak{B})}(\mathbf{a}) & = & r^{\mathfrak{B}}(h(\mathbf{a})) \quad \text{(Definition of } r^{h^{-1}(\mathfrak{B})}) \\ & \leq & r^{\mathfrak{C}}(h(\mathbf{a})) \quad (\mathfrak{B} \lesssim \mathfrak{C}) \\ & = & r^{\mathfrak{A}}(\mathbf{a}). \quad (h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{C}) \end{array}$$

Thus, $h^{-1}(\mathfrak{B}) \lesssim \mathfrak{A}$. Moreover, by definition, $h^{-1}(\mathfrak{B}) \twoheadrightarrow_s \mathfrak{B}$. So we get $\mathfrak{A} \in \mathbb{FE}(\mathsf{K})$.

(b) Suppose that $\mathfrak{A} \in \mathbb{F}\mathbb{R}(\mathsf{K})$. Then $\mathfrak{C} \lesssim \mathfrak{A}$ and there exists $h : \mathfrak{B} \twoheadrightarrow_s \mathfrak{C}$, for some $\mathfrak{B} \in \mathsf{K}$. In this case $\mathfrak{B} \lesssim h^{-1}(\mathfrak{A})$ and $h : h^{-1}(\mathfrak{A}) \twoheadrightarrow_s \mathfrak{A}$. So we get $\mathfrak{A} \in \mathbb{R}\mathbb{F}(\mathsf{K})$.

To conclude this part, we must show that $\mathbb{R}\mathbb{F} = \mathbb{H}$. One direction is straightforward. Suppose $h : \mathfrak{C} \twoheadrightarrow_s \mathfrak{A}$ and that $\mathfrak{B} \lesssim \mathfrak{C}$, for some $\mathfrak{B} \in \mathsf{K}$. Then $h : \mathfrak{B} \twoheadrightarrow \mathfrak{A}$ is an epimorphism and, hence $\mathfrak{A} \in \mathbb{H}(\mathsf{K})$. Assume, conversely, that $\mathfrak{A} \in \mathbb{H}(\mathsf{K})$. Hence, there exists an epimorphism $h : \mathfrak{B} \twoheadrightarrow \mathfrak{A}$, with $\mathfrak{B} \in \mathsf{K}$. We have seen that such an epimorphism factors.

$$\begin{tikzcd}
\mathfrak{B} \arrow[rr, twoheadrightarrow, "h"] \arrow[dr, "i"'] & & \mathfrak{A} \\
& h^{-1}(\mathfrak{A}) \arrow[ur, twoheadrightarrow, "s", "\hat{h}"'] &
\end{tikzcd}$$

In the diagram $\mathfrak{B} \lesssim h^{-1}(\mathfrak{A})$ and $\hat{h} : h^{-1}(\mathfrak{A}) \twoheadrightarrow_s \mathfrak{A}$. Hence, $\mathfrak{A} \in \mathbb{R}\mathbb{F}(\mathsf{K})$.

(c) Suppose that $\mathfrak{A} \in \mathbb{F}\mathbb{S}(\mathsf{K})$. Hence, we have $\mathfrak{C} \lesssim \mathfrak{A}$ and $\mathfrak{C} \subseteq \mathfrak{B}$, for some $\mathfrak{B} \in \mathsf{K}$. We define the $\mathcal{L}$-structure $\mathfrak{D} = \langle \mathcal{B}, R^{\mathfrak{D}} \rangle$, such that $R^{\mathfrak{D}} = \{r^{\mathfrak{D}} : r \in R\}$ is given by setting, for all $r \in R$, with $\rho(r) = n$, and all $b_1, \dots, b_n \in B$,

$$r^{\mathfrak{D}}(\mathbf{b}) = \begin{cases} r^{\mathfrak{A}}(\mathbf{b}), & \text{if } \mathbf{b} \in C, \\ r^{\mathfrak{B}}(\mathbf{b}), & \text{if } \mathbf{b} \notin C. \end{cases}$$

We have $\mathfrak{B} \lesssim \mathfrak{D}$, since, by definition, the underlying algebras are identical and, for all $r \in R$, $r^{\mathfrak{B}} \leq r^{\mathfrak{D}}$. Next, we show that $\mathfrak{A} \subseteq \mathfrak{D}$. First, note that $E^{\mathfrak{A}} = E^{\mathfrak{D}} \upharpoonright_C$, since, for all $a_1, a_2 \in A$,

$$\begin{array}{rcll}
E^{\mathfrak{A}}(a_1, a_2) & = & E^{\mathfrak{C}}(a_1, a_2) & (\mathfrak{C} \lesssim \mathfrak{A}) \\
& = & E^{\mathfrak{B}}(a_1, a_2) & (\mathfrak{C} \subseteq \mathfrak{B}) \\
& = & E^{\mathfrak{D}}(a_1, a_2). & (\mathfrak{D} = \langle \mathcal{B}, R^{\mathfrak{D}} \rangle)
\end{array}$$

Moreover, by definition of $r^{\mathfrak{D}}$, we have, for all $r \in R$, with $\rho(r) = n$, and all $a_1, \dots, a_n \in A$,

$$r^{\mathfrak{D}}(\mathbf{a}) = r^{\mathfrak{A}}(\mathbf{a}).$$

We conclude that $\mathfrak{A} \subseteq \mathfrak{D}$ and, hence, $\mathfrak{A} \in \mathbb{S}\mathbb{F}(\mathsf{K})$.

■

Next, we explore how the image and preimage operators under reductive morphisms $\mathbb{R}$ and $\mathbb{E}$, respectively, interact with the model reduction operator $\mathbb{L}$ and with each other.

Lemma 151 *(a)* $\mathbb{E}\mathbb{L} = \mathbb{E}\mathbb{R} = \mathbb{R}\mathbb{E}$*;*

(b) $\mathbb{L}\mathbb{E} = \mathbb{L}\mathbb{R} = \mathbb{R}\mathbb{L} = \mathbb{L} \leq \mathbb{E}\mathbb{L}$.

Proof:

(a) Suppose that $\mathfrak{A} \in \mathbb{RE}(\mathsf{K})$. Then there exist $h : \mathfrak{C} \twoheadrightarrow_s \mathfrak{A}$ and $g : \mathfrak{C} \twoheadrightarrow_s \mathfrak{B}$, with $\mathfrak{B} \in \mathsf{K}$. Then, by Theorem 146, $\mathfrak{C}^* \cong \mathfrak{A}^*$ and $\mathfrak{C}^* \cong \mathfrak{B}^*$. Hence, there exist $\mathfrak{A} \twoheadrightarrow_s \mathfrak{C}^*$ and $\mathfrak{B} \twoheadrightarrow_s \mathfrak{C}^*$. These show that $\mathfrak{A} \in \mathbb{ER}(\mathsf{K})$.

Suppose, conversely, that $\mathfrak{A} \in \mathbb{ER}(\mathsf{K})$. Then, there exist $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{C}$ and $g : \mathfrak{B} \twoheadrightarrow_s \mathfrak{C}$, with $\mathfrak{B} \in \mathsf{K}$. Let $\mathbf{F}$ be an algebra of $\mathcal{L}$-terms, such that there exists algebra homomorphisms $k : \mathbf{F} \twoheadrightarrow \mathbf{A}$ and $f : \mathbf{F} \twoheadrightarrow \mathbf{B}$, such that the rectangle commutes.

$$\begin{tikzcd}
\mathbf{F} \arrow[r, two heads, "f"] \arrow[d, two heads, "k"'] & \mathbf{B} \arrow[d, two heads, "g"] \\
\mathbf{A} \arrow[r, two heads, "h"'] & \mathbf{C}
\end{tikzcd}$$

Then, clearly, $\langle \mathcal{F}, \{\bot\}_{r\in R}\rangle$, with $\mathcal{F} = \langle \mathbf{F}, \Delta^{\mathbf{F}}\rangle$, is an $\mathcal{L}$-structure and

$$h \circ k = g \circ f : \langle \mathcal{F}, \{\bot\}_{r\in R}\rangle \to \mathfrak{C}$$

is a morphism. Let $\mathfrak{F} = (h \circ k)^{-1}(\mathfrak{C})$. Then, for all $r \in R$, with $\rho(r) = n$, and all $t_1, \dots, t_n \in F$,

$$\begin{array}{rcl}
r^{(h\circ k)^{-1}(\mathfrak{C})}(t_1,\dots,t_n) & = & r^{\mathfrak{C}}(h(k(t_1)),\dots,h(k(t_n))) \\
 & & \quad (\text{Definition of } r^{(h\circ k)^{-1}(\mathfrak{C})}) \\
 & = & r^{\mathfrak{A}}(k(t_1),\dots,k(t_n)) \\
 & & \quad (h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{C})
\end{array}$$

and, similarly,

$$\begin{array}{rcl}
r^{(h\circ k)^{-1}(\mathfrak{C})}(t_1,\dots,t_n) & = & r^{(g\circ f)^{-1}(\mathfrak{C})}(t_1,\dots,t_n) \\
 & = & r^{\mathfrak{C}}(g(f(t_1)),\dots,g(f(t_n))) \\
 & = & r^{\mathfrak{B}}(f(t_1),\dots,f(t_n)).
\end{array}$$

Hence, $k : \mathfrak{F} \twoheadrightarrow_s \mathfrak{A}$ and $f : \mathfrak{F} \twoheadrightarrow_s \mathfrak{B}$. This shows that $\mathfrak{A} \in \mathbb{RE}(\mathsf{K})$.

Finally, notice that $\mathbb{L} \leq \mathbb{R}$. Hence $\mathbb{EL} \leq \mathbb{ER}$. So it suffices to show the reverse inclusion. Assume $\mathfrak{A} \in \mathbb{ER}(\mathsf{K})$. Then, there exist $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{C}$ and $g : \mathfrak{B} \twoheadrightarrow_s \mathfrak{C}$, with $\mathfrak{B} \in \mathsf{K}$. Thus, there exist $\mathfrak{A} \twoheadrightarrow_s \mathfrak{C}^*$ and $\mathfrak{B} \twoheadrightarrow_s \mathfrak{C}^*$. This shows that $\mathfrak{A} \in \mathbb{EL}(\mathsf{K})$.

(b) Clearly, $\mathbb{L} \leq \mathbb{LE}$, $\mathbb{L} \leq \mathbb{LR}$, $\mathbb{L} \leq \mathbb{RL}$ and $\mathbb{L} \leq \mathbb{EL}$. So we must show that the reverse inclusions of the first three also hold.

Suppose $\mathfrak{A} \in \mathbb{LE}(\mathsf{K})$. Then $\mathfrak{A} \cong \mathfrak{C}^*$ and there exists $h : \mathfrak{C} \twoheadrightarrow_s \mathfrak{B}$, for some $\mathfrak{B} \in \mathsf{K}$. Then, by Theorem 146, $\mathfrak{A} \cong \mathfrak{C}^* \cong \mathfrak{B}^*$. Hence, $\mathfrak{A} \in \mathbb{L}(\mathsf{K})$.

Let $\mathfrak{A} \in \mathbb{LR}(\mathsf{K})$. Then $\mathfrak{A} \cong \mathfrak{C}^*$ and there exists $h : \mathfrak{B} \twoheadrightarrow_s \mathfrak{C}$, for some $\mathfrak{B} \in \mathsf{K}$. Using again Theorem 146, we get $\mathfrak{A} \cong \mathfrak{C}^* \cong \mathfrak{B}^*$. Thus, $\mathfrak{A} \in \mathbb{L}(\mathsf{K})$.

Finally, suppose $\mathfrak{A} \in \mathbb{RL}(\mathsf{K})$. Then there exist $h : \mathfrak{C} \twoheadrightarrow_s \mathfrak{A}$ and $\mathfrak{B} \in \mathsf{K}$, such that $\mathfrak{C} \cong \mathfrak{B}^*$. Note that $\mathfrak{C}^* \cong (\mathfrak{B}^*)^* \cong \mathfrak{B}^* \cong \mathfrak{C}$. Thus, by Lemma 131, $\mathrm{Ker}(h) = E^{\mathfrak{C}}$. Thus $\mathfrak{A} \overset{h}{\cong} \mathfrak{C} \cong \mathfrak{B}^*$, showing that $\mathfrak{A} \in \mathbb{L}(\mathsf{K})$.

■

4.12 The Diagram Lemma

This section revisits some standard constructions from model theory. However, they are viewed under the light of languages without equality and in view of the interpretations in G adopted for the relation symbols.

Consider an $\mathcal{L}$-structure $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$. We let $\mathcal{L}_A$ be the expansion of $\mathcal{L}$ resulting by adding to $\mathcal{L}$ new individual constants c_a, for $a \in A$. One uses $\bar{a}$ to denote the sequence of all elements of A in some predetermined order and $\bar{c}$ for the sequence of the corresponding constants. Further, $\mathcal{L}_A$-structures are denoted by $(\mathfrak{B}, b_a)_{a\in A}$, where $\mathfrak{B}$ is an $\mathcal{L}$-structure and $b_a \in B$, for every $a \in A$.

Let $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ be an $\mathcal{L}$-structure. The **diagram** $D(\mathfrak{A})$ of $\mathfrak{A}$ is the collection

$$D(\mathfrak{A}) = \{D^g(\mathfrak{A}) : g \in G\},$$

where $D^g(\mathfrak{A})$ is the set of all atomic sentences φ over $\mathcal{L}_A$ satisfying

$$\varphi^{(\mathfrak{A},a)_{a\in A}} = g.$$

Sometimes we denote this by $(\mathfrak{A}, a)_{a\in A} \vDash_g \varphi$. Moreover, we abbreviate by $(\mathfrak{A}, a)_{a\in A} \vDash D(\mathfrak{A})$ the statement $(\mathfrak{A}, a)_{a\in A} \vDash_g D^g(\mathfrak{A})$, for all $g \in G$.

Define

$$L(\mathfrak{A}) = \{L^g(\mathfrak{A}) : g \in G\},$$

where $L^g(\mathfrak{A})$ is the set of all infinitary conjunctions of Leibniz sentences over $\mathcal{L}_A$

$$\bigwedge_{\psi \text{ Leibniz}} \psi(t(\bar{c}), t'(\bar{c})),$$

such that

$$\Omega(\mathfrak{A})(t^{\mathbf{A}}(\bar{a}), t'^{\mathbf{A}}(\bar{a})) = g.$$

The **Leibniz diagram** $D_\ell(\mathfrak{A})$ of $\mathfrak{A}$, is the collection

$$D_\ell(\mathfrak{A}) = \{D_\ell^g(\mathfrak{A}) : g \in G\},$$

resulting by adding $L(\mathfrak{A})$ to $D(\mathfrak{A})$, that is,

$$D_\ell^g(\mathfrak{A}) = D^g(\mathfrak{A}) \cup L^g(\mathfrak{A}).$$

Lemma 152 *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ *and* $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ *be* $\mathcal{L}$*-structures and let* $h : A \to B$*. If* $(\mathfrak{B}, h(a))_{a \in A} \vDash D_\ell(\mathfrak{A})$*, then* $h^* : \mathfrak{A}^* \rightarrowtail_s \mathfrak{B}^*$*.*

Proof: Suppose $(\mathfrak{B}, h(a))_{a \in A} \vDash_g D_\ell^g(\mathfrak{A})$, for all $g \in G$.

We start by showing that $h^* : A^* \to B^*$ is well-defined. Suppose that, for some $a, a' \in A$, $a^* = a'^*$, i.e., $\Omega(\mathfrak{A})(a, a') = \top$. By Theorem 135, we have

$$\bigwedge_{\psi \text{ Leibniz}} \psi(c_a, c_{a'}) \in D_\ell^\top(\mathfrak{A}).$$

By hypothesis, $\bigwedge_\psi \psi^{\mathfrak{B}}(h(a), h(a')) = \top$. By Theorem 135,

$$\Omega(\mathfrak{B})(h(a), h(a')) = \top.$$

Hence, $h(a)^* = h(a')^*$. Thus, by definition of h^*, $h^*(a^*) = h^*(a'^*)$.

Next, we show that $h^* : \mathbf{A}^* \to \mathbf{B}^*$ is a homomorphism. Let $f \in F$, with $\rho(f) = n$, and $a_1, \ldots, a_n \in A$. Because of the interpretation of the constants in $(\mathfrak{A}, a)_{a \in A}$, we have

$$\bigwedge_{\psi \text{ Leibniz}} \psi(c_{f^{\mathbf{A}}(\mathbf{a})}, f(c_{a_1}, \ldots, c_{a_n})) \in D_\ell^\top(\mathfrak{A}).$$

By hypothesis, $\bigwedge_\psi \psi^{\mathfrak{B}}(h(f^{\mathbf{A}}(\mathbf{a})), f^{\mathbf{B}}(h(\mathbf{a}))) = \top$. Thus, by Theorem 135,

$$\Omega(\mathfrak{B})(h(f^{\mathbf{A}}(\mathbf{a})), f^{\mathbf{B}}(h(\mathbf{a}))) = \top.$$

So $h^* : \mathbf{A}^* \to \mathbf{B}^*$ is a homomorphism.

To see that it is a G-morphism, suppose that $a, a' \in A$. Notice that, since $(\mathfrak{B}, h(a))_{a \in A} \vDash D_\ell(\mathfrak{A})$, we have, $\Omega(\mathfrak{A})(a, a') = \Omega(\mathfrak{B})(h(a), h(a'))$. Thus, we obtain

$$\begin{array}{lll}
E^{\mathfrak{A}^*}(a^*, a'^*) & = & \overline{\Omega(\mathfrak{A})}(a^*, a'^*) \quad (\text{Definition of } E^{\mathfrak{A}^*}) \\
 & = & \Omega(\mathfrak{A})(a, a') \quad (\text{Definition of } \overline{\Omega(\mathfrak{A})}) \\
 & = & \Omega(\mathfrak{B})(h(a), h(a')) \quad (\text{Comment above}) \\
 & = & \overline{\Omega(\mathfrak{B})}(h(a)^*, h(a')^*) \quad (\text{Definition of } \overline{\Omega(\mathfrak{B})}) \\
 & = & E^{\mathfrak{B}^*}(h^*(a^*), h^*(a'^*)). \quad (\text{Definition of } E^{\mathfrak{B}^*} \text{ and } h^*)
\end{array}$$

Let us see now that h^* is a strict morphism from $\mathfrak{A}^*$ to $\mathfrak{B}^*$. Suppose $r \in R$, with $\rho(r) = n$, and $a_1, \ldots, a_n \in A$, such that $r^{\mathfrak{A}}(a_1, \ldots, a_n) = g$. Then we have $r(c_{a_1}, \ldots, c_{a_n}) \in D_\ell^g(\mathfrak{A})$. Thus, by hypothesis, $r^{\mathfrak{B}}(h(a_1), \ldots, h(a_n)) = g$. This proves that, for all $r \in R$, with $\rho(r) = n$, and all $a_1, \ldots, a_n \in A$,

$$\begin{array}{lll}
r^{\mathfrak{A}^*}(\mathbf{a}^*) & = & r^{\mathfrak{A}}(\mathbf{a}) \quad (\text{Definition of } r^{\mathfrak{A}^*}) \\
 & = & r^{\mathfrak{B}}(h(\mathbf{a})) \quad (\text{Shown above}) \\
 & = & r^{\mathfrak{B}^*}(h(\mathbf{a})^*) \quad (\text{Definition of } r^{\mathfrak{B}^*}) \\
 & = & r^{\mathfrak{B}^*}(h^*(\mathbf{a}^*)), \quad (\text{Definition of } h^*)
\end{array}$$

that is, $h^* : \mathfrak{A}^* \to_s \mathfrak{B}^*$.

From the fact that, for all $a, a' \in A$,

$$E^{\mathfrak{A}^*}(a^*, a'^*) = E^{\mathfrak{B}^*}(h^*(a^*), h^*(a'^*)),$$

which was shown above, it follows that $h^* : A^* \rightarrowtail B^*$ is an embedding. ■

In the reverse direction, we obtain

Lemma 153 *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}} \rangle$ *and* $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}} \rangle$ *be* $\mathcal{L}$*-structures and let* $h : \mathfrak{A} \to \mathfrak{B}$*, such that* $h^* : \mathfrak{A}^* \rightarrowtail_s \mathfrak{B}^*$*. Then* $(\mathfrak{B}, h(a))_{a \in A} \vDash D_\ell(\mathfrak{A})$*.*

Proof: Assume that $h : \mathfrak{A} \to \mathfrak{B}$ and that $h^* : \mathfrak{A}^* \rightarrowtail_s \mathfrak{B}^*$.

Suppose, first, that $r \in R$, with $\rho(r) = n$, and $t_1(c_{\mathbf{a}}), \ldots, t_n(c_{\mathbf{a}})$ are closed terms in $\mathcal{L}_A$, such that $r(t_1, \ldots, t_n) \in D^g(\mathfrak{A})$. Thus, $r^{\mathfrak{A}}(t_1^{\mathbf{A}}(\mathbf{a}), \ldots, t_n^{\mathbf{A}}(\mathbf{a})) = g$. Now we compute

$$\begin{array}{rcl}
r^{\mathfrak{B}}(t_1^{\mathbf{B}}(h(\mathbf{a})), \ldots, t_n^{\mathbf{B}}(h(\mathbf{a}))) & = & r^{\mathfrak{B}^*}(t_1^{\mathbf{B}^*}(h(\mathbf{a})^*), \ldots, t_n^{\mathbf{B}^*}(h(\mathbf{a})^*)) \\
 & & \quad \text{(Definition of } r^{\mathfrak{B}^*}) \\
 & = & r^{\mathfrak{B}^*}(t_1^{\mathbf{B}^*}(h^*(\mathbf{a}^*)), \ldots, t_n^{\mathbf{B}^*}(h^*(\mathbf{a}^*))) \\
 & & \quad \text{(Definition of } h^*) \\
 & = & r^{\mathfrak{B}^*}(h^*(t_1^{\mathbf{A}^*}(\mathbf{a}^*)), \ldots, h^*(t_n^{\mathbf{A}^*}(\mathbf{a}^*))) \\
 & & \quad (h^* : \mathbf{A}^* \rightarrowtail \mathbf{B}^*) \\
 & = & r^{\mathfrak{A}^*}(t_1^{\mathbf{A}^*}(\mathbf{a}^*), \ldots, t_n^{\mathbf{A}^*}(\mathbf{a}^*)) \\
 & & \quad (h^* : \mathfrak{A}^* \rightarrowtail_s \mathfrak{B}^*) \\
 & = & r^{\mathfrak{A}}(t_1^{\mathbf{A}}(\mathbf{a}), \ldots, t_n^{\mathbf{A}}(\mathbf{a})) \\
 & & \quad \text{(Definition of } r^{\mathfrak{A}^*}) \\
 & = & g. \quad \text{(Hypothesis)}
\end{array}$$

Hence, $(\mathfrak{B}, h(a))_{a \in A} \vDash D(\mathfrak{A})$

Finally, suppose $\bigwedge_{\psi \text{ Leibniz}} \psi(t, t') \in L^g(\mathfrak{A})$. Then, by the definition of $L^g(\mathfrak{A})$,

$$\Omega(\mathfrak{A})(t^{\mathbf{A}}(\mathbf{a}), t'^{\mathbf{A}}(\mathbf{a})) = g.$$

Since $h^* : \mathfrak{A}^* \rightarrowtail_s \mathfrak{B}^*$, we get

$$\Omega(\mathfrak{B})(h(t^{\mathbf{A}}(\mathbf{a})), h(t'^{\mathbf{A}}(\mathbf{a}))) = g.$$

Since $h : \mathfrak{A} \to \mathfrak{B}$,

$$\Omega(\mathfrak{B})(t^{\mathbf{B}}(h(\mathbf{a})), t'^{\mathbf{B}}(h(\mathbf{a}))) = g.$$

Hence, by Theorem 135,

$$\bigwedge_{\psi \text{ Leibniz}} \psi^{\mathfrak{B}}(t^{\mathbf{B}}(h(\mathbf{a})), t'^{\mathbf{B}}(h(\mathbf{a}))) = g.$$

This shows that $(\mathfrak{B}, h(a))_{a \in A} \vDash L(\mathfrak{A})$. We conclude that $(\mathfrak{B}, h(a))_{a \in A} \vDash D_\ell(\mathfrak{A})$. ■

4.13 The Reduction Operator Lemma

In this section we focus on the commutativity properties of the reduction operator $\mathbb{L}$ with respect to other class operators. These are crucial in deriving characterizations of reduced classes of models from corresponding characterizations of full classes (see treatment in [23]).

Lemma 154 *Let* $\mathfrak{A} = \langle \mathbf{A}, E^{\mathfrak{A}}, R^{\mathfrak{A}}\rangle$ *and* $\mathfrak{B} = \langle \mathbf{B}, E^{\mathfrak{B}}, R^{\mathfrak{B}}\rangle$ *be* $\mathcal{L}$*-structures, such that* $\mathfrak{A} \subseteq \mathfrak{B}$*, and* $\Theta \in \mathrm{Gon}(\mathfrak{B})$*. Let* $\Theta_A = \Theta{\restriction_A}$ *(which, by Lemma 132, is a* G*-congruence on* $\mathfrak{A}$*) and define* $h : A/\hat{\Theta}_A \to B/\hat{\Theta}$ *by setting, for all* $a \in A$*,*

$$h(a/\hat{\Theta}_A) = a/\hat{\Theta}.$$

Then $h : \mathfrak{A}/\Theta_A \rightarrowtail_s \mathfrak{B}/\Theta$*.*

Proof: First, h is well-defined. If $a, a' \in A$, such that $\langle a, a'\rangle \in \hat{\Theta}_A$, then, by definition, $\langle a, a'\rangle \in \hat{\Theta}$, whence $h(a/\hat{\Theta}_A) = h(a'/\hat{\Theta}_A)$.

Further, $h : \mathbf{A}/\Theta_A \to \mathbf{B}/\Theta$ is an algebra homomorphism. For $f \in F$, with $\rho(f) = n$, and $a_1, \ldots, a_n \in A$, we have

$$\begin{aligned}
h(f^{\mathbf{A}/\Theta_A}(\mathbf{a}/\hat{\Theta}_A)) &= h(f^{\mathbf{A}}(\mathbf{a})/\hat{\Theta}_A) \quad (\text{Definition of } f^{\mathbf{A}/\Theta_A})\\
&= f^{\mathbf{A}}(\mathbf{a})/\hat{\Theta} \quad (\text{Definition of } h)\\
&= f^{\mathbf{B}}(\mathbf{a})/\hat{\Theta} \quad (\mathfrak{A} \subseteq \mathfrak{B})\\
&= f^{\mathbf{B}/\Theta}(\mathbf{a}/\hat{\Theta}) \quad (\text{Definition of } f^{\mathbf{B}/\Theta})\\
&= f^{\mathbf{B}/\Theta}(h(\mathbf{a}/\hat{\Theta}_A)). \quad (\text{Definition of } h)
\end{aligned}$$

Next, let $a, a' \in A$. Then we have

$$\begin{aligned}
E^{\mathfrak{A}/\Theta_A}(a/\hat{\Theta}_A, a'/\hat{\Theta}_A) &= \bar{\Theta}_A(a/\hat{\Theta}_A, a'/\hat{\Theta}_A) \quad (\text{Definition of } E^{\mathfrak{A}/\Theta_A})\\
&= \Theta_A(a, a') \quad (\text{Definition of } \bar{\Theta}_A)\\
&= \Theta(a, a') \quad (\text{Definition of } \Theta_A)\\
&= \bar{\Theta}(a/\hat{\Theta}, a'/\hat{\Theta}) \quad (\text{Definition of } \bar{\Theta})\\
&= E^{\mathfrak{B}/\Theta}(a/\hat{\Theta}, a'/\hat{\Theta}) \quad (\text{Definition of } E^{\mathfrak{B}/\Theta})\\
&= E^{\mathfrak{B}/\Theta}(h(a/\hat{\Theta}_A), h(a'/\hat{\Theta}_A)). \quad (\text{Definition of } h)
\end{aligned}$$

Hence, h is a morphism of G-algebras. Next, we show that h is a strict homomorphism of $\mathcal{L}$-structures. Let $r \in R$, with $\rho(r) = n$, and $a_1, \ldots, a_n \in n$. Then

$$\begin{aligned}
r^{\mathfrak{A}/\Theta_A}(\mathbf{a}/\hat{\Theta}_A) &= r^{\mathfrak{A}}(\mathbf{a}) \quad (\text{Definition of } r^{\mathfrak{A}/\Theta_A})\\
&= r^{\mathfrak{B}}(\mathbf{a}) \quad (\mathfrak{A} \subseteq \mathfrak{B})\\
&= r^{\mathfrak{B}/\Theta}(\mathbf{a}/\hat{\Theta}) \quad (\text{Definition of } r^{\mathfrak{B}/\Theta})\\
&= r^{\mathfrak{B}/\Theta}(h(\mathbf{a}/\hat{\Theta}_A)). \quad (\text{Definition of } h)
\end{aligned}$$

Since, as was shown above, for all $a, a' \in A$,

$$E^{\mathfrak{A}/\Theta_A}(a/\hat{\Theta}_A, a'/\hat{\Theta}_A) = E^{\mathfrak{B}/\Theta}(h(a/\hat{\Theta}_A), h(a'/\hat{\Theta}_A)),$$

we get that $h : \mathfrak{A}/\theta_A \rightarrowtail_s \mathfrak{B}/\theta$. ■

Theorem 155 (Reduction Operator Lemma) *We have* $\mathbb{LS} = \mathbb{LSL}$ *and* $\mathbb{LP} = \mathbb{LPL}$*, that is,* $\mathbb{S}^* = \mathbb{S}^*\mathbb{L}$ *and* $\mathbb{P}^* = \mathbb{P}^*\mathbb{L}$*.*

Proof: Suppose $\mathfrak{A} \in \mathbb{S}^*(\mathsf{K})$. Then $\mathfrak{A} \cong \mathfrak{C}^*$, for some $\mathfrak{C}$, such that $\mathfrak{C} \subseteq \mathfrak{B}$, for some $\mathfrak{B} \in \mathsf{K}$. Consider $\Omega(\mathfrak{B})$ and define $\Omega(\mathfrak{B})_C = \Omega(\mathfrak{B})\restriction_C$. By Lemma 154, $h : \mathfrak{C}/\Omega(\mathfrak{B})_C \rightarrowtail_s \mathfrak{B}^*$. By Theorem 141, $h : \mathfrak{C}/\Omega(\mathfrak{B})_C \cong h(\mathfrak{C}/\Omega(\mathfrak{B})_C)$. Moreover, by the definition of $h(\mathfrak{C}/\Omega(\mathfrak{B})_C)$, $h(\mathfrak{C}/\Omega(\mathfrak{B})_C) \subseteq \mathfrak{B}^*$. By Theorem 146, $(\mathfrak{C}/\Omega(\mathfrak{B})_C)^* \cong \mathfrak{C}^*$. So we have

$$\mathfrak{A} \cong \mathfrak{C}^* \cong (\mathfrak{C}/\Omega(\mathfrak{B})_C)^* \quad \text{and} \quad \mathfrak{C}/\Omega(\mathfrak{B})_C \cong h(\mathfrak{C}/\Omega(\mathfrak{B})_C) \subseteq \mathfrak{B}^*.$$

Therefore, we obtain $\mathfrak{A} \in \mathbb{S}^*\mathbb{L}(\mathsf{K})$.

Suppose, conversely, that $\mathfrak{A} \in \mathbb{S}^*\mathbb{L}(\mathsf{K})$. Then $\mathfrak{A} \cong \mathfrak{C}^*$, with $\mathfrak{C} \subseteq \mathfrak{B}^*$, for some $\mathfrak{B} \in \mathsf{K}$. Consider the canonical projection $\pi : \mathfrak{B} \twoheadrightarrow_s \mathfrak{B}^*$. Then, by Lemma 125, $\pi^{-1}(\mathfrak{C}) \subseteq \mathfrak{B}$ and, moreover, $\pi\restriction_{\pi^{-1}(\mathfrak{C})} : \pi^{-1}(\mathfrak{C}) \twoheadrightarrow_s \mathfrak{C}$. Hence, by Theorem 146, $\mathfrak{A} \cong \mathfrak{C}^* \cong \pi^{-1}(\mathfrak{C})^* \in \mathbb{S}^*(\mathsf{K})$.

For the second equality, the key property to be proven is that, for every collection $\mathfrak{A}_i$, $i \in I$,

$$\left(\prod_{i\in I}\mathfrak{A}_i\right)^* \cong \left(\prod_{i\in I}\mathfrak{A}_i^*\right)^*.$$

Suppose this isomorphism has been shown. Then we have

$$\begin{array}{lll}\mathfrak{A} \in \mathbb{P}^*(\mathsf{K}) & \text{iff} & \mathfrak{A} \cong (\prod_{i\in I}\mathfrak{A}_i)^*, \text{ for some } \mathfrak{A}_i \in \mathsf{K}, i \in I,\\ & \text{iff} & \mathfrak{A} \cong (\prod_{i\in I}\mathfrak{A}_i^*)^*, \text{ for some } \mathfrak{A}_i \in \mathsf{K}, i \in I,\\ & \text{iff} & \mathfrak{A} \in \mathbb{P}^*\mathbb{L}(\mathsf{K}).\end{array}$$

Thus, the displayed isomorphism suffices to show $\mathbb{P}^* = \mathbb{P}^*\mathbb{L}$. So we turn to proving the displayed isomorphism. We first define

$$h : \prod_{i\in I}\mathfrak{A}_i^* \twoheadrightarrow_s \left(\prod_{i\in I}\mathfrak{A}_i\right)^*.$$

For all $\mathbf{a} = \langle a_i : i \in I\rangle \in \prod_{I\in I}\mathfrak{A}_i$, let

$$h(\langle a_i^* : i \in I\rangle) = \langle a_i : i \in I\rangle^*.$$

We abbreviate this as $h(\mathbf{a}^*) = (\mathbf{a})^*$. To see that this is well defined, suppose $\mathbf{a}^* = \mathbf{b}^*$, i.e., $a_i^* = b_i^*$, $i \in I$. Consider an arbitrary atomic $\mathcal{L}$-formula $\varphi(x, z_1, \dots, z_k)$ and elements $\mathbf{c}_1, \dots, \mathbf{c}_k \in \prod_{i\in I} A_i$. We obtain

$$\begin{array}{ll}\multicolumn{2}{l}{\varphi^{\prod\mathfrak{A}_i}(\mathbf{a}, \mathbf{c}_1, \dots, \mathbf{c}_k) = g}\\ \text{iff} & \bigwedge_{i\in I}\varphi^{\mathfrak{A}_i}(a_i, c_{1i}, \dots, c_{ki}) = g \quad (\text{Definition of } \varphi^{\prod\mathfrak{A}_i})\\ \text{iff} & \bigwedge_{i\in I}\varphi^{\mathfrak{A}_i^*}(a_i^*, c_{1i}^*, \dots, c_{ki}^*) = g \quad (\text{Definition of } \varphi^{\mathfrak{A}_i^*})\\ \text{iff} & \bigwedge_{i\in I}\varphi^{\mathfrak{A}_i^*}(b_i^*, c_{1i}^*, \dots, c_{ki}^*) = g \quad (a_i^* = b_i^*, i \in I)\\ \text{iff} & \bigwedge_{i\in I}\varphi^{\mathfrak{A}_i}(b_i, c_{1i}, \dots, c_{ki}) = g \quad (\text{Definition of } \varphi^{\mathfrak{A}_i^*})\\ \text{iff} & \varphi^{\prod\mathfrak{A}_i}(\mathbf{b}, \mathbf{c}_1, \dots, \mathbf{c}_k) = g. \quad (\text{Definition of } \varphi^{\prod\mathfrak{A}_i})\end{array}$$

By Theorem 135, $(\mathbf{a})^* = (\mathbf{b})^*$, showing that h is well defined.

Next, we show that h is an algebra homomorphism. We have, for all $f \in F$, with $\rho(f) = n$, and all $\mathbf{a}_1, \dots, \mathbf{a}_n \in \prod_{i \in I} A_i$,

$$\begin{aligned}
&h(f^{\Pi \mathfrak{A}_i^*}(\mathbf{a}_1^*, \dots, \mathbf{a}_n^*))\\
&\quad = h(\langle f^{\mathfrak{A}_i^*}(a_{1i}^*, \dots, a_{ni}^*) : i \in I\rangle) \quad \text{(Definition of } f^{\Pi \mathfrak{A}_i^*})\\
&\quad = h(\langle f^{\mathfrak{A}_i}(a_{1i}, \dots, a_{ni})^* : i \in I\rangle) \quad \text{(Definition of } f^{\mathfrak{A}_i^*})\\
&\quad = \langle f^{\mathfrak{A}_i}(a_{1i}, \dots, a_{ni}) : i \in I\rangle^* \quad \text{(Definition of } h)\\
&\quad = f^{\Pi \mathfrak{A}_i}(\mathbf{a}_1, \dots, \mathbf{a}_n)^* \quad \text{(Definition of } f^{\Pi \mathfrak{A}_i})\\
&\quad = f^{(\Pi \mathfrak{A}_i)^*}((\mathbf{a}_1)^*, \dots, (\mathbf{a}_n)^*) \quad \text{(Definition of } f^{(\Pi \mathfrak{A}_i)^*})\\
&\quad = f^{(\Pi \mathfrak{A}_i)^*}(h(\mathbf{a}_1^*), \dots, h(\mathbf{a}_n^*)). \quad \text{(Definition of } h)
\end{aligned}$$

To see that h is a G-algebra morphism, let $\mathbf{a}, \mathbf{b} \in \prod_{i \in I} A_i$. Then

$$\begin{aligned}
E^{\Pi \mathfrak{A}_i^*}(\mathbf{a}^*, \mathbf{b}^*) &= \textstyle\bigwedge_{i \in I} E^{\mathfrak{A}_i^*}(a_i^*, b_i^*) \quad \text{(Definition of } E^{\Pi \mathfrak{A}_i^*})\\
&= \textstyle\bigwedge_{i \in I} E^{\mathfrak{A}_i}(a_i, b_i) \quad \text{(Definition of } E^{\mathfrak{A}_i^*})\\
&= E^{\Pi \mathfrak{A}_i}(\mathbf{a}, \mathbf{b}) \quad \text{(Definition of } E^{\Pi \mathfrak{A}_i})\\
&= E^{(\Pi \mathfrak{A}_i)^*}((\mathbf{a})^*, (\mathbf{b})^*) \quad \text{(Definition of } E^{(\Pi \mathfrak{A}_i)^*})\\
&= E^{(\Pi \mathfrak{A}_i)^*}(h(\mathbf{a}^*), h(\mathbf{b}^*)). \quad \text{(Definition of } h)
\end{aligned}$$

Let us, finally, show that h is a strict morphism. Consider $r \in R$, with $\rho(r) = n$, and $\mathbf{a}_1, \dots, \mathbf{a}_n \in \prod_{i \in I} A_i$. Then

$$\begin{aligned}
&r^{(\Pi \mathfrak{A}_i)^*}(h(\mathbf{a}_1^*), \dots, h(\mathbf{a}_n^*))\\
&\quad = r^{(\Pi \mathfrak{A}_i)^*}((\mathbf{a}_1)^*, \dots, (\mathbf{a}_n)^*) \quad \text{(Definition of } h)\\
&\quad = r^{\Pi \mathfrak{A}_i}(\mathbf{a}_1, \dots, \mathbf{a}_n) \quad \text{(Definition of } r^{(\Pi \mathfrak{A}_i)^*})\\
&\quad = \textstyle\bigwedge_{i \in I} r^{\mathfrak{A}_i}(a_{1i}, \dots, a_{ni}) \quad \text{(Definition of } r^{\Pi \mathfrak{A}_i})\\
&\quad = \textstyle\bigwedge_{i \in I} r^{\mathfrak{A}_i^*}(a_{1i}^*, \dots, a_{ni}^*) \quad \text{(Definition of } r^{\mathfrak{A}_i^*})\\
&\quad = r^{\Pi \mathfrak{A}_i^*}(\mathbf{a}_1^*, \dots, \mathbf{a}_n^*) \quad \text{(Definition of } r^{\Pi \mathfrak{A}_i^*})
\end{aligned}$$

By Theorem 146, $h^* : (\prod_{i \in I} \mathfrak{A}_i^* / \mathcal{F})^* \cong (\prod_{i \in I} \mathfrak{A}_i / \mathcal{F})^*$.

■

4.14 Universal Atomic Classes

Let K be a class of $\mathcal{L}$-structures. K is a **universal atomic class** if $\mathsf{K} = \mathrm{Mod}(\Gamma)$, for some set $\Gamma = \{\Gamma_g : g \in G\}$ of atomic $\mathcal{L}$-formulas, where Γ_g represents formulas that should be evaluated to $\geq g$. Let $\mathrm{Atm}(\mathsf{K}) = \{\mathrm{Atm}_g(\mathsf{K}) : g \in G\}$ be the set of all atomic $\mathcal{L}$-formulas evaluated to $\geq g$ in all members of K. Then K is a universal atomic class if and only if $\mathsf{K} = \mathrm{Mod}(\mathrm{Atm}(\mathsf{K}))$.

Lemma 156 *Let K be a class of $\mathcal{L}$-structures. If K is universal atomic, then it is closed under $\mathbb{H}$, $\mathbb{E}$, $\mathbb{S}$ and $\mathbb{P}$.*

Proof: Since $\mathsf{K} = \mathrm{Mod}(\mathrm{Atm}(\mathsf{K}))$, to show that a structure is in K, it suffices to show that it "satisfies" an arbitrary atomic formula $r(t_1(\bar{x}), \dots, t_n(\bar{x}))$ in $\mathrm{Atm}_g(\mathsf{K})$, for all $g \in G$.

Suppose, first, that $\mathfrak{A} \in \mathbb{H}(\mathsf{K})$. Then, there exists $\mathfrak{B} \in \mathsf{K}$ and $\mathfrak{B} \overset{h}{\twoheadrightarrow} \mathfrak{A}$. Let $r(t_1(\bar{x}), \dots, t_n(\bar{x})) \in \mathrm{Atm}_g(\mathsf{K})$ and $\mathbf{a}$ in A. Since h is surjective, there exists $\mathbf{b}$ in B, such that $h(\mathbf{b}) = \mathbf{a}$. We obtain

$$\begin{array}{rcl} r^{\mathfrak{A}}(t_1^{\mathbf{A}}(\mathbf{a}), \dots, t_n^{\mathbf{A}}(\mathbf{a})) & = & r^{\mathfrak{A}}(t_1^{\mathbf{A}}(h(\mathbf{b})), \dots, t_n^{\mathbf{A}}(h(\mathbf{b}))) \\ & = & r^{\mathfrak{A}}(h(t_1^{\mathbf{B}}(\mathbf{b})), \dots, h(t_n^{\mathbf{A}}(\mathbf{b}))) \quad (h : \mathbf{B} \to \mathbf{A}) \\ & \geq & r^{\mathfrak{B}}(t_1^{\mathbf{B}}(\mathbf{b}), \dots, t_n^{\mathbf{B}}(\mathbf{b})). \quad (h : \mathfrak{B} \to \mathfrak{A}) \\ & \geq & g. \quad (\mathfrak{B} \in \mathsf{K}) \end{array}$$

Hence $\mathfrak{A} \in \mathsf{K}$ and K is closed under morphic images.

Suppose, next, that $\mathfrak{A} \in \mathbb{E}(\mathsf{K})$. Then, there exists $\mathfrak{B} \in \mathsf{K}$ and $\mathfrak{A} \overset{h}{\twoheadrightarrow}_s \mathfrak{B}$. Let $r(t_1(\bar{x}), \dots, t_n(\bar{x})) \in \mathrm{Atm}_g(\mathsf{K})$ and $\mathbf{a}$ in A. We get

$$\begin{array}{rcl} r^{\mathfrak{A}}(t_1^{\mathbf{A}}(\mathbf{a}), \dots, t_n^{\mathbf{A}}(\mathbf{a})) & = & r^{\mathfrak{B}}(h(t_1^{\mathbf{A}}(\mathbf{a})), \dots, h(t_n^{\mathbf{A}}(\mathbf{a}))) \quad (h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}) \\ & = & r^{\mathfrak{B}}(t_1^{\mathbf{B}}(h(\mathbf{a})), \dots, t_n^{\mathbf{B}}(h(\mathbf{a}))) \quad (h : \mathbf{A} \to \mathbf{B}) \\ & \geq & g. \quad (\mathfrak{B} \in \mathsf{K}) \end{array}$$

Hence $\mathfrak{A} \in \mathsf{K}$ and K is closed under expansions.

Suppose, now, that $\mathfrak{A} \in \mathbb{S}(\mathsf{K})$. Then, there exists $\mathfrak{B} \in \mathsf{K}$ and $\mathfrak{A} \subseteq \mathfrak{B}$. We have, for all $r(t_1(\bar{x}), \dots, t_n(\bar{x})) \in \mathrm{Atm}_g(\mathsf{K})$ and $\mathbf{a}$ in A,

$$\begin{array}{rcl} r^{\mathfrak{A}}(t_1^{\mathbf{A}}(\mathbf{a}), \dots, t_n^{\mathbf{A}}(\mathbf{a})) & = & r^{\mathfrak{B}}(t_1^{\mathbf{B}}(\mathbf{a}), \dots, t_n^{\mathbf{B}}(\mathbf{a})) \quad (\mathfrak{A} \subseteq \mathfrak{B}) \\ & \geq & g. \quad (\mathfrak{B} \in \mathsf{K}) \end{array}$$

Hence $\mathfrak{A} \in \mathsf{K}$ and K is closed under substructures.

Suppose, finally, that $\mathfrak{A} \in \mathbb{P}(\mathsf{K})$. Then, there exist $\mathfrak{A}_i \in \mathsf{K}$, $i \in I$, such that $\mathfrak{A} = \prod_{i \in I} \mathfrak{A}_i$. We have, for all $r(t_1(\bar{x}), \dots, t_n(\bar{x})) \in \mathrm{Atm}_g(\mathsf{K})$ and $\mathbf{a} = \langle \mathbf{a}_i : i \in I \rangle$ in A,

$$\begin{array}{rcl} r^{\mathfrak{A}}(t_1^{\mathbf{A}}(\mathbf{a}), \dots, t_n^{\mathbf{A}}(\mathbf{a})) & = & \bigwedge_{i \in I} r^{\mathfrak{A}_i}(t_1^{\mathbf{A}_i}(\mathbf{a}_i), \dots, t_n^{\mathbf{A}_i}(\mathbf{a}_i)) \quad (\text{Definition of } r^{\mathfrak{A}}) \\ & \geq & g. \quad (\mathfrak{A}_i \in \mathsf{K},\ i \in I) \end{array}$$

Hence $\mathfrak{A} \in \mathsf{K}$ and K is closed under products. ■

Theorem 157 *Let K be a class of $\mathcal{L}$-structures. The following statements are equivalent:*

(i) K *is a universal atomic class;*

(ii) K *is closed under* $\mathbb{H}$, $\mathbb{E}$, $\mathbb{S}$ *and* $\mathbb{P}$*;*

(iii) $\mathsf{K} = \mathbb{HESP}(\mathsf{K}')$, *for some class* K' *of* $\mathcal{L}$*-structures.*

Proof:

(i)⇒(ii) By Lemma 156.

(ii)⇒(iii) This is clear by taking $\mathsf{K}' = \mathsf{K}$.

(iii)⇒(i) We have that

$$\mathsf{K} \subseteq \mathrm{Mod}(\mathrm{Atm}(\mathsf{K})) = \mathrm{Mod}(\mathrm{Atm}(\mathbb{HESP}(\mathsf{K}'))) \subseteq \mathrm{Mod}(\mathrm{Atm}(\mathsf{K}')).$$

For the reverse inclusion, suppose $\mathfrak{A} \in \mathrm{Mod}(\mathrm{Atm}(\mathsf{K}'))$. To show that $\mathfrak{A} \in \mathsf{K}$, it suffices, by hypothesis, to show that $\mathfrak{A} \in \mathbb{HESP}(\mathsf{K}')$. Let $\Delta = \{\Delta_g : g \in G\}$, where Δ_g is the set of atomic $\mathcal{L}_A$-sentences $\varphi(c_{\mathbf{a}})$, such that $\varphi^{(\mathfrak{A},a)_{a\in A}} \ngeq g$.

Claim: If $\varphi(c_{\mathbf{a}}) \in \Delta_g$, there exist $\mathfrak{B}_\varphi \in \mathsf{K}'$ and $\{b_{a,\varphi} : a \in A\} \subseteq B_\varphi$, such that

$$\varphi^{(\mathfrak{B}_\varphi, b_{a,\varphi})_{a\in A}} \ngeq g.$$

Suppose to the contrary. Then, for all $\mathfrak{B} \in \mathsf{K}'$ and all $\mathbf{b}$ in B, $\varphi^{\mathfrak{B}}(\mathbf{b}) \geq g$. Hence, $\varphi(\bar{x}) \in \mathrm{Atm}_g(\mathsf{K}')$. Thus, since $\mathfrak{A} \in \mathrm{Mod}(\mathrm{Atm}(\mathsf{K}'))$, $\varphi^{(\mathfrak{A},a)_{a\in A}} \geq g$. This contradicts the hypothesis $\varphi \in \Delta_g$.

Define

$$\begin{array}{rcl} \mathfrak{B} & = & \prod_{\varphi\in\Delta} \mathfrak{B}_\varphi; \\ b_a & = & \langle b_{a,\varphi} : \varphi \in \Delta\rangle, \quad a \in A; \\ \mathfrak{C} & = & \mathfrak{B}\restriction_{\{b_a : a\in A\}}. \end{array}$$

By construction, $\mathfrak{C} \in \mathbb{SP}(\mathsf{K}')$. Also by construction, for all $g \in G$ and all $\varphi \in \Delta_g$, $\varphi^{(\mathfrak{B},b_a)_{a\in A}} \ngeq g$. Thus, for all $g \in G$ and all $\varphi \in \Delta_g$, $\varphi^{(\mathfrak{C},b_a)_{a\in A}} \ngeq g$. Let $V = \{x_a : a \in A\}$ and consider $\mathbf{T} := \mathbf{Tm}_{\mathcal{L}}(V)$, the absolutely free $\mathcal{L}$-algebra generated by V and let $\mathcal{T} = \langle \mathbf{T}, \Delta_{\mathbf{T}}\rangle$. Let $h : \mathcal{T} \to \mathcal{C}$ be specified by

$$h(x_a) = b_a, \quad a \in A.$$

Consider the preimage $h^{-1}(\mathfrak{C})$ of $\mathfrak{C}$ under h. Then $h : h^{-1}(\mathfrak{C}) \twoheadrightarrow_s \mathfrak{C}$.

Claim: $g : \mathcal{T} \to \mathcal{A}$, specified by $x_a \mapsto a$, defines a surjective homomorphism $g : h^{-1}(\mathfrak{C}) \twoheadrightarrow \mathfrak{A}$.

Clearly, $g : \mathcal{T} \to \mathcal{A}$ is a surjective morphism of G-algebras. To see that it is a morphism of structures, consider $r \in R$, with $\rho(r) = n$, and $t_1(x_{a_1}, \dots, x_{a_k}), \dots, t_n(x_{a_1}, \dots, x_{a_k})$ $\mathcal{L}$-terms in k variables. Then

$$\begin{array}{l} r^{h^{-1}(\mathfrak{C})}(t_1(x_{a_1},\dots,x_{a_k}),\dots,t_n(x_{a_1},\dots,x_{a_k})) \\ \qquad = r^{\mathfrak{C}}(t_1^{\mathbf{C}}(b_{a_1},\dots,b_{a_k}),\dots,t_n^{\mathbf{C}}(b_{a_1},\dots,b_{a_k})) \\ \qquad\quad (\text{Definition of } r^{h^{-1}(\mathfrak{C})}) \\ \qquad \leq r^{\mathfrak{A}}(t_1^{\mathbf{A}}(a_1,\dots,a_k),\dots,t_n^{\mathbf{A}}(a_1,\dots,a_k)) \\ \qquad\quad (\varphi \in \Delta_g \text{ implies } \varphi^{(\mathfrak{C},b_a)_{a\in A}} \ngeq g) \\ \qquad = r^{\mathfrak{A}}(g(t_1(x_{a_1},\dots,x_{a_k})),\dots,g(t(x_{a_1},\dots,x_{a_k})). \\ \qquad\quad (\text{Definition of } g) \end{array}$$

This shows that $g : h^{-1}(\mathfrak{C}) \twoheadrightarrow \mathfrak{A}$. We have the following formation.

$$\begin{tikzcd}
h^{-1}(\mathfrak{C}) \arrow[r, two heads, "s"] \arrow[d, two heads] & \mathfrak{C} \arrow[d, "\subseteq"] \\
\mathfrak{A} & \prod_{\varphi \in \Delta} \mathfrak{B}_\varphi
\end{tikzcd}$$

Since $\mathfrak{A}$ is a homomorphic image of $h^{-1}(\mathfrak{C})$, $h^{-1}(\mathfrak{C})$ is an extension of $\mathfrak{C}$ and $\mathfrak{C} \in \mathbb{SP}(\mathsf{K}')$, we get that $\mathfrak{A} \in \mathbb{HESP}(\mathsf{K}')$. Therefore, by the hypothesis, $\mathfrak{A} \in \mathsf{K}$.

■

Remark: The preceding result yields an analog of Birkhoff's Variety Theorem when one specializes to languages having G-equality. In that case reductive homomorphisms are exactly isomorphisms.

Let K be a class of $\mathcal{L}$-algebras. The **(full) universal atomic class generated by** K, denoted K^V, is the class

$$\mathsf{K}^V = \mathrm{Mod}(\mathrm{Atm}(\mathsf{K})).$$

The **reduced universal atomic class generated by** K is the class

$$\mathsf{K}^{V*} = \mathbb{L}(\mathrm{Mod}(\mathrm{Atm}(\mathsf{K}))).$$

Lemma 158 *Let* K *be a class of* $\mathcal{L}$*-structures.*

(a) $\mathsf{K}^V = \mathbb{HESP}(\mathsf{K})$*;*

(b) $\mathsf{K}^{V*} = \mathbb{F}^*\mathbb{ESP}(\mathsf{K})$.

Proof:

(a) By the proof of Theorem 157, we get $\mathrm{Mod}(\mathrm{Atm}(\mathsf{K})) = \mathbb{HESP}(\mathsf{K})$. So, by definition, $\mathsf{K}^V = \mathbb{HESP}(\mathsf{K})$.

(b) It suffices to show that $\mathbb{LHESP} = \mathbb{F}^*\mathbb{ESP}$. We have

$$\begin{array}{rcll}
\mathbb{LHESP} & = & \mathbb{LRFESP} & \text{(by Lemma 150)} \\
 & = & \mathbb{LFESP} & \text{(by Lemma 151)} \\
 & = & \mathbb{F}^*\mathbb{ESP}. &
\end{array}$$

This proves Part (b).

■

Open Questions

There are many problems that we have not addresed concerning the framework developed here (or, possibly, some modified version of it, if more appropriate and/or convenient). To mention the few very obvious ones, inspired by classical model theory and the adaptation in [23], how could one work to obtain analogs of the characterization of classes axiomatized (in some way) by arbitrary first-order sentences, by universal sentences or by universal Horn sentences?