Chapter 3

Graded Algebraic Semantics for Graded Logics

3.1 Introduction

Our goal in this chapter is to extend the results of Font and Jansana [28] in order to apply to arbitrary graded logics. The setting is inspired by the work of Diaconescu on the formalization of many-valued logics as institutions [21]. In order to accomplish this, we first develop a machinery involving graded models (G-models) and graded congruences (G-congruences). An attempt at a theory on the same premises, paralleling the work of Blok and Pigozzi rather than the one of Font and Jansana was presented in Chapter 2.

We now provide an outline of the contents section-by section.

In Section 3.2, we define *G-algebras* and develop the machinery needed to deal with those algebras, much like the basic machinery of Universal Algebra that allows manipulating ordinary algebras. More precisely, we introduce *G-morphisms* and *G-congruences* and prove analogs of the Homomorphism, the Second Isomorphism and the Correspondence Theorems. We also introduce *subalgebras* and *direct products* and define *subdirect products*. We conclude with a characterization theorem for subdirect products involving quotient G-algebras, akin to the well known characterization of subdirect products of ordinary algebras involving quotient algebras.

In Section 3.3, we adapt the notion of logical filter and logical matrix to the graded context, thus obtaining the notions of *G-filter* and of *G-matrix*, respectively. We also define the key concept of the *Leibniz G-congruence* of a G-matrix and prove an analog of the well known characterization theorem of Blok and Pigozzi (Theorem 1.5 of [6]). We show that Leibniz G-congruences commute with inverse surjective G-morphisms. We define *reduced G-matrices*. The Leibniz G-congruence of a G-matrix allows passing from a given G-matrix to its *reduction*. Similarly, a *reduced class* is a class of G-matrices obtained by another class by applying the process of reduction to all its members.

In Section 3.4, *graded G-logics* are introduced, essentially as closure-like operators on the set of all functions from an underlying G-algebra to the grade set G. That is, they are operators satisfying the properties of inflationarity, monotonicity, idempotency and translation (an analog of structurality) with respect to the ordering induced by G. As with ordinary closure operators, they turn out to be in one-to-one correspondence with the corresponding collections of *closed functions* from the underlying G-algebra into G. One may also define an ordering on G-logics over the same underlying G-algebra by comparing the action of their corresponding G-operators or, equivalently, by comparing (in the reverse inclusion ordering) the collections of their closed functions.

In Section 3.5, *logical G-congruences* are introduced. A *logical G-congruence* on a G-logic is a G-congruence on the underlying G-algebra that is compatible with every closed G-set, or function, of the G-logic. The greatest such congruence is called the *Tarski G-congruence* of the G-logic. Tarski

G-congruences have a characterization induced by the characterization of Leibniz G-congruences and by the close relationship that governs Leibniz and Tarski G-congruences, both paralleling the standard ones in the context of classical Algebraic Logic. This characterization allows one to immediately conclude that the Tarski operator is monotone on G-logics over the same underlying G-algebra.

In Section 3.6, we discuss *logical G-morphisms*. These are G-morphisms between underlying G-algebras that preserve the logical structure. If the morphism is surjective and both preserves and reflects the logical structure, it is termed a *bilogical G-morphism*. We provide several characterizing conditions for a surjective G-morphism to be a bilogical G-morphism along the lines of the ones presented by Font and Jansana in Proposition 1.4 of [28]. From the characterizing properties one may infer that bilogical G-morphisms establish a correspondence of closed G-sets of the bimorphic G-logics and also isomorphisms of the lattices of corresponding G-logics over the underlying G-algebras. These are also properties known to hold in the traditional context (Proposition 1.5 and Corollary 1.6 of [28]). Also known in the conventional framework is the property that, roughly speaking, Tarski G-congruences are preserved under inverse bilogical G-morphisms. The section closes with a quick look at (*logical*) *G-isomorphisms*, which are bijective logical G-morphisms whose inverses are also logical G-morphisms. They are characterized as being bilogical G-morphisms which are G-algebra isomorphisms.

In Section 3.7, we use logical G-congruences to define *quotients* of G-logics. These quotients and accompanying *quotient G-morphisms* are used to establish analogs of the Homomorphism Theorems of Universal Algebra for G-logics. Analogs are obtained of the *Homomorphism*, of the *Second Isomorphism* and of the *Correspondence Theorems*. We also define *reduced G-logics* and the *reduction* of a G-logic, which is the quotient of the G-logic by its Tarski G-congruence. Several results involving reductions are established. E.g., it is shown that the reduction of a quotient G-logic is isomorphic to the reduction of the original G-logic and that two G-logics related via a bilogical G-morphism have isomorphic reductions.

In Section 3.8, we introduce *sentential G-logics*, that is, G-logics whose underlying G-algebras are formula algebras. These can be interpreted, as in the ordinary theory, either via G-matrices or via G-logic models (G-models, for simplicity). We explore both types and study some basic properties, in particular interactions with bilogical G-morphisms and the effect of reductions. Finally we discuss *completeness* of a sentential G-logic with respect to special classes of G-models.

Section 3.9 is devoted to a more in-depth study of *G-models* of a sentential G-logic. These models play the role that abstract logics, serving as models of sentential logics, play in the traditional setting of [28]. The property of being a G-model is preserved under bilogical G-morphisms. Several completeness results are revisited. The section closes with a characterization of G-models as those G-logics whose families of closed G-sets consist entirely of filters of the G-logic. This is a standard result, well known in the context of sentential logics (Proposition 2.7 of [28]), and serves as the main connecting thread between G-matrix models and G-logic models of a given sentential G-logic.

Section 3.10 focuses on a special class of G-models, named *full G-models*. These are G-models whose reductions have as collection of closed G-sets the entire collection of G-filters on the quotient G-algebra. It is shown that any G-model whose collection of closed G-sets are of this form is in fact a full G-model. Such models are called *basic full G-models*. The class of all full G-models is closed in both the forward and backward directions (that is, both under direct and inverse images) under bilogical G-morphisms. Exploiting these results, the class of all full G-models is characterized as the smallest class of G-logics containing all basic full G-models and closed under both images and preimages of bilogical G-morphisms. This parallels the situation in the traditional setting (Corollary 2.13 of [28]).

In Section 3.11, $\mathbb{S}$-*algebras* are introduced. They constitute the G-algebraic reducts of reduced full models of a sentential G-logic $\mathbb{S}$. The class of $\mathbb{S}$-algebras may be characterized, without reference to full models, as the class of G-algebraic reducts of all reduced G-models of $\mathbb{S}$. Moreover, it is an abstract class, i.e., closed under isomorphisms. Other characterizations are also provided, relating, e.g., $\mathbb{S}$-algebras with full G-models and bilogical G-morphisms. The section revisits the issue of completeness. It is shown that a sentential G-logic $\mathbb{S}$ is complete with respect to the classes of full G-models, basic full G-models and reduced full G-models. These completeness results refine the standard coarser results asserting completeness with respect to the classes of all G-models and of all reduced G-models. The section concludes by explicating the relations between the classes $\mathrm{Alg}(\mathbb{S})$ of $\mathbb{S}$-algebras and of $\mathrm{Alg}^*(\mathbb{S})$ of G-algebraic reducts of reduced G-matrix models of $\mathbb{S}$ and by comparing corresponding classes for sentential G-logics over the same logical signature.

Section 3.12 is the one detailing the work specifically targeted towards obtaining an analog of the Isomorphism Theorem 2.30 of Font and Jansana [28]. A sentential G-logic $\mathbb{S}$ is given and a G-algebra $\mathcal{A}$ is fixed. For any G-congruence Θ on $\mathcal{A}$, one considers the quotient G-morphism $\mathcal{A} \to \mathcal{A}/\Theta$ and the full G-model $\langle \mathcal{A}/\Theta, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)\rangle$, whose closed functions consist of all $\mathbb{S}$-filters on the quotient G-algebra. This model generates via the quotient morphism a full G-model $\widetilde{H}_{\mathcal{A}}(\Theta)$ on $\mathcal{A}$. The Isomorphism Theorem establishes an isomorphism between the lattice of all full G-models of $\mathbb{S}$ on $\mathcal{A}$ and the lattice of all $\mathrm{Alg}(\mathbb{S})$-G-congruences on $\mathcal{A}$. Moreover, this isomorphism is given by the Tarski operator and its inverse is the operator $\Theta \mapsto \widetilde{H}_{\mathcal{A}}(\Theta)$, described above.

In the last three sections, Sections 3.13, 3.14 and 3.15, we introduce specific classes of G-logics in the algebraic hierarchy. Section 3.13 deals with *protoalgebraic G-logics*, which were cursorily introduced in Section 2.7 and are characterized by the monotonicity of the Leibniz operator. They form the class corresponding to the protoalgebraic sentential logics of Blok and Pigozzi [5] (see also [15]). Section 3.15, on the other hand, introduces *weakly algebraizable* and *algebraizable G-logics*, corresponding to weakly algebraizable [16] and algebraizable [6, 33, 34] deductive systems. They form a narrower class than protoalgebraic logics and require that, in addition to being monotone, the Leibniz operator be injective and injective plus join continuous, respectively. The study of injectivity in the context of monotonicity, which passes through the notion of a *Leibniz G-filter* of a protoalgebraic logic, is the subject of Section 3.14. The study in Sections 3.13 and 3.15 of the few classes of the Leibniz hierarchy of G-logics does not go in depth, since they are only meant to indicate how the machinery developed in earlier sections may be used in this setting. Only a few of their basic properties, closely related or inferred by the definitions and known from the traditional context, are revisited and adapted to the setting of G-logics.

3.2 Graded Algebras

We fix again a complete lattice $\mathbf{G} = \langle G, \leq \rangle$ (sometimes with additional structure, as needed). Recall that, for any set A, the set G^A of functions $X : A \to G$, is ordered by

$$X \leq Y \quad \text{iff} \quad X(a) \leq Y(a), \text{ for all } a \in A.$$

As was the case in Chapter 2, this ordering will also play a critical role in the sequel.

Let $\mathcal{L}$ be a fixed (but arbitrary) language, consisting of logical connectives or algebraic operation symbols, depending on the point of view, with attached arities. We work with algebras $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$ of similarity type $\mathcal{L}$. The set of all homomorphisms from $\mathbf{A}$ to $\mathbf{B}$ is denoted $\mathrm{Hom}(\mathbf{A}, \mathbf{B})$. We write $h : \mathbf{A} \to \mathbf{B}$ to signify that $h \in \mathrm{Hom}(\mathbf{A}, \mathbf{B})$.

Recall that a *graded congruence* or *G-congruence* Θ on $\mathbf{A}$ is a function

$$\Theta : A^2 \to G,$$

such that, for all $a, b, c \in A$, all operation symbols $\lambda \in \mathcal{L}$, of arity n, and all $a_1, b_1, \ldots, a_n, b_n \in A$,

(**Reflexivity**) $\Theta(a, a) = \top$;

(**Symmetry**) $\Theta(a, b) = \Theta(b, a)$;

(**Transitivity**) $\Theta(a, b) \wedge \Theta(b, c) \leq \Theta(a, c)$;

(**Congruence**) $\Theta(a_1, b_1) \wedge \cdots \wedge \Theta(a_n, b_n) \leq \Theta(\lambda^{\mathbf{A}}(a_1, \ldots, a_n), \lambda^{\mathbf{A}}(b_1, \ldots, b_n))$.

The set of all G-congruences of $\mathbf{A}$ is denoted by $\mathrm{Gon}(\mathbf{A})$. It is naturally ordered by setting, for all $\Theta, \Theta' \in \mathrm{Gon}(\mathbf{A})$,

$$\Theta \leq \Theta' \quad \text{iff} \quad \Theta(a,b) \leq \Theta'(a,b), \text{ for all } a,b \in A.$$

The set $\mathrm{Gon}(\mathbf{A})$, equipped with $\leq$, forms a complete lattice, denoted

$$\mathbf{Gon}(\mathbf{A}) = \langle \mathrm{Gon}(\mathbf{A}), \leq \rangle.$$

Its least element is the G-congruence $\Delta_{\mathbf{A}} : A^2 \to G$, with

$$\Delta_{\mathbf{A}}(a,b) = \begin{cases} \top, & \text{if } a = b, \\ \bot, & \text{if } a \neq b, \end{cases}$$

and its largest element is the G-congruence $\nabla_{\mathbf{A}} : A^2 \to G$, with

$$\nabla_{\mathbf{A}}(a,b) = \top, \text{ for all } a,b \in A.$$

A G-congruence Θ on $\mathbf{A}$ is called **reduced** if, for all $a,b \in A$,

$$\Theta(a,b) = \top \quad \text{if and only if} \quad a = b.$$

Consider an arbitrary G-congruence Θ on $\mathbf{A}$. Recall, from Section 2.5, the stratified congruence $\hat{\Theta} = \{\hat{\Theta}_g : g \in G\}$ associated with G. Here, we use extensively the $\top$-stratum $\hat{\Theta}_\top$ of $\hat{\Theta}$. So by slightly abusing (or, rather, overloading) notation, we set

$$\hat{\Theta} := \hat{\Theta}_\top = \{\langle a,b\rangle \in A^2 : \Theta(a,b) = \top\}.$$

By Proposition 10, $\hat{\Theta}$ is a congruence on $\mathbf{A}$. Construct the quotient $\mathbf{A}/\hat{\Theta}$ and define on it a function

$$\bar{\Theta} : (A/\hat{\Theta})^2 \to G$$

by

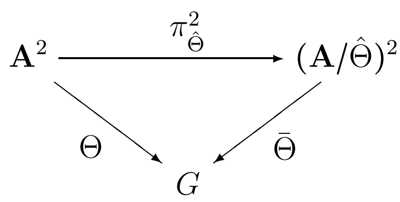
$$\bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) = \Theta(a,b).$$

The function $\bar{\Theta}$ is well defined. Consider $a,a',b,b' \in A$, such that $\langle a,a'\rangle \in \hat{\Theta}$ and $\langle b,b'\rangle \in \hat{\Theta}$. Then

$$\begin{array}{rcl} \Theta(a',b') & = & \top \wedge \Theta(a',b') \wedge \top \\ & = & \Theta(a,a') \wedge \Theta(a',b') \wedge \Theta(b',b) \\ & & (\langle a,a'\rangle \in \hat{\Theta} \text{ and } \langle b,b'\rangle \in \hat{\Theta}) \\ & \leq & \Theta(a,b), \quad \text{(Transitivity)} \end{array}$$

whence, by symmetry, $\Theta(a,b) = \Theta(a',b')$.

Proposition 50 *Let* $\mathbf{A}$ *be an algebra and* Θ *a* G*-congruence on* $\mathbf{A}$*. Then* $\bar{\Theta}$ *is a reduced* G*-congruence on* $\mathbf{A}/\hat{\Theta}$ *and*



$$\Theta = \bar{\Theta} \circ \pi^2_{\hat{\Theta}},$$

where $\pi_{\hat{\Theta}} : \mathbf{A} \to \mathbf{A}/\hat{\Theta}$ *is the natural quotient homomorphism.*

Proof: That $\bar{\Theta}$ is a G-congruence is a consequence of the fact that Θ is a G-congruence. E.g., for Transitivity, given $a, b, c \in A$, we get

$$\begin{array}{rcl} \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) \wedge \bar{\Theta}(b/\hat{\Theta}, c/\hat{\Theta}) & = & \Theta(a,b) \wedge \Theta(b,c) \quad \text{(Definition of } \bar{\Theta}) \\ & \leq & \Theta(a,c) \quad \text{(Transitivity)} \\ & = & \bar{\Theta}(a/\hat{\Theta}, c/\hat{\Theta}). \quad \text{(Definition of } \bar{\Theta}) \end{array}$$

To see that $\bar{\Theta}$ is reduced, let $a, b \in A$. Then

$$\begin{array}{rll} \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) = \top & \text{iff} & \Theta(a,b) = \top \quad \text{(Definition of } \bar{\Theta}) \\ & \text{iff} & \langle a, b \rangle \in \hat{\Theta} \quad \text{(Definition of } \hat{\Theta}) \\ & \text{iff} & a/\hat{\Theta} = b/\hat{\Theta}. \end{array}$$

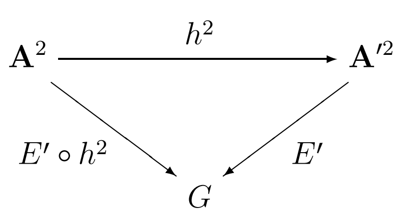
Finally, for all $a, b \in A$, $\bar{\Theta}(\pi_{\hat{\Theta}}(a), \pi_{\hat{\Theta}}(b)) = \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) = \Theta(a,b)$. ■

A **reduced graded algebra** or **reduced** G**-algebra** is a pair $\mathcal{A} = \langle \mathbf{A}, E \rangle$, where:

- $\mathbf{A}$ is an $\mathcal{L}$-algebra;
- E is a reduced G-congruence on $\mathbf{A}$.

Since in this chapter, as opposed to Chapter 2, we are going to be dealing exclusively with reduced G-algebras, to simplify discussion we again (hopefully not perilously) overload terminology and refer to reduced G-algebras simply as G**-algebras**. Note that every $\mathcal{L}$-algebra $\mathbf{A}$ may be viewed as a G-algebra, namely, the G-algebra $\mathcal{A} = \langle \mathbf{A}, \Delta_{\mathbf{A}} \rangle$.

Let $\mathcal{A} = \langle \mathbf{A}, E \rangle$ and $\mathcal{A}' = \langle \mathbf{A}', E' \rangle$ be G-algebras. A G**-algebra morphism** or G**-morphism** $h : \mathcal{A} \to \mathcal{A}'$ is an algebra homomorphism $h : \mathbf{A} \to \mathbf{A}'$, such that



$$E \leq E' \circ h^2.$$

Note that $E' \circ h^2$ is a G-congruence on $\mathbf{A}$. E.g., for Transitivity, for all $a, b, c \in A$,

$$E'(h(a), h(b)) \wedge E'(h(b), h(c)) \leq E'(h(a), h(c))$$

follows by the transitivity of E' on $\mathbf{A}'$.

A G-congruence Θ on $\mathbf{A}$ is said to be a **G-congruence** on $\mathcal{A} = \langle \mathbf{A}, E \rangle$ if

$$E \leq \Theta.$$

We denote the set of all G-congruences on $\mathcal{A}$ by $\mathrm{Gon}(\mathcal{A})$.

Proposition 51 *Let $\mathcal{A} = \langle \mathbf{A}, E \rangle$ be a G-algebra. The set $\mathrm{Gon}(\mathcal{A})$ is a principal filter of $\mathbf{Gon}(\mathbf{A})$ and, hence, $\mathbf{Gon}(\mathcal{A}) = \langle \mathrm{Gon}(\mathcal{A}), \leq \rangle$ is a complete lattice.*

Proof: By definition

$$\mathrm{Gon}(\mathcal{A}) = \{\Theta \in \mathrm{Gon}(\mathbf{A}) : E \leq \Theta\}.$$

Thus, $\mathrm{Gon}(\mathcal{A})$ is clearly a principal filter of $\mathbf{Gon}(\mathbf{A})$. ∎

Let $\mathcal{A} = \langle \mathbf{A}, E \rangle$ be a G-algebra and $\Theta \in \mathrm{Gon}(\mathcal{A})$. The **quotient G-algebra** $\mathcal{A}/\Theta$ of $\mathcal{A}$ by Θ is defined by

$$\mathcal{A}/\Theta = \langle \mathbf{A}/\hat{\Theta}, \bar{\Theta} \rangle.$$

Proposition 50 ensures that this is a well-defined G-algebra. Moreover, it shows that

$$\pi_\Theta : \mathcal{A} \to \mathcal{A}/\Theta$$

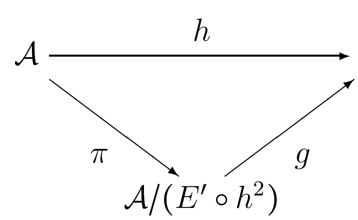
is a G-morphism. It is called the **quotient G-morphism**.

The preceding machinery allows us to obtain, for G-algebras, G-congruences and G-morphisms analogs of the well known Homomorphism Theorems of Universal Algebra [9, 38, 2]. First, we revisit the Homomorphism Theorem.

Theorem 52 (Homomorphism) *Let $\mathcal{A} = \langle \mathbf{A}, E \rangle$ and $\mathcal{A}' = \langle \mathbf{A}', E' \rangle$ be G-algebras and $h : \mathcal{A} \to \mathcal{A}'$ a surjective G-morphism. Then $E' \circ h^2$ is a G-congruence on $\mathcal{A}$ and*

$$\mathcal{A}/(E' \circ h^2) \cong \mathcal{A}'$$

by means of a unique G-isomorphism g satisfying commutativity of



where $\pi : \mathcal{A} \to \mathcal{A}/(E' \circ h^2)$ denotes the quotient G-morphism.

Proof: We have, for all $a, b \in A$,

$$\begin{array}{rll}(E' \circ h^2)(a,b) = \top & \text{iff} & E'(h(a),h(b)) = \top \quad \text{(Composition)}\\ & \text{iff} & h(a) = h(b) \quad (E' \text{ reduced})\\ & \text{iff} & \langle a,b\rangle \in \mathrm{Ker}(h). \quad (\text{Definition of } \mathrm{Ker}(h))\end{array}$$

Therefore, by the Homomorphism Theorem of Universal Algebra, there exists a unique isomorphism

$$\begin{array}{rrcl} g: & \mathbf{A}/\widehat{E' \circ h^2} & \longrightarrow & \mathbf{A}';\\ & a/\widehat{E' \circ h^2} & \longmapsto & h(a).\end{array}$$

It suffices now to show that this algebra isomorphism is also a G-algebra isomorphism, i.e., that it satisfies $E' \circ g^2 = \overline{E' \circ h^2}$. We have, for all $a, b \in A$,

$$\begin{array}{rcl}(E' \circ g^2)(a/\widehat{E' \circ h^2}, b/\widehat{E' \circ h^2}) & = & E'(g(a/\widehat{E' \circ h^2}), g(b/\widehat{E' \circ h^2}))\\ & & \quad \text{(Composition)}\\ & = & E'(h(a),h(b))\\ & & \quad (\text{Definition of } g)\\ & = & \overline{E' \circ h^2}(a/\widehat{E' \circ h^2}, b/\widehat{E' \circ h^2}).\\ & & \quad (\text{Definition of } \overline{E' \circ h^2})\end{array}$$

Therefore, $g : \mathcal{A}/(E' \circ h^2) \cong \mathcal{A}'$. ■

The analog of the Second Isomorphism Theorem comes next. Suppose $\mathcal{A} = \langle \mathbf{A}, E\rangle$ is a G-algebra and $\Theta, \Theta' \in \mathrm{Gon}(\mathcal{A})$, such that $\Theta \le \Theta'$. On $\mathcal{A}/\Theta = \langle \mathbf{A}/\hat{\Theta}, \bar{\Theta}\rangle$, we define the function

$$\Theta'/\Theta : (A/\hat{\Theta})^2 \to G,$$

given, for all $a, b \in A$, by

$$\Theta'/\Theta(a/\hat{\Theta}, b/\hat{\Theta}) = \Theta'(a,b).$$

This is well defined, since, for all $a, a', b, b' \in A$, such that $\langle a, a'\rangle \in \hat{\Theta}$ and $\langle b, b'\rangle \in \hat{\Theta}$, we have $\Theta(a,a') = \top$ and $\Theta(b,b') = \top$, and, hence,

$$\begin{array}{rcl}\Theta'(a,b) & = & \top \wedge \Theta'(a,b) \wedge \top\\ & = & \Theta(a',a) \wedge \Theta'(a,b) \wedge \Theta(b,b')\\ & \le & \Theta'(a',a) \wedge \Theta'(a,b) \wedge \Theta'(b,b')\\ & \le & \Theta'(a',b').\end{array}$$

Thus, by symmetry, $\Theta'(a,b) = \Theta'(a',b')$.

Theorem 53 (Second Isomorphism) *Let* $\mathcal{A} = \langle \mathbf{A}, E\rangle$ *be a* G*-algebra and* $\Theta, \Theta' \in \mathrm{Gon}(\mathcal{A})$*, such that* $\Theta \le \Theta'$*. Then* $\Theta'/\Theta \in \mathrm{Gon}(\mathcal{A}/\Theta)$ *and*

$$(\mathcal{A}/\Theta)/(\Theta'/\Theta) \cong \mathcal{A}/\Theta'$$

via $(a/\hat{\Theta})/\widehat{\Theta'/\Theta} \mapsto a/\hat{\Theta}'$.

Proof: We first show that Θ'/Θ is a G-congruence on $\mathcal{A}/\Theta$. This is essentially a direct consequence of the fact that Θ' is a G-congruence on $\mathcal{A}$. We have, for all $a,b,c \in A$, all n-ary $\lambda \in \mathcal{L}$ and all $a_1,b_1,\ldots,a_n,b_n \in A$:

- For Reflexivity,

$$\begin{array}{rcll}\Theta'/\Theta(a/\hat{\Theta},a/\hat{\Theta}) & = & \Theta'(a,a) & (\text{Definition of } \Theta'/\Theta)\\ & = & \top. & (\Theta' \in \mathrm{Gon}(\mathcal{A}))\end{array}$$

- For Symmetry,

$$\begin{array}{rcll}\Theta'/\Theta(a/\hat{\Theta},b/\hat{\Theta}) & = & \Theta'(a,b) & (\text{Definition of } \Theta'/\Theta)\\ & = & \Theta'(b,a) & (\Theta' \in \mathrm{Gon}(\mathcal{A}))\\ & = & \Theta'/\Theta(b/\hat{\Theta},a/\hat{\Theta}). & (\text{Definition of } \Theta'/\Theta)\end{array}$$

- For Transitivity,

$$\begin{array}{rcl}\Theta'/\Theta(a/\hat{\Theta},b/\hat{\Theta}) \wedge \Theta'/\Theta(b/\hat{\Theta},c/\hat{\Theta}) & = & \Theta'(a,b) \wedge \Theta'(b,c)\\ & & \quad (\text{Definition of } \Theta'/\Theta)\\ & \leq & \Theta'(a,c)\\ & & \quad (\Theta' \in \mathrm{Gon}(\mathcal{A}))\\ & = & \Theta'/\Theta(a/\hat{\Theta},c/\hat{\Theta}).\\ & & \quad (\text{Definition of } \Theta'/\Theta)\end{array}$$

- Finally, for Congruence,

$$\begin{array}{l}\bigwedge_{i=1}^n \Theta'/\Theta(a_i/\hat{\Theta},b_i/\hat{\Theta})\\ \qquad = \bigwedge_{i=1}^n \Theta'(a_i,b_i)\\ \qquad\quad (\text{Definition of } \Theta'/\Theta)\\ \qquad \leq \Theta'(\lambda^{\mathbf{A}}(a_1,\ldots,a_n),\lambda^{\mathbf{A}}(b_1,\ldots,b_n))\\ \qquad\quad (\Theta' \in \mathrm{Gon}(\mathcal{A}))\\ \qquad = \Theta'/\Theta(\lambda^{\mathbf{A}}(a_1,\ldots,a_n)/\hat{\Theta},\lambda^{\mathbf{A}}(b_1,\ldots,b_n)/\hat{\Theta})\\ \qquad\quad (\text{Definition of } \Theta'/\Theta)\\ \qquad = \Theta'/\Theta(\lambda^{\mathbf{A}/\hat{\Theta}}(a_1/\hat{\Theta},\ldots,a_n/\hat{\Theta}),\lambda^{\mathbf{A}/\hat{\Theta}}(b_1/\hat{\Theta},\ldots,b_n/\hat{\Theta})).\\ \qquad\quad (\text{Definition of } \lambda^{\mathbf{A}/\hat{\Theta}})\end{array}$$

Additionally, for all $a,b \in A$,

$$\begin{array}{rcll}\bar{\Theta}(a/\hat{\Theta},b/\hat{\Theta}) & = & \Theta(a,b) & (\text{Definition of } \bar{\Theta})\\ & \leq & \Theta'(a,b) & (\text{Hypothesis})\\ & = & \Theta'/\Theta(a/\hat{\Theta},b/\hat{\Theta}). & (\text{Definition of } \Theta'/\Theta)\end{array}$$

Now we conclude that $\Theta'/\Theta \in \mathrm{Gon}(\mathcal{A}/\Theta)$. Next, for all $a,b \in A$, we have

$$\begin{array}{rll}\langle a/\hat{\Theta},b/\hat{\Theta}\rangle \in \widehat{\Theta'/\Theta} & \text{iff} & \Theta'/\Theta(a/\hat{\Theta},b/\hat{\Theta}) = \top \quad (\text{Definition of } \widehat{\Theta'/\Theta})\\ & \text{iff} & \Theta'(a,b) = \top \quad (\text{Definition of } \Theta'/\Theta)\\ & \text{iff} & \langle a,b\rangle \in \hat{\Theta}' \quad (\text{Definition of } \hat{\Theta}')\\ & \text{iff} & \langle a/\hat{\Theta},b/\hat{\Theta}\rangle \in \hat{\Theta}'/\hat{\Theta}. \quad (\text{Definition of } \hat{\Theta}'/\hat{\Theta})\end{array}$$

This proves that $\widehat{\Theta'/\Theta} = \hat{\Theta}'/\hat{\Theta}$. Thus, by the Second Isomorphism Theorem of Universal Algebra,

$$(a/\hat{\Theta})(\hat{\Theta}'/\hat{\Theta}) \mapsto a/\hat{\Theta}'$$

is an isomorphism $(\mathbf{A}/\hat{\Theta})/\widehat{\Theta'/\Theta} \cong \mathbf{A}/\hat{\Theta}'$.

It only remains to show that this algebra isomorphism is actually a G-algebra isomorphism. We have, for all $a, b \in A$,

$$\begin{array}{rcl}\overline{\Theta'/\Theta}((a/\hat{\Theta})/\widehat{\Theta'/\Theta}, (b/\hat{\Theta})/\widehat{\Theta'/\Theta}) & = & \Theta'/\Theta(a/\hat{\Theta}, b/\hat{\Theta}) \\ & = & \Theta'(a, b) \\ & = & \bar{\Theta}'(a/\hat{\Theta}', b/\hat{\Theta}').\end{array}$$

Hence $(\mathcal{A}/\Theta)/(\Theta'/\Theta) \cong \mathcal{A}/\Theta'$ via $(a/\hat{\Theta})(\hat{\Theta}'/\hat{\Theta}) \mapsto a/\hat{\Theta}'$. ∎

We may now formulate an analog of the Correspondence Theorem.

Theorem 54 (Correspondence Theorem) *Let $\mathcal{A} = \langle \mathbf{A}, E \rangle$ be a G-algebra and $\Theta \in \mathrm{Gon}(\mathcal{A})$. The segment $[\Theta, \nabla_{\mathcal{A}}]$ of $\mathbf{Gon}(\mathcal{A})$ is isomorphic to the lattice $\mathbf{Gon}(\mathcal{A}/\Theta)$ by the mapping $\Theta' \mapsto \Theta'/\Theta$.*

Proof: Suppose $\Theta' \in \mathrm{Gon}(\mathcal{A})$ is such that $\Theta \le \Theta'$. By the Second Isomorphism Theorem 53, we have $\Theta'/\Theta \in \mathrm{Gon}(\mathcal{A}/\Theta)$. Conversely, consider $\Theta'' \in \mathrm{Gon}(\mathcal{A}/\Theta)$. Define $\tilde{\Theta}'' : A^2 \to G$ by setting, for all $a, b \in A$,

$$\tilde{\Theta}''(a, b) = \Theta''(a/\hat{\Theta}, b/\hat{\Theta}).$$

We show that $\tilde{\Theta}'' \in \mathrm{Gon}(\mathbf{A})$. We have, for all $a, b, c \in A$, all n-ary $\lambda \in \mathcal{L}$ and all $a_1, b_1, \ldots, a_n, b_n \in A$:

- For Reflexivity,

$$\begin{array}{rcl}\tilde{\Theta}''(a, a) & = & \Theta''(a/\hat{\Theta}, a/\hat{\Theta}) \quad (\text{Definition of } \tilde{\Theta}'') \\ & = & \top. \quad (\Theta'' \in \mathrm{Gon}(\mathcal{A}/\Theta))\end{array}$$

- For Symmetry,

$$\begin{array}{rcl}\tilde{\Theta}''(a, b) & = & \Theta''(a/\hat{\Theta}, b/\hat{\Theta}) \quad (\text{Definition of } \tilde{\Theta}'') \\ & = & \Theta''(b/\hat{\Theta}, a/\hat{\Theta}) \quad (\Theta'' \in \mathrm{Gon}(\mathcal{A}/\Theta)) \\ & = & \tilde{\Theta}''(b, a). \quad (\text{Definition of } \tilde{\Theta}'')\end{array}$$

- For Transitivity,

$$\begin{array}{rcl}\tilde{\Theta}''(a, b) \wedge \tilde{\Theta}''(b, c) & = & \Theta''(a/\hat{\Theta}, b/\hat{\Theta}) \wedge \Theta''(b/\hat{\Theta}, c/\hat{\Theta}) \\ & & \quad (\text{Definition of } \tilde{\Theta}'') \\ & \le & \Theta''(a/\hat{\Theta}, c/\hat{\Theta}) \quad (\Theta'' \in \mathrm{Gon}(\mathcal{A}/\Theta)) \\ & = & \tilde{\Theta}''(a, c). \quad (\text{Definition of } \tilde{\Theta}'')\end{array}$$

- For Congruence,

$$\begin{array}{rcl}\bigwedge_{i=1}^n \tilde{\Theta}''(a_i,b_i) & = & \bigwedge_{i=1}^n \Theta''(a_i/\hat{\Theta},b_i/\hat{\Theta}) \quad (\text{Definition of } \tilde{\Theta}'')\\ & \le & \Theta''(\lambda^{\mathbf{A}/\hat{\Theta}}(a_1/\hat{\Theta},\dots,a_n/\hat{\Theta}),\lambda^{\mathbf{A}/\hat{\Theta}}(b_1/\hat{\Theta},\dots,b_n/\hat{\Theta}))\\ & & \quad (\Theta'' \in \mathrm{Gon}(\mathcal{A}/\Theta))\\ & = & \Theta''(\lambda^{\mathbf{A}}(a_1,\dots,a_n)/\hat{\Theta},\lambda^{\mathbf{A}}(b_1,\dots,b_n)/\hat{\Theta})\\ & & \quad (\text{Definition of } \lambda^{\mathbf{A}/\hat{\Theta}})\\ & = & \tilde{\Theta}''(\lambda^{\mathbf{A}}(a_1,\dots,a_n),\lambda^{\mathbf{A}}(b_1,\dots,b_n)).\\ & & \quad (\text{Definition of } \tilde{\Theta}'')\end{array}$$

Next we show $\Theta' \in \mathrm{Gon}(\mathcal{A})$. We have, for all $a,b \in A$,

$$\begin{array}{rcl}\Theta(a,b) & = & \bar{\Theta}(a/\hat{\Theta},b/\hat{\Theta}) \quad (\text{Definition of } \bar{\Theta})\\ & \le & \Theta''(a/\hat{\Theta},b/\hat{\Theta}) \quad (\Theta'' \in \mathrm{Gon}(\mathcal{A}/\Theta))\\ & = & \tilde{\Theta}''(a,b). \quad (\text{Definition of } \tilde{\Theta}'')\end{array}$$

The two mapping above are easily seen to be order preserving and inverses of one another. Indeed, for all $\Theta' \in \mathrm{Gon}(\mathcal{A})$, with $\Theta \le \Theta'$, and all $a,b \in A$,

$$\begin{array}{rcl}\widetilde{\Theta'/\Theta}(a,b) & = & \Theta'/\Theta(a/\hat{\Theta},b/\hat{\Theta}) \quad (\text{Definition of } \widetilde{\Theta'/\Theta})\\ & = & \Theta'(a,b). \quad (\text{Definition of } \Theta'/\Theta)\end{array}$$

And, for all $\Theta'' \in \mathrm{Gon}(\mathcal{A}/\Theta)$ and all $a,b \in A$,

$$\begin{array}{rcl}\tilde{\Theta}''/\Theta(a/\hat{\Theta},b/\hat{\Theta}) & = & \tilde{\Theta}''(a,b) \quad (\text{Definition of } \tilde{\Theta}''/\Theta)\\ & = & \Theta''(a/\hat{\Theta},b/\hat{\Theta}). \quad (\text{Definition of } \tilde{\Theta}'')\end{array}$$

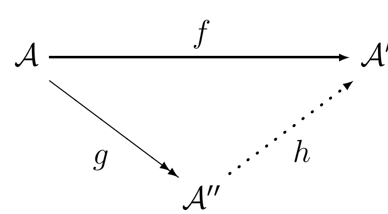
This establishes the claimed correspondence. ∎

We also present a fill-in lemma for G-algebras that will come in handy in the sequel.

Lemma 55 *Let $\mathcal{A} = \langle \mathbf{A}, E\rangle$, $\mathcal{A}' = \langle \mathbf{A}', E'\rangle$ and $\mathcal{A}'' = \langle \mathbf{A}'', E''\rangle$ be G-algebras, $f : \mathcal{A} \to \mathcal{A}'$ a G-morphism and $g : \mathcal{A} \to \mathcal{A}''$ a surjective G-morphism, such that*

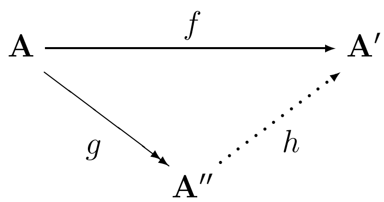
$$E'' \circ g^2 \le E' \circ f^2.$$

Then, there exists a unique G-morphism $h : \mathcal{A}'' \to \mathcal{A}'$, such that

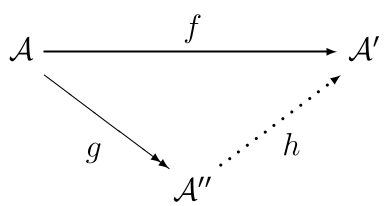


$$h \circ g = f.$$

Proof: Notice that $E'' \circ g^2 \leq E' \circ f^2$ implies $\widehat{E'' \circ g^2} \subseteq \widehat{E' \circ f^2}$, i.e., $\mathrm{Ker}(g) \subseteq \mathrm{Ker}(f)$. Thus, there exists a unique homomorphism $h : \mathbf{A}'' \to \mathbf{A}'$ which makes the following triangle commute.



This gives rise to a commutative diagram of G-algebras.



In fact, for all $a'', b'' \in A''$, we have

$$\begin{array}{rcll} E''(a'', b'') & = & E''(g(a), g(b)) & (g \text{ surjective}) \\ & \leq & E'(f(a), f(b)) & (\text{Hypothesis}) \\ & = & E'(h(g(a)), h(g(b))) & (h \circ g = f) \\ & = & E'(h(a''), h(b'')). & (\text{Choice of } a, b) \end{array}$$

Hence $h : \mathcal{A}'' \to \mathcal{A}'$ is a G-morphism. ■

Another operation on G-algebras that will be needed later is that of taking subdirect products. Let $\mathcal{A}_i = \langle \mathbf{A}_i, E_i \rangle$, $i \in I$, be a collection of G-algebras. The **direct product** of the $\mathcal{A}_i$, $i \in I$, is the G-algebra

$$\prod_{i \in I} \mathcal{A}_i = \left\langle \prod_{i \in I} \mathbf{A}_i, \bigwedge_{i \in I} E_i \right\rangle,$$

where $\prod_{i \in I} \mathbf{A}_i$ is the product $\mathcal{L}$-algebra of the $\mathbf{A}_i$, $i \in I$, and $\bigwedge_{i \in I} E_i$ is the function $\bigwedge_{i \in I} E_i : \prod_{i \in I} A_i \to G$ defined, for all $a_i, b_i \in A_i$, $i \in I$, by

$$\bigwedge_{i \in I} E_i(\bar{a}, \bar{b}) = \bigwedge_{i \in I} E_i(a_i, b_i).$$

We note that this is well defined, i.e., $\bigwedge_{i \in I} E_i$ is a reduced G-congruence on $\prod_{i \in I} \mathbf{A}_i$. E.g., to show Transitivity, let $a_i, b_i, c_i \in A_i$, $i \in I$. Then

$$\begin{array}{rcl} \bigwedge_{i \in I} E_i(\bar{a}, \bar{b}) \wedge \bigwedge_{i \in I} E_i(\bar{b}, \bar{c}) & = & \bigwedge_{i \in I} E_i(a_i, b_i) \wedge \bigwedge_{i \in I} E_i(b_i, c_i) \\ & = & \bigwedge_{i \in I}(E_i(a_i, b_i) \wedge E_i(b_i, c_i)) \\ & \leq & \bigwedge_{i \in I} E_i(a_i, c_i) \\ & = & \bigwedge_{i \in I} E_i(\bar{a}, \bar{c}). \end{array}$$

To show it is reduced, let $a_i, b_i \in A_i$, $i \in I$. Then

$$\begin{array}{rll} \bigwedge_{i\in I} E_i(\bar{a},\bar{b}) = \top & \text{iff} & \bigwedge_{i\in I} E_i(a_i,b_i) = \top \\ & \text{iff} & E_i(a_i,b_i) = \top,\ i \in I, \\ & \text{iff} & a_i = b_i,\ i \in I, \\ & \text{iff} & \bar{a} = \bar{b}. \end{array}$$

Note that the projection $\pi_i : \prod_{i\in I} \mathbf{A}_i \to \mathbf{A}_i$ becomes a G-morphism

$$\pi_i : \prod_{i\in I} \mathcal{A}_i \to \mathcal{A}_i.$$

Indeed, for all $a_i, b_i \in A_i$, $i \in I$,

$$\bigwedge_{i\in I} E_i(\bar{a},\bar{b}) = \bigwedge_{i\in I} E_i(a_i,b_i) \le E_i(a_i,b_i) = (E_i \circ \pi_i)(\bar{a},\bar{b}).$$

$\pi_i : \prod_{i\in I} \mathcal{A}_i \to \mathcal{A}_i$ is called the **i-th projection G-morphism**.

Let $\mathcal{A} = \langle \mathbf{A}, E\rangle$ and $\mathcal{A}' = \langle \mathbf{A}', E'\rangle$ be G-algebras. $\mathcal{A}$ is said to be a **subalgebra** of $\mathcal{A}'$, written $\mathcal{A} \le \mathcal{A}'$, if $\mathbf{A}$ is a subalgebra of $\mathbf{A}'$ and E is the restriction of E' on A^2, i.e., for all $a, b \in A$,

$$E(a,b) = E'(a,b).$$

A G-algebra $\mathcal{A} = \langle \mathbf{A}, E\rangle$ is called a **subdirect product** of the G-algebras $\mathcal{A}_i$, $i \in I$, written $\mathcal{A} \subseteq_{\mathsf{sd}} \prod_{i\in I} \mathcal{A}_i$, if it is a subalgebra of $\prod_{i\in I} \mathcal{A}_i$, such that, for all $i \in I$, $\pi_i(A)$ is surjective, where $\pi_i : \prod_{i\in I} \mathcal{A}_i \to \mathcal{A}_i$ denotes the i-th projection G-morphism. An embedding $j : \mathcal{A} \to \prod_{i\in I} \mathcal{A}_i$ is a **subdirect embedding**, written $j : \mathcal{A} \hookrightarrow_{\mathsf{sd}} \prod_{i\in I} \mathcal{A}_i$, if $j(\mathcal{A})$ is a subdirect product of the $\mathcal{A}_i$, $i \in I$.

We have the following result paralleling the one applicable to subdirect products of ordinary algebras.

Proposition 56 *Let $\mathcal{A} = \langle \mathbf{A}, E\rangle$ be a G-algebra and $\Theta, \Theta_i \in \mathrm{Gon}(\mathcal{A})$, $i \in I$. Then*

$$\begin{array}{rrcl} j : & \mathcal{A}/\Theta & \longrightarrow & \prod_{i\in I} \mathcal{A}/\Theta_i; \\ & a/\hat{\Theta} & \longmapsto & \langle a/\hat{\Theta}_i : i \in I\rangle, \end{array}$$

is a subdirect embedding if and only if $\Theta = \bigwedge_{i\in I} \Theta_i$.

Proof: Suppose, first, that $\Theta = \bigwedge_{i\in I} \Theta_i$. Then, for all $a, b \in A$,

$$\begin{array}{rll} \langle a,b\rangle \in \hat{\Theta} & \text{iff} & \Theta(a,b) = \top \quad (\text{Definition of } \hat{\Theta}) \\ & \text{iff} & \bigwedge_{i\in I} \Theta_i(a,b) = \top \quad (\text{Hypothesis}) \\ & \text{iff} & \Theta_i(a,b) = \top,\ i \in I, \quad (\text{Property of } \bigwedge) \\ & \text{iff} & \langle a,b\rangle \in \hat{\Theta}_i,\ i \in I. \quad (\text{Definition of } \hat{\Theta}_i) \end{array}$$

Thus, j is an algebra embedding. Moreover, for all $a,b \in A$,

$$\begin{array}{rcll}\bar{\Theta}(a/\hat{\Theta},b/\hat{\Theta}) & = & \Theta(a,b) & \text{(Definition of } \bar{\Theta})\\ & = & \bigwedge_{i\in I}\Theta_i(a,b) & \text{(Hypothesis)}\\ & = & \bigwedge_{i\in I}\bar{\Theta}_i(a/\hat{\Theta}_i,b/\hat{\Theta}_i) & \text{(Definition of } \bar{\Theta}_i)\\ & = & \bigwedge_{i\in I}\bar{\Theta}_i\circ j(a/\hat{\Theta},b/\hat{\Theta}). & \text{(Definition of } j)\end{array}$$

Hence, j is a G-algebra embedding. Finally, it is clear that $\pi_i\circ j$ is surjective, for all $i \in I$. So j is a subdirect embedding.

Suppose, conversely, that j is a subdirect embedding. It follows that, for all $a,b \in A$, we have

$$\langle a,b\rangle \in \hat{\Theta} \quad \text{iff} \quad \langle a,b\rangle \in \hat{\Theta}_i,\ i\in I,$$
$$\bar{\Theta}(a/\hat{\Theta},b/\hat{\Theta}) = \bigwedge_{i\in I}\bar{\Theta}_i(a/\hat{\Theta}_i,b/\hat{\Theta}_i).$$

It is easy to see that these two conditions ensure that $\Theta = \bigwedge_{i\in I}\Theta_i$. For all $a,b \in A$,

$$\begin{array}{rcll}\Theta(a,b) & = & \bar{\Theta}(a/\hat{\Theta},b/\hat{\Theta}) & \text{(Definition of } \bar{\Theta})\\ & = & \bigwedge_{i\in I}\bar{\Theta}_i(a/\hat{\Theta}_i,b/\hat{\Theta}_i) & \text{(Display above)}\\ & = & \bigwedge_{i\in I}\Theta_i(a,b). & \text{(Definition of } \bar{\Theta}_i)\end{array}$$

Thus, j is a subdirect embedding if and only if $\Theta = \bigwedge\Theta_i$. ■

Given a class K of G-algebras and a G-algebra $\mathcal{A} = \langle \mathbf{A},E\rangle$ not necessarily in K, a G-congruence $\Theta \in \mathrm{Gon}(\mathcal{A})$ is called a **K-G-congruence** if $\mathcal{A}/\Theta \in \mathsf{K}$. We let $\mathrm{Gon}_{\mathsf{K}}(\mathcal{A})$ denote the set of all K-G-congruences on the G-algebra $\mathcal{A}$. This set ordered by $\leq$ is not generally a lattice.

The absolutely free $\mathcal{L}$-algebra is denoted by $\mathbf{Fm}_{\mathcal{L}}(V) = \langle \mathrm{Fm}_{\mathcal{L}}(V),\mathcal{L}\rangle$, where V is a fixed countably infinite set of variables. This algebra may also be represented as a G-algbera $\mathcal{F}m_{\mathcal{L}}(V) = \langle \mathbf{Fm}_{\mathcal{L}}(V),\Delta_{\mathbf{Fm}_{\mathcal{L}}(V)}\rangle$ by attaching the reduced G-congruence that has value $\top$ on the diagonal and $\bot$ elsewhere. Formulas are denoted by $\varphi,\psi,\chi,\ldots$ and equations, formally pairs of formulas, by $\langle\varphi,\psi\rangle$ or $\varphi\approx\psi$.

Given a G-algebra $\mathcal{A} = \langle \mathbf{A},E\rangle$, an **interpretation** in $\mathcal{A}$ is a G-morphism $h : \mathcal{F}m_{\mathcal{L}}(V) \to \mathcal{A}$. Note that such morphisms coincide with homomorphisms $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$, because of the G-congruence of $\mathcal{F}m_{\mathcal{L}}(V)$. They are completely specified by their values on the variables. Accordingly, given a formula φ, we write $\varphi(\bar{x})$ to mean that the variables appearing in φ are among those listed in $\bar{x}$. Moreover, we write $\varphi^{\mathbf{A}}(\bar{a})$ to denote the element of A resulting by interpreting the variables in $\bar{x}$ by the corresponding elements in $\bar{a}$. This notation is naturally extended to sets of formulas, which are denoted by $\Phi,\Psi,\Gamma,\ldots$.

A substitution $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$ is an endomorphism of $\mathbf{Fm}_{\mathcal{L}}(V)$.

3.3 Graded Matrices

Let $\mathbf{A}$ be an $\mathcal{L}$-algebra. We saw that a *graded filter* or *G-filter* on $\mathbf{A}$ is a function

$$F : A \to G.$$

Recall from Section 2.6 that, given a G-congruence E on $\mathbf{A}$ and a G-filter F on $\mathbf{A}$, E is said to be *compatible with* F, written E comp F, if, for all $a, b \in A$,

$$E(a,b) \wedge F(a) \leq F(b).$$

If E is compatible with F, then F is said to be a *G-filter of* the G-algebra $\mathcal{A} = \langle \mathbf{A}, E \rangle$. In this chapter, we enhance slightly the notion of G-matrix in comparison to the definition we gave in Section 2.4. Here a **graded matrix** or G-**matrix** $\mathfrak{A} = \langle \mathcal{A}, F \rangle$ is a pair consisting of:

- A G-algebra $\mathcal{A} = \langle \mathbf{A}, E \rangle$;
- A G-filter F on $\mathcal{A}$.

A G-**congruence on** $\mathfrak{A}$ is a G-congruence Θ on $\mathcal{A}$ that is compatible with F. That is, it has to satisfy:

- $\Theta \geq E$;
- Θ comp F.

We write $\mathrm{Gon}(\mathfrak{A})$ for the set of all G-congruences on the G-matrix $\mathfrak{A}$.

Let $\mathfrak{A} = \langle \mathcal{A}, F \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, be a G-matrix and $\Theta \in \mathrm{Gon}(\mathfrak{A})$. Consider the quotient G-algebra $\mathcal{A}/\Theta = \langle \mathbf{A}/\hat{\Theta}, \bar{\Theta} \rangle$ and on $\mathcal{A}/\Theta$ define the function $\bar{F} : \mathcal{A}/\hat{\Theta} \to G$ by setting, for all $a \in A$,

$$\bar{F}(a/\hat{\Theta}) = F(a).$$

This function is well defined. If $a, b \in A$, such that $\langle a, a' \rangle \in \hat{\Theta}$, then

$$\begin{array}{rcll} F(a) & = & \top \wedge F(a) & \\ & = & \Theta(a,a') \wedge F(a) & (\langle a, a' \rangle \in \hat{\Theta}) \\ & \leq & F(a'), & (\Theta \in \mathrm{Gon}(\mathfrak{A})) \end{array}$$

whence, by symmetry, $F(a) = F(a')$.

The **quotient** G-**matrix** $\mathfrak{A}/\Theta$ **of** $\mathfrak{A}$ **by** Θ is the G-matrix

$$\mathfrak{A}/\Theta = \langle \mathcal{A}/\Theta, \bar{F} \rangle.$$

To see that this is well defined it must be shown that $\bar{\Theta}$ comp $\bar{F}$. Suppose $a, b \in A$. Then

$$\begin{array}{rcll} \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) \wedge \bar{F}(a/\hat{\Theta}) & = & \Theta(a,b) \wedge F(a) & (\text{Definitions of } \bar{\Theta} \text{ and } \bar{F}) \\ & \leq & F(b) & (\Theta \in \mathrm{Gon}(\mathfrak{A})) \\ & = & \bar{F}(b/\hat{\Theta}). & (\text{Definition of } \bar{F}) \end{array}$$

We can show that, under certain conditions on G, the collection $\mathrm{Gon}(\mathfrak{A})$ of all G-congruences on $\mathfrak{A}$ is a principal ideal of $\mathbf{Gon}(\mathcal{A})$. This was essentially done in Proposition 15. We shall restrict attention to those lattices for which this is the case and, hence, we shall always assume the existence of a largest G-congruence in $\mathrm{Gon}(\mathfrak{A})$. This maximum element is called the **Leibniz G-congruence** of the G-matrix $\mathfrak{A}$, or of the G-filter F on $\mathcal{A}$, and is denoted by $\Omega(\mathfrak{A})$ or $\Omega_{\mathcal{A}}(F)$. The well-known characterization of Blok and Pigozzi (see Page 11 of [6]) of the ordinary Leibniz congruence is generalized as follows. Note that this is a slight improvement over Theorem 16, since our G-matrices here are over G-algebras and not simply over algebras, as was the case in Chapter 2. However, the statement and the proof technique, modulo some details, are almost identical. We give it again for the sake of completeness.

Theorem 57 *Let $\mathfrak{A} = \langle\mathcal{A}, F\rangle$ be a G-matrix. We have, for all $a, b \in A$,*

$$\Omega_{\mathcal{A}}(F)(a,b) = \bigwedge_{\substack{\varphi(x,\bar{z})\in \mathrm{Fm}_{\mathcal{L}}(V)\\ \bar{c}\in A}} F(\varphi^{\mathbf{A}}(a,\bar{c})) \leftrightarrow F(\varphi^{\mathbf{A}}(b,\bar{c})).$$

Proof: Define, for all $a, b \in A$,

$$\Theta(a,b) = \bigwedge_{\substack{\varphi(x,\bar{z})\in \mathrm{Fm}_{\mathcal{L}}(V)\\ \bar{c}\in A}} F(\varphi^{\mathbf{A}}(a,\bar{c})) \leftrightarrow F(\varphi^{\mathbf{A}}(b,\bar{c})).$$

Our goal is to show that $\Omega_{\mathcal{A}}(F) = \Theta$. For the inequality left to right, suppose $a, b \in A$, $\varphi(x,\bar{z}) \in \mathrm{Fm}_{\mathcal{L}}(V)$ and $\bar{c} \in A$. Then

$$\begin{array}{rcl}\Omega_{\mathcal{A}}(F)(a,b) & \leq & \Omega_{\mathcal{A}}(F)(\varphi^{\mathbf{A}}(a,\bar{c}), \varphi^{\mathbf{A}}(b,\bar{c})) \\ & & \quad (\Omega_{\mathcal{A}}(F) \text{ a } G\text{-congruence}) \\ & \leq & F(\varphi^{\mathbf{A}}(a,\bar{c})) \leftrightarrow F(\varphi^{\mathbf{A}}(b,\bar{c})). \\ & & \quad (\Omega_{\mathcal{A}}(F) \text{ compatible with } F)\end{array}$$

Since $\varphi(x,\bar{z}) \in \mathrm{Fm}_{\mathcal{L}}(V)$ and $\bar{c} \in A$ were arbitrary,

$$\Omega_{\mathcal{A}}(F)(a,b) \leq \bigwedge_{\substack{\varphi(x,\bar{z})\in \mathrm{Fm}_{\mathcal{L}}(V)\\ \bar{c}\in A}} F(\varphi^{\mathbf{A}}(a,\bar{c})) \leftrightarrow F(\varphi^{\mathbf{A}}(b,\bar{c})) = \Theta(a,b).$$

By definition, E is also a G-congruence on $\mathbf{A}$ that is compatible with F. Hence, using the same reasoning, with E in place of $\Omega_{\mathcal{A}}(F)$, we obtain that $E \leq \Theta$.

For the reverse inequality, taking into account the property of $\Omega_{\mathcal{A}}(F)$ as the largest G-congruence on $\mathcal{A}$ compatible with F, it suffices to show that Θ is a G-congruence on $\mathcal{A}$ compatible with F. We know that $E \leq \Theta$.

- For all $a \in A$,

$$\Theta(a,a) = \bigwedge_{\varphi,\bar{c}} F(\varphi^{\mathbf{A}}(a,\bar{c})) \leftrightarrow F(\varphi^{\mathbf{A}}(a,\bar{c})) = \top.$$

- For all $a, b \in A$,

$$\begin{array}{rcl}\Theta(a,b) & = & \bigwedge_{\varphi,\bar{c}} F(\varphi^{\mathbf{A}}(a,\bar{c})) \leftrightarrow F(\varphi^{\mathbf{A}}(b,\bar{c})) \\ & = & \bigwedge_{\varphi,\bar{c}} F(\varphi^{\mathbf{A}}(b,\bar{c})) \leftrightarrow F(\varphi^{\mathbf{A}}(a,\bar{c})) \\ & = & \Theta(b,a).\end{array}$$

- For all $a, b, c \in A$,

$$\begin{array}{rcl}\Theta(a,b) \wedge \Theta(b,c) & = & \bigwedge_{\varphi,\bar{e}} F(\varphi^{\mathbf{A}}(a,\bar{e})) \leftrightarrow F(\varphi^{\mathbf{A}}(b,\bar{e})) \\ & & \wedge \bigwedge_{\varphi,\bar{e}} F(\varphi^{\mathbf{A}}(b,\bar{e})) \leftrightarrow F(\varphi^{\mathbf{A}}(c,\bar{e})) \\ & \leq & \bigwedge_{\varphi,\bar{e}}(F(\varphi^{\mathbf{A}}(a,\bar{e})) \leftrightarrow F(\varphi^{\mathbf{A}}(b,\bar{e})) \\ & & \quad \wedge F(\varphi^{\mathbf{A}}(b,\bar{e})) \leftrightarrow F(\varphi^{\mathbf{A}}(c,\bar{e}))) \\ & \leq & \bigwedge_{\varphi,\bar{e}} F(\varphi^{\mathbf{A}}(a,\bar{e})) \leftrightarrow F(\varphi^{\mathbf{A}}(c,\bar{e})) \\ & = & \Theta(a,c).\end{array}$$

- Suppose λ is an n-ary operation symbol in $\mathcal{L}$ and $a_1, b_1, \ldots, a_n, b_n \in \mathcal{L}$. Let us write, for convenience and brevity, $a_{i\ldots j}$ so signify the tuple $a_i, a_{i+1}, \ldots, a_j$. Then we have

$$\begin{array}{rcl}\bigwedge_{i=1}^n \Theta(a_i,b_i) & = & \bigwedge_{i=1}^n \bigwedge_{\varphi,\bar{c}} F(\varphi^{\mathbf{A}}(a_i,\bar{c})) \leftrightarrow F(\varphi^{\mathbf{A}}(b_i,\bar{c})) \\ & \leq & \bigwedge_{i=1}^n \bigwedge_{\varphi,\bar{c}} F(\varphi^{\mathbf{A}}(\lambda^{\mathbf{A}}(b_{1\ldots i-1}, a_{i\ldots n}),\bar{c})) \\ & & \qquad \leftrightarrow F(\varphi^{\mathbf{A}}(\lambda^{\mathbf{A}}(b_{1\ldots i}, a_{i+1\ldots n}),\bar{c})) \\ & \leq & \bigwedge_{\varphi,\bar{c}} \bigwedge_{i=1}^n F(\varphi^{\mathbf{A}}(\lambda^{\mathbf{A}}(b_{1\ldots i-1}, a_{i\ldots n}),\bar{c})) \\ & & \qquad \leftrightarrow F(\varphi^{\mathbf{A}}(\lambda^{\mathbf{A}}(b_{1\ldots i}, a_{i+1\ldots n}),\bar{c})) \\ & \leq & \bigwedge_{\varphi,\bar{c}} F(\varphi^{\mathbf{A}}(\lambda^{\mathbf{A}}(a_1,\ldots,a_n),\bar{c})) \\ & & \qquad \leftrightarrow F(\varphi^{\mathbf{A}}(\lambda^{\mathbf{A}}(b_1,\ldots,b_n),\bar{c})) \\ & = & \Theta(\lambda^{\mathbf{A}}(a_1,\ldots,a_n), \lambda^{\mathbf{A}}(b_1,\ldots,b_n)).\end{array}$$

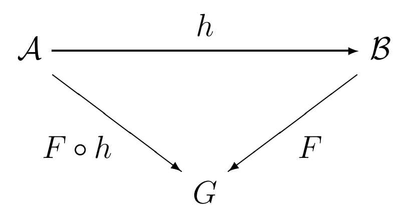
Finally, taking $x \in \mathrm{Fm}_{\mathcal{L}}(V)$ for φ in the definition of Θ, we have, for all $a, b \in A$,

$$\Theta(a,b) \leq F(a) \leftrightarrow F(b),$$

whence Θ is compatible with F. We conclude that $\Theta \leq \Omega_{\mathcal{A}}(F)$. ■

The following result is an analog of the well known property of the traditional theory asserting that the Leibniz operator commutes with inverse surjective homomorphisms.

Theorem 58 *Let $\mathcal{A} = \langle \mathbf{A}, D \rangle$ and $\mathcal{B} = \langle \mathbf{B}, E \rangle$ be G-algebras and $h : \mathcal{A} \to \mathcal{B}$ a surjective G-morphism. For every G-filter F on $\mathcal{B}$, $F \circ h$ is a G-filter on $\mathcal{A}$ and*



$$\Omega_{\mathcal{B}}(F) \circ h^2 = \Omega_{\mathcal{A}}(F \circ h).$$

Proof: We must show that D comp $F \circ h$. We have, for all $a, b \in A$,

$$\begin{array}{rcll} D(a,b) \wedge F(h(a)) & \leq & E(h(a),h(b)) \wedge F(h(a)) & (h : \mathcal{A} \to \mathcal{B}) \\ & \leq & F(h(b)). & (F\ G\text{-filter on } \mathcal{B}) \end{array}$$

For the last equality, let $a, b \in A$. Then, we have

$$\begin{array}{rcl} \Omega_{\mathcal{B}}(F)(h(a),h(b)) & = & \bigwedge_{\varphi,\vec{d}} F(\varphi^{\mathbf{B}}(h(a),\vec{d})) \leftrightarrow F(\varphi^{\mathbf{B}}(h(b),\vec{d})) \\ & & \text{(Theorem 57)} \\ & = & \bigwedge_{\varphi,\vec{c}} F(\varphi^{\mathbf{B}}(h(a),h(\vec{c}))) \leftrightarrow F(\varphi^{\mathbf{B}}(h(b),h(\vec{c}))) \\ & & (h \text{ surjective}) \\ & = & \bigwedge_{\varphi,\vec{c}} F(h(\varphi^{\mathbf{A}}(a,\vec{c})) \leftrightarrow F(h(\varphi^{\mathbf{A}}(b,\vec{c}))) \\ & & (h : \mathbf{A} \to \mathbf{B}) \\ & = & \Omega_{\mathcal{A}}(F \circ h)(a,b). \quad \text{(Theorem 57)} \end{array}$$

Therefore, $\Omega_{\mathcal{B}}(F) \circ h^2 = \Omega_{\mathcal{A}}(F \circ h)$. ∎

The mapping $F \mapsto \Omega_{\mathcal{A}}(F)$ is called the **Leibniz operator** of the G-algebra $\mathcal{A}$.

We say a G-matrix $\mathfrak{A} = \langle \mathcal{A}, F \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, is **reduced** when its only G-matrix congruence is E.

Proposition 59 *Let $\mathfrak{A} = \langle \mathcal{A}, F \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, be a G-matrix. Then $\mathfrak{A}^* := \mathfrak{A}/\Omega_{\mathcal{A}}(F)$ is reduced.*

Proof: To prove that $\mathfrak{A}^*$ is reduced, we need to show that $\Omega_{\mathcal{A}/\Omega_{\mathcal{A}}(F)}(\bar{F})$ is equal to $\bar{\Omega}_{\mathcal{A}}(F)$. Let $a, b \in A$. We have

$$\begin{array}{l} \Omega_{\mathcal{A}/\Omega_{\mathcal{A}}(F)}(\bar{F})(a/\hat{\Omega}_{\mathcal{A}}(F), b/\hat{\Omega}_{\mathcal{A}}(F)) \\ \quad = (\Omega_{\mathcal{A}/\Omega_{\mathcal{A}}(F)}(\bar{F}) \circ \pi^2_{\Omega_{\mathcal{A}}(F)})(a,b) \quad (\text{Definition of } \pi_{\Omega_{\mathcal{A}}(F)}) \\ \quad = \Omega_{\mathcal{A}}(\bar{F} \circ \pi_{\Omega_{\mathcal{A}}(F)})(a,b) \quad \text{(Theorem 58)} \\ \quad = \Omega_{\mathcal{A}}(F)(a,b) \quad (\text{Definition of } \bar{F}) \\ \quad = \bar{\Omega}_{\mathcal{A}}(F)(a/\hat{\Omega}_{\mathcal{A}}(F), b/\hat{\Omega}_{\mathcal{A}}(F)). \quad \text{(Proposition 50)} \end{array}$$

This shows that $\mathfrak{A}^*$ is reduced. ∎

$\mathfrak{A}^*$ is called the **reduction** of $\mathfrak{A}$. Given a class M of G-matrices, we let

$$\mathsf{M}^* = \{\mathfrak{A}^* : \mathfrak{A} \in \mathsf{M}\}.$$

3.4 Graded Logics

Recall the notion of *G-logic* from Section 2.2. Here, we generalize this notion to include logics that are not necessarily over (G-sets on) the formula algebra but on an arbitrary G-algebra. This comes in handy when discussing models, which is the central topic of this chapter.

Let $\mathcal{A} = \langle \mathbf{A}, E \rangle$ be a G-algebra. By a (**closure**) G-**operator** on $\mathcal{A}$, we mean a mapping $C : G^A \to G^A$, such that the following axioms are satisfied:

Inflationarity $X \le C(X)$, for all $X : A \to G$;

Monotonicity $C(X) \le C(Y)$, for all $X, Y : A \to G$, with $X \le Y$;

Idempotency $C(C(X)) = C(X)$, for all $X : A \to G$;

Translation $C(X \circ h) \le C(X) \circ h$, for all $X : A \to G$ and all $h : \mathbf{A} \to \mathbf{A}$.

Given a G-operator C on $\mathcal{A}$, we say that a function $X : A \to G$, or a G-set on A, is **closed**, or that it is a **theory** of C, if

$$C(X) = X.$$

By a **graded logic**, or a G-**logic**, we mean a pair $\mathbb{L} = \langle \mathcal{A}, C \rangle$, where $\mathcal{A} = \langle \mathbf{A}, E \rangle$ is a G-algebra and C is a G-operator on $\mathbf{A}$, such that E is compatible with every closed function X. We denote by $\mathcal{C} = \mathrm{Cl}(\mathbb{L}) = \mathrm{Cl}(\mathrm{C})$ the collection of all closed functions, or closed G-sets, of $\mathbb{L}$. Note that the partial order $\le$ on G^A is inherited by the set of closed functions of $\mathbb{L}$.

Lemma 60 *Let $\mathbb{L} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, be a G-logic. Then, for all $X : A \to G$,*

$$C(X) = \bigwedge \{Y : X \le Y \in \mathrm{Cl}(\mathbb{L})\}.$$

Proof: If $X \le Y \in \mathrm{Cl}(\mathbb{L})$, then, by Monotonicity and closure,

$$C(X) \le C(Y) = Y.$$

Hence, we get $C(X) \le \bigwedge \{Y : X \le Y \in \mathrm{Cl}(\mathbb{L})\}$.

Conversely, since, by Inflationarity and Idempotency, $X \le C(X) \in \mathrm{Cl}(\mathbb{L})$, we get

$$\bigwedge \{Y : X \le Y \in \mathrm{Cl}(\mathbb{L})\} \le C(X).$$

Therefore, $C(X) = \bigwedge \{Y : X \le Y \in \mathrm{Cl}(\mathbb{L})\}$. ∎

Let $\mathbb{L} = \langle \mathcal{A}, C \rangle$ and $\mathbb{L}' = \langle \mathcal{A}, C' \rangle$ be two G-logics over the same G-algebra $\mathcal{A} = \langle \mathbf{A}, E \rangle$. We say that $\mathbb{L}$ is **weaker than** $\mathbb{L}'$ or that $\mathbb{L}'$ is **stronger than** $\mathbb{L}$ or that $\mathbb{L}'$ is an **extension** of $\mathbb{L}$, written $\mathbb{L} \le \mathbb{L}'$, if, for all $X \in G^A$,

$$C(X) \le C'(X).$$

We denote the collection of all G-logics on a G-algebra $\mathcal{A}$ by $\mathrm{Log}_{\mathbf{G}}(\mathcal{A})$. This set forms the ordered structure $\mathbf{Log}_{\mathbf{G}}(\mathcal{A}) = \langle \mathrm{Log}_{\mathbf{G}}(\mathcal{A}), \le \rangle$.

As is the case with ordinary logics, the extension relation between G-logics is characterized by the reverse containment relation between their closed G-sets.

Proposition 61 *Let $\mathcal{A} = \langle \mathbf{A}, E \rangle$ be a G-algebra and $\mathbb{L} = \langle \mathcal{A}, C \rangle$ and $\mathbb{L}' = \langle \mathcal{A}, C' \rangle$ be G-logics over $\mathcal{A}$. Then,*

$$\mathbb{L} \le \mathbb{L}' \quad \textit{iff} \quad \mathrm{Cl}(\mathbb{L}') \subseteq \mathrm{Cl}(\mathbb{L}).$$

Proof: Suppose, first, that $\mathbb{L} \leq \mathbb{L}'$ and let $X \in \mathrm{Cl}(\mathbb{L}')$. To see that $X \in \mathrm{Cl}(\mathbb{L})$, compute

$$C(X) \leq C'(X) = X.$$

Hence, $\mathrm{Cl}(\mathbb{L}') \subseteq \mathrm{Cl}(\mathbb{L})$.

Suppose, conversely, that $\mathrm{Cl}(\mathbb{L}') \subseteq \mathrm{Cl}(\mathbb{L})$ and let $X : A \to G$. Then, using Lemma 60, we get

$$C(X) = \bigwedge\{Y : X \leq Y \in \mathrm{Cl}(\mathbb{L})\} \leq \bigwedge\{Y : X \leq Y \in \mathrm{Cl}(\mathbb{L}')\} = C'(X).$$

Therefore, $\mathbb{L} \leq \mathbb{L}'$. ∎

3.5 Logical Congruences

Let $\mathbb{L} = \langle \mathcal{A}, C\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, be a G-logic. A **logical G-congruence** of $\mathbb{L}$ is a G-congruence Θ on $\mathcal{A}$, such that, for all $X : A \to G$ and all $a, b \in A$,

$$\Theta(a,b) \wedge C(X)(a) \leq C(X)(b).$$

This condition is equivalent to the assertion that Θ is compatible with every closed function in $\mathbb{L}$, i.e., it satisfies, for every $X \in \mathrm{Cl}(\mathbb{L})$ and all $a, b \in A$,

$$\Theta(a,b) \wedge X(a) \leq X(b).$$

The collection of all logical G-congruences of $\mathbb{L}$ is denoted by $\mathrm{Gon}(\mathbb{L})$. From the characterization above, we see that

$$\mathrm{Gon}(\mathbb{L}) = \bigcap\{\mathrm{Gon}(\langle \mathcal{A}, \mathrm{X}\rangle) : \mathrm{X} \in \mathrm{Cl}(\mathbb{L})\}.$$

The set $\mathrm{Gon}(\mathbb{L})$, ordered by $\leq$, is a complete lattice, $\mathbf{Gon}(\mathbb{L}) = \langle \mathrm{Gon}(\mathbb{L}), \leq\rangle$, and a principal ideal of the lattice $\mathbf{Gon}(\mathcal{A})$. We term its generator the **Tarski G-congruence** of $\mathbb{L}$ and denote it by $\widetilde{\Omega}(\mathbb{L})$ or $\widetilde{\Omega}_{\mathcal{A}}(C)$,

$$\widetilde{\Omega}(\mathbb{L}) = \max \mathrm{Gon}(\mathbb{L}).$$

The **Tarski operator on $\mathcal{A}$** is the mapping that associates $\widetilde{\Omega}_{\mathcal{A}}(C)$ to a G-logic $\mathbb{L} = \langle \mathcal{A}, C\rangle$ on $\mathcal{A}$,

$$\widetilde{\Omega}_{\mathcal{A}} : \langle \mathcal{A}, C\rangle \mapsto \widetilde{\Omega}_{\mathcal{A}}(C).$$

From the definition, we have

$$\mathrm{Gon}(\mathbb{L}) = \{\Theta \in \mathrm{Gon}(\mathcal{A}) : \Theta \leq \widetilde{\Omega}(\mathbb{L})\}.$$

Furthermore, for any $\mathbb{L} = \langle \mathcal{A}, C\rangle$, we have

$$\widetilde{\Omega}(\mathbb{L}) = \bigwedge \{\Omega_{\mathcal{A}}(X) : X \in \mathrm{Cl}(\mathbb{L})\}.$$

Directly from Theorem 57 we get the following characterization of the Tarski G-congruence of a G-logic.

Proposition 62 *Let $\mathbb{L} = \langle \mathcal{A}, C\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, be a G-logic. Then, for all $a, b \in A$,*

$$\widetilde{\Omega}(\mathbb{L})(a,b) = \bigwedge_{\substack{X \in \mathrm{Cl}(\mathbb{L}) \\ \varphi(x,\bar{z}) \in \mathrm{Fm}_{\mathcal{L}}(\mathrm{V}) \\ \bar{c} \in A}} X(\varphi^{\mathbf{A}}(a,\bar{c})) \leftrightarrow X(\varphi^{\mathbf{A}}(b,\bar{c})).$$

Proof: Indeed, for every $a, b \in A$, we have

$$\begin{aligned} \widetilde{\Omega}(\mathbb{L})(a,b) &= \bigwedge_{X \in \mathrm{Cl}(\mathbb{L})} \Omega_{\mathcal{A}}(X)(a,b) \\ &\quad \text{(Definition of } \widetilde{\Omega}(\mathbb{L})) \\ &= \bigwedge_{X \in \mathrm{Cl}(\mathbb{L})} \bigwedge_{\substack{\varphi(x,\bar{z}) \in \mathrm{Fm}_{\mathcal{L}}(\mathrm{V}) \\ \bar{c} \in A}} X(\varphi^{\mathbf{A}}(a,\bar{c})) \leftrightarrow X(\varphi^{\mathbf{A}}(b,\bar{c})). \\ &\quad \text{(Theorem 57)} \end{aligned}$$

■

From Proposition 62, it is easily inferred that the Tarski operator on a given G-algebra is monotone with respect to extension ($\leq$) on G-logics and $\leq$ on G-congruences.

Proposition 63 *Let $\mathcal{A} = \langle \mathbf{A}, E\rangle$ be a G-algebra. The Tarski operator $\widetilde{\Omega}_{\mathcal{A}}$ is order preserving on $\mathcal{A}$, i.e., if $\mathbb{L} = \langle \mathcal{A}, C\rangle$ and $\mathbb{L}' = \langle \mathcal{A}, C'\rangle$ are two G-logics on $\mathcal{A}$,*

$$\mathbb{L} \leq \mathbb{L}' \quad \textit{implies} \quad \widetilde{\Omega}(\mathbb{L}) \leq \widetilde{\Omega}(\mathbb{L}').$$

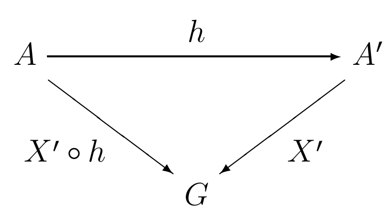
Proof: We have

$$\begin{aligned} \mathbb{L} \leq \mathbb{L}' \quad &\text{iff} \quad \mathrm{Cl}(\mathbb{L}') \subseteq \mathrm{Cl}(\mathbb{L}) \quad \text{(Proposition 61)} \\ &\text{implies} \quad \widetilde{\Omega}(\mathbb{L}) \leq \widetilde{\Omega}(\mathbb{L}'). \quad \text{(Proposition 62)} \end{aligned}$$

■

3.6 Logical Morphisms

Let $\mathbb{L} = \langle \mathcal{A}, C\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, and $\mathbb{L}' = \langle \mathcal{A}', C'\rangle$, with $\mathcal{A}' = \langle \mathbf{A}', E'\rangle$, be G-logics. A **logical G-morphism** $h : \mathbb{L} \to \mathbb{L}'$ is a G-morphism $h : \mathcal{A} \to \mathcal{A}'$, such that, for all $X' \in G^{A'}$,



$$X' \in \mathrm{Cl}(\mathbb{L}') \quad \text{implies} \quad X' \circ h \in \mathrm{Cl}(\mathbb{L}).$$

This may be equivalently written as

$$\mathrm{Cl}(\mathbb{L}') \circ h = \{X' \circ h : X' \in \mathrm{Cl}(\mathbb{L}')\} \subseteq \mathrm{Cl}(\mathbb{L}).$$

Note that this definition makes sense, since for all $a, b \in A$,

$$\begin{array}{rcll} E(a,b) \wedge X'(h(a)) & \leq & E'(h(a),h(b)) \wedge X'(h(a)) & (E \leq E' \circ h^2) \\ & \leq & X'(h(b)), & (E' \text{ comp } X') \end{array}$$

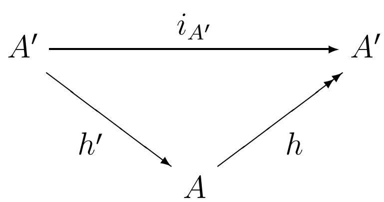
that is E is compatible with $X' \circ h$, for all $X' \in \mathrm{Cl}(\mathbb{L}')$.

Given a logical G-morphism $h : \mathbb{L} \to \mathbb{L}'$, we say that $\mathbb{L}$ is **projectively generated from $\mathbb{L}'$ by h** if

$$\mathrm{Cl}(\mathbb{L}) = \mathrm{Cl}(\mathbb{L}') \circ h.$$

We call $h : \mathbb{L} \to \mathbb{L}'$ a **bilogical G-morphism**, written $h : \mathbb{L} \to_b \mathbb{L}'$, if it is surjective and projectively generates $\mathbb{L}$ from $\mathbb{L}'$.

Recall that given a surjective mapping $h : A \to A'$, there exists a (non uniquely defined, in general) mapping $h' : A' \to A$, such that $h \circ h' = i_{A'}$.



Such a mapping $h' : A' \to A$ is called a **section of** h.

The following proposition, an analog of Proposition 1.4 of [28], provides several characterizations of bilogical G-morphisms.

Proposition 64 *Let $\mathbb{L} = \langle \mathcal{A}, C\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, and $\mathbb{L}' = \langle \mathcal{A}', C'\rangle$, with $\mathcal{A}' = \langle \mathbf{A}', E'\rangle$, be G-logics and $h : \mathcal{A} \to \mathcal{A}'$ a surjective morphism, with a section $h' : A' \to A$. Then the following are equivalent:*

(i) $h : \mathbb{L} \to_b \mathbb{L}'$ *is a bilogical G-morphism;*

(ii) $\mathrm{Cl}(\mathbb{L}) \subseteq \mathrm{Cl}(\mathbb{L}') \circ h$ *and for all* $X' \in G^{A'}$, $C(X' \circ h) = C'(X') \circ h$;

(iii) For all $X' \in G^{A'}$, $C'(X') = C(X' \circ h) \circ h'$ *and* $E' \circ h^2 \in \mathrm{Gon}(\mathbb{L})$;

(iv) $\mathrm{Cl}(\mathbb{L}') = \mathrm{Cl}(\mathbb{L}) \circ h'$ *and* $E' \circ h^2 \in \mathrm{Gon}(\mathbb{L})$;

(v) $\mathrm{Cl}(\mathbb{L}) = \mathrm{Cl}(\mathbb{L}') \circ h$.

Proof:

(i)⇒(ii) By definition $\mathrm{Cl}(\mathbb{L}) = \mathrm{Cl}(\mathbb{L}') \circ h$. In particular, $\mathrm{Cl}(\mathbb{L}) \subseteq \mathrm{Cl}(\mathbb{L}') \circ h$. Moreover, for all $X' \in G^{A'}$, we have

$$\begin{aligned} C'(X') \circ h &= \bigwedge\{Y' : X' \le Y' \in \mathrm{Cl}(\mathbb{L}')\} \circ h \\ &\quad\ \text{(Lemma 60)} \\ &= \bigwedge\{Y' \circ h : X' \circ h \le Y' \circ h \text{ and } Y' \in \mathrm{Cl}(\mathbb{L}')\} \\ &\quad\ (h \text{ surjective}) \\ &= \bigwedge\{Y : X' \circ h \le Y \in \mathrm{Cl}(\mathbb{L})\} \\ &\quad\ (\mathrm{Cl}(\mathbb{L}) = \mathrm{Cl}(\mathbb{L}') \circ h) \\ &= C(X' \circ h). \quad \text{(Lemma 60)} \end{aligned}$$

(ii)⇒(iii) By hypothesis, for all $X' \in G^{A'}$, $C(X' \circ h) = C'(X') \circ h$. Composing with h' on the right, we get $C'(X') = C(X' \circ h) \circ h'$. Next, since E' is compatible with every closed G-set in $\mathbb{L}'$, we have that, for all $a, b \in A$ and all $X' \in G^{A'}$,

$$E'(h(a), h(b)) \wedge C'(X')(h(a)) \le C'(X')(h(b)).$$

Hence, by hypothesis, for all $X \in G^A$,

$$E'(h(a), h(b)) \wedge C(X)(a) \le C(X)(b).$$

This proves that $E' \circ h^2 \in \mathrm{Gon}(\mathbb{L})$.

(iii)⇒(iv) Suppose, first, that $X' \in \mathrm{Cl}(\mathbb{L}')$. Then

$$X' = C'(X') = C(X' \circ h) \circ h' \in \mathrm{Cl}(\mathbb{L}) \circ h'.$$

Conversely, observe that, for all $a \in A$, if $b = h'(h(a))$, then $h(a) = h(b)$ and, hence, $E'(h(a), h(b)) = \top$. Thus, since $E' \circ h^2 \in \mathrm{Gon}(\mathbb{L})$, for all $X \in \mathrm{Cl}(\mathbb{L})$, $X(a) = X(b) = X(h'(h(a)))$, i.e., $X \circ h' \circ h = X$. Thus, if $X \in \mathrm{Cl}(\mathbb{L})$, then

$$\begin{aligned} X \circ h' &= C(X) \circ h' \quad \text{(Idempotency)} \\ &= C(X \circ h' \circ h) \circ h' \quad (X = X \circ h' \circ h) \\ &= C'(X \circ h') \quad \text{(Hypothesis)} \\ &\in \mathrm{Cl}(\mathbb{L}'). \end{aligned}$$

(iv)⇒(v) Suppose $X' \in \mathrm{Cl}(\mathbb{L}')$. Then we have, by hypothesis,

$$\begin{aligned} X' \circ h &= (X \circ h') \circ h, \text{ for some } X \in \mathrm{Cl}(\mathbb{L}), \\ &= X \in \mathrm{Cl}(\mathbb{L}). \quad (E' \circ h^2 \in \mathrm{Gon}(\mathbb{L})) \end{aligned}$$

If, conversely, $X \in \mathrm{Cl}(\mathbb{L})$, then $X \circ h' \in \mathrm{Cl}(\mathbb{L}')$ and, again, taking into account that $E' \circ h^2 \in \mathrm{Gon}(\mathbb{L})$,

$$X = (X \circ h') \circ h \in \mathrm{Cl}(\mathbb{L}') \circ h.$$

(v)⇒(i) By definition.

■

Statements (iv) and (v) of Proposition 64 allow us to establish an order isomorphism between lattices of closed G-sets of two logically bimorphic G-logics.

Proposition 65 *Let $\mathbb{L} = \langle \mathcal{A}, C\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, and $\mathbb{L}' = \langle \mathcal{A}', C'\rangle$, with $\mathcal{A}' = \langle \mathbf{A}', E'\rangle$, be G-logics and $h : \mathcal{A} \to \mathcal{A}'$ a surjective morphism. Then $h : \mathbb{L} \to \mathbb{L}'$ is a bilogical G-morphism if and only if the latices $\langle \mathrm{Cl}(\mathbb{L}), \leq\rangle$ and $\langle \mathrm{Cl}(\mathbb{L}'), \leq\rangle$ of closed functions of $\mathbb{L}$ and $\mathbb{L}'$, respectively, are isomorphic under*

$$X \mapsto X \circ h' \quad \text{and} \quad Y \mapsto Y \circ h,$$

where $h' : A' \to A$ is a section of h.

Proof: The fact that the two maps are bijections is given by Proposition 64. That they are order preserving is clear from their definitions. ■

Moreover, the isomorphism of Proposition 65 extends to isomorphisms between lattices of logics on the two underlying G-algebras containing logics whose sets of closed functions correspond under the bilogical G-morphism.

Corollary 66 *Let $\mathbb{L} = \langle \mathcal{A}, C\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, and $\mathbb{L}' = \langle \mathcal{A}', C'\rangle$, with $\mathcal{A}' = \langle \mathbf{A}', E'\rangle$, be G-logics and $h : \mathbb{L} \to_b \mathbb{L}'$ a bilogical G-morphism. Then h induces an isomorphism*

$$\mathbf{Log_G}(\mathcal{A})^{\mathbb{L}} \cong \mathbf{Log_G}(\mathcal{A}')^{\mathbb{L}'}$$

between the lattice of all G-logics on $\mathcal{A}$ extending $\mathbb{L}$ and the lattice of all G-logics on $\mathcal{A}'$ extending $\mathbb{L}'$.

Proof: Directly from the isomorphism of Proposition 65. ■

Another property of bilogical G-morphisms is that they establish a close connection between the Tarski G-congruences of the bimorphically related G-logics. Roughly speaking, this is the property asserting that Tarski congruences are invariant under inverse bilogical morphisms, well-known in the traditional framework (see, e.g., Proposition 1.7 of [28]).

Proposition 67 *Let $\mathbb{L} = \langle \mathcal{A}, C\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, and $\mathbb{L}' = \langle \mathcal{A}', C'\rangle$, with $\mathcal{A}' = \langle \mathbf{A}', E'\rangle$, be G-logics and $h : \mathbb{L} \to_b \mathbb{L}'$ a bilogical G-morphism. Then*

$$\widetilde{\Omega}(\mathbb{L}) = \widetilde{\Omega}(\mathbb{L}') \circ h^2.$$

Proof: Using Proposition 62, we have, for all $a, b \in A$,

$$\begin{aligned}
&\widetilde{\Omega}(\mathbb{L}')(h(a), h(b)) \\
&\quad = \bigwedge_{\substack{X' \in \mathrm{Cl}(\mathbb{L}') \\ \varphi(x,\bar{z}) \in \mathrm{Fm}_{\mathcal{L}}(\mathrm{V}) \\ \bar{c}' \in A'}} X'(\varphi^{\mathbf{A}'}(h(a), \bar{c}')) \leftrightarrow X'(\varphi^{\mathbf{A}'}(h(b), \bar{c}')) \\
&\qquad \text{(Proposition 62)} \\
&\quad = \bigwedge_{\substack{X' \in \mathrm{Cl}(\mathbb{L}') \\ \varphi(x,\bar{z}) \in \mathrm{Fm}_{\mathcal{L}}(\mathrm{V}) \\ \bar{c} \in A}} X'(\varphi^{\mathbf{A}'}(h(a), h(\bar{c}))) \leftrightarrow X'(\varphi^{\mathbf{A}'}(h(b), h(\bar{c}))) \\
&\qquad (h \text{ surjective}) \\
&\quad = \bigwedge_{\substack{X' \in \mathrm{Cl}(\mathbb{L}') \\ \varphi(x,\bar{z}) \in \mathrm{Fm}_{\mathcal{L}}(\mathrm{V}) \\ \bar{c} \in A}} X'(h(\varphi^{\mathbf{A}}(a, \bar{c}))) \leftrightarrow X'(h(\varphi^{\mathbf{A}}(b, \bar{c}))) \\
&\qquad (h : \mathbf{A} \to \mathbf{A}') \\
&\quad = \bigwedge_{\substack{X \in \mathrm{Cl}(\mathbb{L}) \\ \varphi(x,\bar{z}) \in \mathrm{Fm}_{\mathcal{L}}(\mathrm{V}) \\ \bar{c} \in A}} X(\varphi^{\mathbf{A}}(a, \bar{c})) \leftrightarrow X(\varphi^{\mathbf{A}}(b, \bar{c})) \quad (h : \mathbb{L} \to_b \mathbb{L}') \\
&\quad = \widetilde{\Omega}(\mathbb{L})(a, b). \quad \text{(Proposition 62)}
\end{aligned}$$

Therefore $\widetilde{\Omega}(\mathbb{L}') \circ h^2 = \widetilde{\Omega}(\mathbb{L})$. ■

Let $\mathbb{L} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, and $\mathbb{L}' = \langle \mathcal{A}', C' \rangle$, with $\mathcal{A}' = \langle \mathbf{A}', E' \rangle$, be G-logics. A logical G-morphism $h : \mathbb{L} \to \mathbb{L}'$ is a (**logical**) G-**isomorphism**, written $h : \mathbb{L} \cong \mathbb{L}'$, if it is bijective and its inverse h^{-1} is also a logical G-morphism $h^{-1} : \mathbb{L}' \to \mathbb{L}$.

Proposition 68 *Let $\mathbb{L} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, and $\mathbb{L}' = \langle \mathcal{A}', C' \rangle$, with $\mathcal{A}' = \langle \mathbf{A}', E' \rangle$, be G-logics and $h : \mathcal{A} \to \mathcal{A}'$ a G-morphism. Then the following are equivalent:*

(i) $h : \mathbb{L} \cong \mathbb{L}'$ *is a G-isomorphism;*

(ii) $h : \mathbb{L} \to_b \mathbb{L}'$, *with $h : \mathcal{A} \to \mathcal{A}'$ an isomorphism;*

(iii) $h : \mathcal{A} \to \mathcal{A}'$ *an isomorphism, with $h^{-1} : \mathbb{L}' \to_b \mathbb{L}$.*

Proof:

(i)⇒(ii) By definition, $\mathrm{Cl}(\mathbb{L}') \circ h \subseteq \mathrm{Cl}(\mathbb{L})$. On the other hand, if $X \in \mathrm{Cl}(\mathbb{L})$, then $X = (X \circ h^{-1}) \circ h$, where $X \circ h^{-1} \in \mathrm{Cl}(\mathbb{L}')$. So $\mathrm{Cl}(\mathbb{L}) \subseteq \mathrm{Cl}(\mathbb{L}') \circ h$. Thus, $\mathrm{Cl}(\mathbb{L}) = \mathrm{Cl}(\mathbb{L}') \circ h$ and $h : \mathbb{L} \to \mathbb{L}'$ is a bilogical G-morphism.

(ii)⇒(iii) This is (i)⇒(iv) of Proposition 64.

(iii)⇒(i) Notice that, if $h : \mathcal{A} \to \mathcal{A}'$ is an isomorphism, then

$$E \circ (h^{-1})^2 \in \mathrm{Gon}(\mathbb{L}') \quad \text{and} \quad E' \circ h^2 \in \mathrm{Gon}(\mathbb{L}).$$

E.g., for all $a', b' \in A'$ and all $X' : A' \to G$,

$$\begin{aligned}
&E(h^{-1}(a'), h^{-1}(b')) \wedge C'(X')(a') \\
&\quad \le E'(h(h^{-1}(a')), h(h^{-1}(b'))) \wedge C'(X')(a') \quad (h : \mathcal{A} \to \mathcal{A}') \\
&\quad = E'(a', b') \wedge C'(X')(a') \quad (h \circ h^{-1} = i_{\mathcal{A}'}) \\
&\quad \le C'(X')(b'). \quad (E' \in \mathrm{Gon}(\mathbb{L}'))
\end{aligned}$$

Hence, this part reduces to (iv)⇒(v) of Proposition 64.

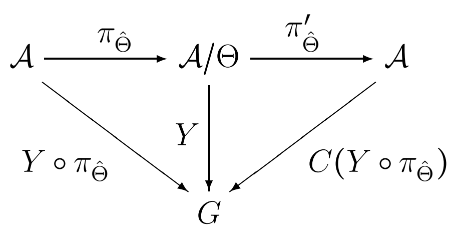
■

3.7 Quotients

Let $\mathbb{L} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, be a G-logic and $\Theta \in \mathrm{Gon}(\mathcal{A})$. If $\Theta \in \mathrm{Gon}(\mathbb{L})$ it is reasonable to define a G-logic structure on the quotient G-algebra $\mathcal{A}/\Theta = \langle \mathbf{A}/\hat{\Theta}, \bar{\Theta} \rangle$. We simply set

$$\mathbb{L}/\Theta = \langle \mathcal{A}/\Theta, C/\Theta \rangle,$$

where, for all $Y : A/\hat{\Theta} \to G$,



$$(C/\Theta)(Y) = C(Y \circ \pi_{\hat{\Theta}}) \circ \pi'_{\hat{\Theta}},$$

with $\pi'_{\hat{\Theta}}$ being a section of $\pi_{\hat{\Theta}}$. This is well defined, since, regardless of the choice of section, we have, for all $a, b \in A$ and all $Y : A/\hat{\Theta} \to G$,

$$\begin{aligned}
&\bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) \wedge (C/\Theta)(Y)(a/\hat{\Theta}) \\
&\quad = \Theta(a, b) \wedge C(Y \circ \pi_{\hat{\Theta}})(\pi'_{\hat{\Theta}}(a/\hat{\Theta})) \quad (\text{Definitions of } \bar{\Theta} \text{ and } C/\Theta) \\
&\quad = \Theta(a, b) \wedge C(Y \circ \pi_{\hat{\Theta}})(a) \quad (\Theta \in \mathrm{Gon}(\mathbb{L})) \\
&\quad \le C(Y \circ \pi_{\hat{\Theta}})(b) \quad (\Theta \in \mathrm{Gon}(\mathbb{L})) \\
&\quad = C(Y \circ \pi_{\hat{\Theta}})(\pi'_{\hat{\Theta}}(b/\hat{\Theta})) \quad (\Theta \in \mathrm{Gon}(\mathbb{L})) \\
&\quad = (C/\Theta)(Y)(b/\hat{\Theta}). \qquad (\text{Definition of } C/\Theta)
\end{aligned}$$

Hence $\bar{\Theta} \in \mathrm{Gon}(\mathbb{L}/\Theta)$. The structure $\mathbb{L}/\Theta$ is termed the **quotient G-logic** of $\mathbb{L}$ by Θ.

We show that, under this definition of $\mathbb{L}/\Theta$, the projection $\pi_{\hat{\Theta}} : \mathbb{L} \to \mathbb{L}/\Theta$ becomes a bilogical G-morphism.

Lemma 69 *Let $\mathbb{L} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, be a G-logic and $\Theta \in \mathrm{Gon}(\mathbb{L})$. The mapping $\pi_{\hat{\Theta}} : \mathcal{A} \to \mathcal{A}/\Theta$ is a bilogical G-morphism $\pi_{\hat{\Theta}} : \mathbb{L} \to_b \mathbb{L}/\Theta$.*

Proof: We must show that $\mathrm{Cl}(\mathbb{L}) = \mathrm{Cl}(\mathbb{L}/\Theta) \circ \pi_{\hat{\Theta}}$. Let us denote again by $\pi'_{\hat{\Theta}}$ a section of $\pi_{\hat{\Theta}}$. Suppose, first, that $Y \in \mathrm{Cl}(\mathbb{L}/\Theta)$. Then

$$\begin{array}{rcl} Y \circ \pi_{\hat{\Theta}} & = & (C/\Theta)(Y) \circ \pi_{\hat{\Theta}} \quad (Y \in \mathrm{Cl}(\mathbb{L}/\Theta)) \\ & = & C(Y \circ \pi_{\hat{\Theta}}) \circ \pi'_{\hat{\Theta}} \circ \pi_{\hat{\Theta}} \quad (\text{Definition of } C/\Theta) \\ & = & C(Y \circ \pi_{\hat{\Theta}}) \quad (\Theta \in \mathrm{Gon}(\mathbb{L})) \\ & \in & \mathrm{Cl}(\mathbb{L}). \end{array}$$

Thus, $\mathrm{Cl}(\mathbb{L}/\Theta) \circ \pi_{\hat{\Theta}} \subseteq \mathrm{Cl}(\mathbb{L})$. Assume, conversely, that $X \in \mathrm{Cl}(\mathbb{L})$. Consider the mapping $X \circ \pi'_{\hat{\Theta}} : A/\hat{\Theta} \to G$. We have

$$\begin{array}{rcl} (C/\Theta)(X \circ \pi'_{\hat{\Theta}}) & = & C(X \circ \pi'_{\hat{\Theta}} \circ \pi_{\hat{\Theta}}) \circ \pi'_{\hat{\Theta}} \quad (\text{Definition of } C/\Theta) \\ & = & C(X) \circ \pi'_{\hat{\Theta}} \quad (\Theta \in \mathrm{Gon}(\mathbb{L})) \\ & = & X \circ \pi'_{\hat{\Theta}}. \quad (X \in \mathrm{Cl}(\mathbb{L})) \end{array}$$

So $X \circ \pi'_{\hat{\Theta}} \in \mathrm{Cl}(\mathbb{L}/\Theta)$ and we obtain

$$X = (X \circ \pi'_{\hat{\Theta}}) \circ \pi_{\hat{\Theta}} \in \mathrm{Cl}(\mathbb{L}/\Theta) \circ \pi_{\hat{\Theta}}.$$

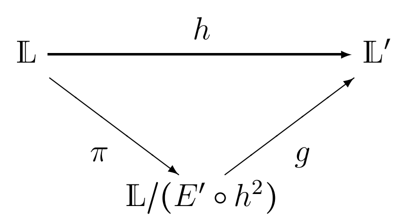
We conclude that $\mathrm{Cl}(\mathbb{L}) \subseteq \mathrm{Cl}(\mathbb{L}/\Theta) \circ \pi_{\hat{\Theta}}$. Thus, $\pi_{\hat{\Theta}} : \mathbb{L} \to_b \mathbb{L}/\Theta$. ■

The next goal is to further abstract the Homomorphism Theorems of Universal Algebra, which were presented for G-algebras in Section 3.2, to G-logics and their quotients. These results form an extension of the corresponding results of Font and Jansana covering quotients of abstract logics in the traditional framework (see Theorems 1.8-1.10 of [28]).

Theorem 70 (Homomorphism) *Let $\mathbb{L} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, and $\mathbb{L}' = \langle \mathcal{A}', C' \rangle$, with $\mathcal{A}' = \langle \mathbf{A}', E' \rangle$, be G-logics and $h : \mathbb{L} \to_b \mathbb{L}'$ a bilogical G-morphism. Then $E' \circ h^2$ is a logical G-congruence of $\mathbb{L}$ and*

$$\mathbb{L}/(E' \circ h^2) \cong \mathbb{L}'$$

by means of a unique G-isomorphism g satisfying commutativity of



where $\pi : \mathbb{L} \to \mathbb{L}/(E' \circ h^2)$ denotes the quotient G-morphism.

Proof: First, by Theorem 52, we know that there exists a unique isomorphism $g : \mathcal{A}/(E' \circ h^2) \to \mathcal{A}'$, such that $g \circ \pi = h$, defined by

$$g(a/\widehat{E' \circ h^2}) = h(a), \quad a \in A.$$

By Proposition 64, $E' \circ h^2 \in \mathrm{Gon}(\mathbb{L})$. Let $\pi' : A/\widehat{E' \circ h^2} \to A$ be a section of π and note that, since $g \circ \pi = h$, we get $g = h \circ \pi'$. We now calculate

$$\begin{array}{rcll} \mathrm{Cl}(\mathbb{L}/(E' \circ h^2)) & = & \mathrm{Cl}(\mathbb{L}) \circ \pi' & \text{(Lemma 69)} \\ & = & \mathrm{Cl}(\mathbb{L}') \circ h \circ \pi' & (h : \mathbb{L} \to_b \mathbb{L}') \\ & = & \mathrm{Cl}(\mathbb{L}') \circ g. & (g = h \circ \pi') \end{array}$$

Therefore, by Proposition 68, g is a G-isomorphism. ∎

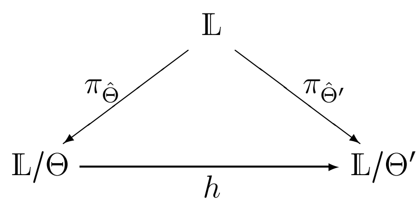
We turn, next, to an analog of the Second Isomorphism Theorem for G-logics. Our intention is to prove it using the Homomorphism Theorem 70.

Theorem 71 (Second Isomorphism) *Let $\mathbb{L} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, be a G-logic and $\Theta, \Theta' \in \mathrm{Gon}(\mathbb{L})$, such that $\Theta \leq \Theta'$. Then $\Theta'/\Theta \in \mathrm{Gon}(\mathbb{L}/\Theta)$ and*

$$(\mathbb{L}/\Theta)/(\Theta'/\Theta) \cong \mathbb{L}/\Theta'$$

via $(a/\hat{\Theta})/\widehat{\Theta'/\Theta} \mapsto a/\hat{\Theta}'$.

Proof: Let us start with the following diagram.



By Lemma 69, $\pi_{\hat{\Theta}} : \mathbb{L} \to_b \mathbb{L}/\Theta$ and $\pi_{\hat{\Theta}'} : \mathbb{L} \to_b \mathbb{L}/\Theta'$ are bilogical morphisms. It is not difficult to show that

$$\begin{array}{rrcl} h: & \mathbb{L}/\Theta & \longrightarrow & \mathbb{L}/\Theta'; \\ & a/\hat{\Theta} & \longmapsto & a/\hat{\Theta}' \end{array}$$

is also a bilogical morphism. Denoting by $\pi'_{\hat{\Theta}}$ a section of $\pi_{\hat{\Theta}}$, we have

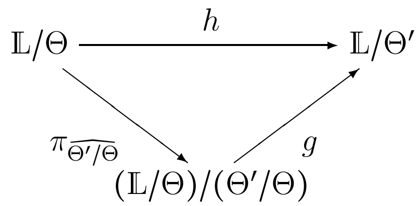
$$\begin{array}{rcll} \mathrm{Cl}(\mathbb{L}/\Theta') \circ h & = & \mathrm{Cl}(\mathbb{L}/\Theta') \circ \pi_{\hat{\Theta}'} \circ \pi'_{\hat{\Theta}} & (h = \pi_{\hat{\Theta}'} \circ \pi'_{\hat{\Theta}}) \\ & = & \mathrm{Cl}(\mathbb{L}) \circ \pi'_{\hat{\Theta}} & (\pi_{\hat{\Theta}'} : \mathbb{L} \to_b \mathbb{L}/\Theta') \\ & = & \mathrm{Cl}(\mathbb{L}/\Theta). & (\pi_{\hat{\Theta}} : \mathbb{L} \to_b \mathbb{L}/\Theta) \end{array}$$

Thus, $h : \mathbb{L}/\Theta \to_b \mathbb{L}/\Theta'$ is indeed a bilogical G-morphism.

Next we show that $\bar{\Theta}' \circ h^2 = \Theta'/\Theta$. We have, for all $a, b \in A$,

$$\begin{array}{rcll} \bar{\Theta}'(h(a/\hat{\Theta}), h(b/\hat{\Theta})) & = & \bar{\Theta}'(a/\hat{\Theta}', b/\hat{\Theta}') & \text{(Definition of } h\text{)} \\ & = & \Theta'(a, b) & \text{(Definition of } \bar{\Theta}'\text{)} \\ & = & \Theta'/\Theta(a/\hat{\Theta}, b/\hat{\Theta}). & \text{(Definition of } \Theta'/\Theta\text{)} \end{array}$$

Finally, we look at the following diagram.



It conforms with the hypotheses of Theorem 70. So we may conclude that $\Theta'/\Theta \in \mathrm{Gon}(\mathbb{L}/\Theta)$ and that

$$\begin{array}{rrcl} g: & (\mathbb{L}/\Theta)/(\Theta'/\Theta) & \longrightarrow & \mathbb{L}/\Theta'; \\ & (a/\hat{\Theta})/\widehat{\Theta'/\Theta} & \longmapsto & a/\hat{\Theta}' \end{array}$$

is the unique G-isomorphism making the diagram commute. ∎

The last theorem in this series that we establish is the analog for G-logics of the Correspondence Theorem of Universal Algebra. A similar generalization for abstract logics is given in Theorem 1.10 of [28].

Theorem 72 (Correspondence Theorem) *Let $\mathbb{L} = \langle \mathcal{A}, C\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, be a G-logic and $\Theta \in \mathrm{Gon}(\mathbb{L})$. The segment $[\Theta, \widetilde{\Omega}(\mathbb{L})]$ of $\mathrm{Gon}(\mathbb{L})$ is isomorphic to the lattice $\mathrm{Gon}(\mathbb{L}/\Theta)$ by the mapping $\Theta' \mapsto \Theta'/\Theta$.*

Proof: Suppose $\Theta' \in \mathrm{Gon}(\mathbb{L})$ is such that $\Theta \leq \Theta' \leq \widetilde{\Omega}(\mathbb{L})$. By the Second Isomorphism Theorem 71, we have $\Theta'/\Theta \in \mathrm{Gon}(\mathbb{L}/\Theta)$. By the Second Isomorphism Theorem 53 for G-algebras, we must show that, if $\Theta \leq \Theta' \in \mathrm{Gon}(\mathcal{A})$, such that $\Theta'/\Theta \in \mathrm{Gon}(\mathbb{L}/\Theta)$, then $\Theta' \in \mathrm{Gon}(\mathbb{L})$. Denoting, again, by $\pi'_{\hat{\Theta}}$ a section of $\pi_{\hat{\Theta}}$, we have, for all $X \in \mathrm{Cl}(\mathbb{L})$ and all $a, b \in A$,

$$\begin{array}{rcl} \Theta'(a,b) \wedge X(a) & = & \Theta'/\Theta(a/\hat{\Theta}, b/\hat{\Theta}) \wedge (X \circ \pi'_{\hat{\Theta}})(a/\hat{\Theta}) \\ & & \quad (\text{Definition of } \Theta'/\Theta \text{ and } \Theta \in \mathrm{Gon}(\mathbb{L})) \\ & \leq & (X \circ \pi'_{\hat{\Theta}})(b/\hat{\Theta}) \quad (\Theta'/\Theta \in \mathrm{Gon}(\mathbb{L}/\Theta)) \\ & = & X(b). \quad (\Theta \in \mathrm{Gon}(\mathbb{L})) \end{array}$$

Hence, $\Theta' \in \mathrm{Gon}(\mathbb{L})$. ∎

We now show that the Tarski G-congruence of a quotient $\mathbb{L}/\Theta$ of a G-logic $\mathbb{L}$ by a logical G-congruence Θ coincides with the quotient by Θ of the Tarski G-congruence of $\mathbb{L}$ itself.

Corollary 73 *Let $\mathbb{L} = \langle \mathcal{A}, C\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, be a G-logic and $\Theta \in \mathrm{Gon}(\mathbb{L})$. Then*

$$\widetilde{\Omega}(\mathbb{L}/\Theta) = \widetilde{\Omega}(\mathbb{L})/\Theta.$$

Proof: Immediate by the Correspondence Theorem 72. ∎

Let $\mathbb{L} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, be a G-logic. It is called **reduced** if it has only one logical G-congruence, i.e., if $\widetilde{\Omega}(\mathbb{L}) = E$. Given a G-logic $\mathbb{L}$, we set

$$\mathbb{L}^* = \mathbb{L}/\widetilde{\Omega}(\mathbb{L})$$

and call $\mathbb{L}^*$ the **reduction** of $\mathbb{L}$. Further, given a class L of G-logics, we set

$$\mathsf{L}^* = \{\mathbb{L}^* : \mathbb{L} \in \mathsf{L}\}.$$

If $\mathbb{L} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, is a G-logic, then, by the preceding corollary, $\mathbb{L}^*$ is always reduced. Moreover, if $\mathbb{L}$ is reduced, it is isomorphic to its reduction and the two G-logics $\mathbb{L}$ and $\mathbb{L}^*$ may be identified. If $\mathbb{L} = \langle \mathcal{A}, C \rangle$, we use the notation

$$\begin{array}{rcl} \mathcal{A}^* & = & \mathbf{A}/\widetilde{\Omega}(\mathbb{L}), \\ C^* & = & C/\widetilde{\Omega}(\mathbb{L}); \\ \mathrm{Cl}(\mathbb{L})^* & = & \mathrm{Cl}(\mathbb{L})/\widetilde{\Omega}(\mathbb{L}). \end{array}$$

We prove that the reduction of a quotient $\mathbb{L}/\Theta$ of a G-logic $\mathbb{L}$ by a logical G-congruence Θ is G-isomorphic with the reduction of $\mathbb{L}$ itself. This forms an analog of Proposition 1.13 of [28].

Proposition 74 *Let $\mathbb{L} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, be a G-logic and $\Theta \in$ $\mathrm{Gon}(\mathbb{L})$. Then*

$$(\mathbb{L}/\Theta)^* \cong \mathbb{L}^*.$$

Proof: We have

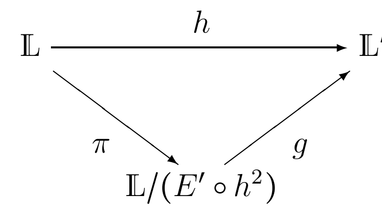
$$\begin{array}{rcl} (\mathbb{L}/\theta)^* & = & (\mathbb{L}/\theta)/\widetilde{\Omega}(\mathbb{L}/\theta) \quad \text{(Definition)} \\ & = & (\mathbb{L}/\theta)/(\widetilde{\Omega}(\mathbb{L})/\theta) \quad \text{(Corollary 73)} \\ & \cong & \mathbb{L}/\widetilde{\Omega}(\mathbb{L}) \quad \text{(Theorem 72)} \\ & = & \mathbb{L}^*. \quad \text{(Definition)} \end{array}$$

∎

More generally, we can show that if two G-logics are related by a bilogical G-morphism, then their corresponding reductions are G-isomorphic. This forms an analog of Proposition 1.14 of [28].

Proposition 75 *Let $\mathbb{L} = \langle \mathcal{A}, C \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, and $\mathbb{L}' = \langle \mathcal{A}', C' \rangle$, with $\mathcal{A}' = \langle \mathbf{A}', E' \rangle$, be G-logics. If there is a bilogical G-morphism $h : \mathbb{L} \to_b \mathbb{L}'$, then $\mathbb{L}^* \cong \mathbb{L}'^*$.*

Proof: Suppose $h : \mathbb{L} \to_b \mathbb{L}'$ is a bilogical G-morphism. By the Homomorphism Theorem 70, there exists a unique G-isomorphism $g : \mathbb{L}/(E' \circ h^2) \to \mathbb{L}'$ that makes the following triangle commute,



where $\pi : \mathbb{L} \to_b \mathbb{L}/(E' \circ h^2)$ is the quotient G-morphism. By Proposition 67,

$$\widetilde{\Omega}(\mathbb{L}/(E' \circ h^2)) = \widetilde{\Omega}(\mathbb{L}') \circ g^2.$$

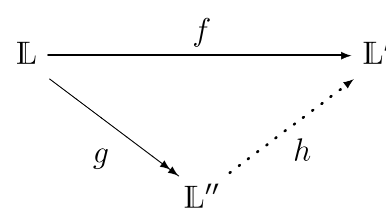
Therefore, as g is a G-isomorphism,

$$(\mathbb{L}/(E' \circ h^2))^* \cong \mathbb{L}'^*.$$

By Proposition 74, $(\mathbb{L}/(E' \circ h^2))^* \cong \mathbb{L}^*$. We conclude that $\mathbb{L}^* \cong \mathbb{L}'^*$. ∎

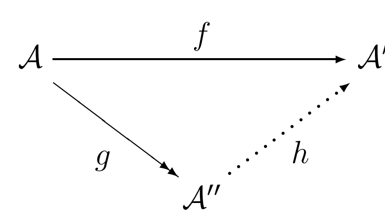
Taking advantage of Lemma 55, a fill-in type of result for G-algebras, we establish a similar result for G-logics, forming an analog of Proposition 1.15 of [28].

Proposition 76 *Let* $\mathbb{L} = \langle \mathcal{A}, C \rangle$, *with* $\mathcal{A} = \langle \mathbf{A}, E \rangle$, $\mathbb{L}' = \langle \mathcal{A}', C' \rangle$, *with* $\mathcal{A}' = \langle \mathbf{A}', E' \rangle$, *and* $\mathbb{L}'' = \langle \mathcal{A}'', C'' \rangle$, *with* $\mathcal{A}'' = \langle \mathbf{A}'', E'' \rangle$, *be* G*-logics, such that* $f : \mathbb{L} \to \mathbb{L}'$ *is a logical* G*-morphism and* $g : \mathbb{L} \to_b \mathbb{L}''$ *is a bilogical* G*-morphism, such that* $E'' \circ g^2 \leq E' \circ f^2$. *Then, there exists a unique logical* G*-morphism* $h : \mathbb{L}'' \to \mathbb{L}'$, *such that* $h \circ g = f$.



Moreover, f *projectively generates* $\mathbb{L}$ *from* $\mathbb{L}'$ *if and only if* h *projectively generates* $\mathbb{L}''$ *from* $\mathbb{L}'$.

Proof: By Lemma 55, there exists a commutative diagram of G-algebras.



We show that $h : \mathbb{L}'' \to \mathbb{L}'$ is a logical G-morphism. Let g' be a section of g. We have

$$\begin{array}{rcl} \mathrm{Cl}(\mathbb{L}') \circ h & = & \mathrm{Cl}(\mathbb{L}') \circ f \circ g' \quad (E'' \circ g^2 \leq E' \circ f^2) \\ & \subseteq & \mathrm{Cl}(\mathbb{L}) \circ g' \quad (f : \mathbb{L} \to \mathbb{L}') \\ & = & \mathrm{Cl}(\mathbb{L}''). \quad (g : \mathbb{L} \to_b \mathbb{L}'') \end{array}$$

If f projectively generates $\mathbb{L}$ from $\mathbb{L}'$, then the inclusion may be replaced by an equality. So, h projectively generates $\mathbb{L}''$ from $\mathbb{L}'$.

Suppose, conversely, that h projectively generates $\mathbb{L}''$ from $\mathbb{L}'$. Then, we have

$$\begin{array}{rcl} \mathrm{Cl}(\mathbb{L}') \circ f & = & \mathrm{Cl}(\mathbb{L}') \circ h \circ g \quad (f = h \circ g) \\ & = & \mathrm{Cl}(\mathbb{L}'') \circ g \quad (\text{Hypothesis}) \\ & = & \mathrm{Cl}(\mathbb{L}). \quad (g : \mathbb{L} \to_b \mathbb{L}'') \end{array}$$

Therefore, f projectively generates $\mathbb{L}$ from $\mathbb{L}'$. ∎

3.8 Sentential Graded Logics

Recall that, in Section 2.2, we introduced G-logics as G-logics acting on G-sets of formulas. Then, in Section 3.4, we generalized this notion to include G-logics over arbitrary G-algebras. Since we now want to refocus on the special G-logics that act on G-sets of formulas, we reintroduce them under the special name of *sentential G-logic* to specifically emphasize the fact the their underlying G-algebra is the G-algebra of formulas over a fixed algebraic type.

A **sentential G-logic** is a pair $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C \rangle$, where

$$\mathcal{F}m_{\mathcal{L}}(V) = \langle \mathbf{Fm}_{\mathcal{L}}(V), \Delta_{\mathbf{Fm}_{\mathcal{L}}(V)} \rangle$$

is the formula G-algebra and $C : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$ a G-operator on $\mathcal{F}m_{\mathcal{L}}(V)$, that is, one that satisfies Inflationarity, Monotonicity, Idempotency and Translation. These are specializations of the corresponding axioms itemized at the beginning of Section 3.4. Even though they were also given explicitly in Section 2.2, we relist them here as well.

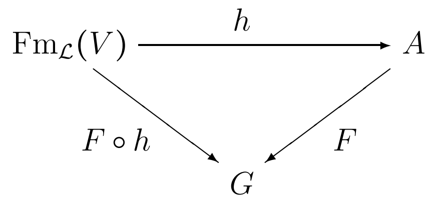
Inflationarity $X \leq C(X)$, for all $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$;

Monotonicity $C(X) \leq C(Y)$, for all $X, Y : \mathrm{Fm}_{\mathcal{L}}(V) \to G$, with $X \leq Y$;

Idempotency $C(C(X)) = C(X)$, for all $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$;

Translation $C(X \circ h) \leq C(X) \circ h$, for all $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$.

A G-matrix $\mathfrak{A} = \langle \mathcal{A}, F\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, is called an $\mathbb{S}$-**matrix** if, for all $h : \mathcal{F}m_{\mathcal{L}}(V) \to \mathcal{A}$ and all $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,



$$X \le F \circ h \quad \text{implies} \quad C(X) \le F \circ h.$$

If $\mathfrak{A} = \langle \mathcal{A}, F\rangle$ is an $\mathbb{S}$-matrix, we call $F : A \to G$ an $\mathbb{S}$-**filter**.

Note that, due to the fact that the reduced G-congruence component of the formula G-algebra is the identity on $\mathbf{Fm}_{\mathcal{L}}(V)$, a G-morphism $h : \mathcal{F}m_{\mathcal{L}}(V) \to \mathcal{A}$ may be identified with a homomorphism $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$. Consequently, the universal quantification "for all $h : \mathcal{F}m_{\mathcal{L}}(V) \to \mathcal{A}$" above may be equivalently replaced by the universal quantification "for all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$". Note, additionally, that the condition defining an $\mathbb{S}$-matrix coincides with the condition $C \le C_{\mathfrak{A}}$, which was used in Section 2.4 to define an $\mathcal{S}$-matrix.

The following is a characterization of $\mathbb{S}$-filters on a G-algebra by means of the closed sets of $\mathbb{S}$.

Lemma 77 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic and $\mathfrak{A} = \langle \mathcal{A}, F\rangle$ a G-matrix. $\mathfrak{A}$ is an $\mathbb{S}$-matrix if and only if for all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$,*

$$F \circ h \in \mathrm{Cl}(\mathbb{S}).$$

Proof: First assume that, for all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$, $F \circ h \in \mathrm{Cl}(\mathbb{S})$. Then, we have, for all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$ and all $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\begin{array}{lll} X \le F \circ h & \text{implies} & C(X) \le C(F \circ h) \quad \text{(Monotonicity)} \\ & \text{iff} & C(X) \le F \circ h. \quad (F \circ h \in \mathrm{Cl}(\mathbb{S})) \end{array}$$

Hence, F is an $\mathbb{S}$-filter.

Suppose, conversely, that F is an $\mathbb{S}$-filter. Then, for all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$, we have $F \circ h \le F \circ h$, whence, by hypothesis, $C(F \circ h) \le F \circ h$. The reverse inequality always holds. So $C(F \circ h) = F \circ h$ and, hence, $F \circ h \in \mathrm{Cl}(\mathbb{S})$. ■

We write $\mathrm{Mat}(\mathbb{S})$ for the collection of all $\mathbb{S}$-matrices. We use $\mathrm{Mat}^*(\mathbb{S})$ for the class $\mathrm{Mat}(\mathbb{S})^*$, i.e., the class of all reductions of members of $\mathrm{Mat}(\mathbb{S})$. We also use $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ to denote the family of $\mathbb{S}$-filters on a G-algebra $\mathcal{A}$.

We have the following characterization of the $\mathbb{S}$-filters on the formula algebra. Here, Translation plays a critical role.

Lemma 78 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic and $\mathfrak{A} = \langle \mathcal{F}m_{\mathcal{L}}(V), F\rangle$ a G-matrix. $\mathfrak{A}$ is an $\mathbb{S}$-matrix if and only if $F \in \mathrm{Cl}(\mathbb{S})$.*

Proof: First, if $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{F}m_{\mathcal{L}}(V))$, then, taking the identity homomorphism on $\mathbf{Fm}_{\mathcal{L}}(V)$ for h in Lemma 77, we get $F \in \mathrm{Cl}(\mathbb{S})$.

Suppose, conversely, that $F \in \mathrm{Cl}(\mathbb{S})$. Then, for all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$ and all $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\begin{array}{lll} X \le F \circ h & \text{implies} & C(X) \le C(F \circ h) \quad \text{(Monotonicity)} \\ & \text{implies} & C(X) \le C(F) \circ h \quad \text{(Translation)} \\ & \text{implies} & C(X) \le F \circ h. \quad (F \in \mathrm{Cl}(\mathbb{S})) \end{array}$$

Therefore, $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{F}m_{\mathcal{L}}(V))$. ■

We call the closed G-sets of $\mathbb{S}$, which, by Lemma 78, coincide with the $\mathbb{S}$-filters on the formula G-algebra $\mathcal{F}m_{\mathcal{L}}(V)$, the **theories of** $\mathbb{S}$ or $\mathbb{S}$-**theories**. The collection of $\mathbb{S}$-theories is denoted by

$$\mathrm{Th}(\mathbb{S}) \coloneqq \mathrm{Cl}(\mathbb{S}) = \mathrm{Fi}_{\mathbb{S}}(\mathcal{F}m_{\mathcal{L}}(V))$$

and the complete lattice they form under $\le$ by $\mathbf{Th}(\mathbb{S}) = \langle \mathrm{Th}(\mathbb{S}), \le \rangle$.

The following proposition details how $\mathbb{S}$-filters and G-morphisms interact. It forms an analog of Proposition 1.19 of [28].

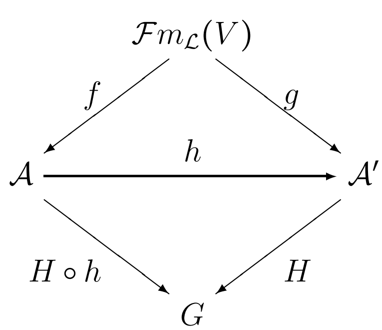
Proposition 79 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C \rangle$ be a sentential G-logic, $\mathcal{A} = \langle \mathbf{A}, E \rangle$ and $\mathcal{A}' = \langle \mathbf{A}', E' \rangle$ be G-algebras and $h : \mathcal{A} \to \mathcal{A}'$ a G-morphism. Suppose $H : A' \to G$.*

(a) If $H \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$, then $H \circ h \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$.

(b) If h is surjective and $H \circ h \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, then $H \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$.

Proof:

(a) We do a little diagram chasing around the diagram shown.



$$\begin{array}{lll} H \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}') & \text{iff} & (\forall g : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}')(H \circ g \in \mathrm{Cl}(\mathbb{S})) \\ & & \quad \text{(Lemma 77)} \\ & \text{implies} & (\forall f : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A})(H \circ h \circ f \in \mathrm{Cl}(\mathbb{S})) \\ & \text{iff} & H \circ f \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}). \quad \text{(Lemma 77)} \end{array}$$

(b) For this part, observe that, if $h : \mathcal{A} \to \mathcal{A}'$ is surjective, then, for all $g : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}'$, there exists $\bar{g} : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$ (playing the role of f in the diagram), such that $h \circ \bar{g} = g$. Thus, we get

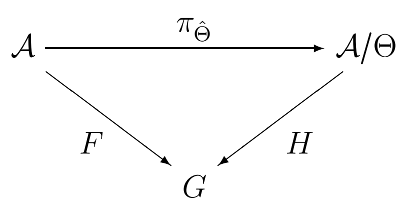
$$\begin{array}{lll} H \circ h \in \mathrm{Fi}_{\$}(\mathcal{A}) & \text{iff} & (\forall f : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A})(H \circ h \circ f \in \mathrm{Cl}(\$)) \\ & & \quad \text{(Lemma 77)} \\ & \text{implies} & (\forall g : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}')(H \circ h \circ \bar{g} \in \mathrm{Cl}(\$)) \\ & \text{iff} & (\forall g : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}')(H \circ g \in \mathrm{Cl}(\$)) \\ & \text{iff} & H \in \mathrm{Fi}_{\$}(\mathcal{A}'). \quad \text{(Lemma 77)} \end{array}$$

■

Proposition 79 implies, in particular, that, given a G-algebra $\mathcal{A}$ and $\Theta \in \mathrm{Gon}(\mathcal{A})$, if $\pi : \mathcal{A} \to \mathcal{A}/\Theta$ is the quotient G-morphism, then, for all $H : A/\hat{\Theta} \to G$, $H \circ \pi \in \mathrm{Fi}_{\$}(\mathcal{A})$ if and only if $H \in \mathrm{Fi}_{\$}(\mathcal{A}/\Theta)$. A partial result in the opposite direction, i.e., starting with $F : A \to G$, which forms an analog of Proposition 1.20 of [28], is given below.

Proposition 80 *Let $\$ = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic, $\mathcal{A} = \langle \mathbf{A}, E\rangle$ be a G-algebra, $F \in \mathrm{Fi}_{\$}(\mathcal{A})$ and $\Theta \in \mathrm{Gon}(\mathcal{A})$. Then Θ is compatible with F, i.e., $\Theta \le \Omega_{\mathcal{A}}(F)$, if and only if $F = H \circ \pi_{\hat{\Theta}}$, for some $H \in \mathrm{Fi}_{\$}(\mathcal{A}/\Theta)$, where $\pi_{\hat{\Theta}} : \mathcal{A} \to \mathcal{A}/\Theta$ is the quotient G-morphism.*

Proof: Let $F \in \mathrm{Fi}_{\$}(\mathcal{A})$. We work with the accompanying diagram.



Suppose, first, that Θ is compatible with F. By compatibility, it is reasonable to define $H : \mathcal{A}/\Theta \to G$ by setting, for all $a \in A$,

$$H(a/\hat{\Theta}) = F(a).$$

Indeed, if $\langle a, b\rangle \in \hat{\Theta}$, then $\Theta(a, b) = \top$, whence,

$$F(a) = \top \wedge F(a) = \Theta(a, b) \wedge F(a) \le F(b)$$

and, by symmetry, $F(a) = F(b)$. Hence H is well defined. Moreover, clearly, $H \circ \pi_{\hat{\Theta}} = F$. Thus, since $\pi_{\hat{\Theta}}$ is surjective, by Proposition 79, Part (b), we get that $H \in \mathrm{Fi}_{\$}(\mathcal{A}/\Theta)$.

Assume, conversely, that $H \in \mathrm{Fi}_{\$}(\mathcal{A}/\Theta)$, such that $F = H \circ \pi_{\hat{\Theta}}$. Then, for all $a, b \in A$,

$$\begin{array}{lll} \Theta(a, b) \wedge F(a) & = & \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) \wedge H(a/\hat{\Theta}) \quad \text{(Definitions of } \bar{\Theta} \text{ and } F) \\ & \le & H(b/\hat{\Theta}) \quad (H \in \mathrm{Fi}_{\$}(\mathcal{A}/\Theta)) \\ & = & F(b). \quad \text{(Definition of } F) \end{array}$$

Hence, Θ is compatible with F. ∎

We look, next, at bilogical G-morphisms between G-logics consisting of full collections of $\mathbb{S}$-filters on corresponding G-algebras. Proposition 81 forms an analog of Proposition 1.21 of [28] for G-logics.

Proposition 81 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic, $\mathcal{A} = \langle \mathbf{A}, E\rangle$ and $\mathcal{A}' = \langle \mathbf{A}', E'\rangle$ be G-algebras and $h : \mathcal{A} \to \mathcal{A}'$ an epimorphism. Then the following are equivalent:*

(i) $h : \langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle \to_b \langle \mathcal{A}', \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')\rangle$ *is a bilogical G-morphism;*

(ii) h induces an isomorphism between the lattices $\mathbf{Fi}_{\mathbb{S}}(\mathcal{A})$ and $\mathbf{Fi}_{\mathbb{S}}(\mathcal{A}')$;

(iii) For all $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ and section h' of h, $F \circ h' \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$ and $E' \circ h^2 \in \mathrm{Gon}(\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle)$.

Proof:

(i)⇒(ii) Suppose $h : \langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle \to_b \langle \mathcal{A}', \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')\rangle$ is a bilogical G-morphism. Then, by definition, $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A}) = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}') \circ h$. So $Y \mapsto Y \circ h$ is a bijection from $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$ to $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. Clearly it is also order preserving and reflecting and, hence, a lattice isomorphism.

(ii)⇒(iii) Let $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. By hypothesis, there exists $Y \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$, such that $F = Y \circ h$. Thus, $F \circ h' = Y \circ h \circ h' = Y \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$.

Assume, next, that $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ and $a, b \in A$. Let $Y \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$ be such that $F = Y \circ h$. Then

$$\begin{array}{rcl}(E' \circ h^2)(a,b) \wedge F(a) & = & E'(h(a),h(b)) \wedge Y(h(a)) \quad (F = Y \circ h)\\ & \le & Y(h(b)) \quad (Y \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}'))\\ & = & F(b). \quad (F = Y \circ h)\end{array}$$

Therefore, $E' \circ h^2 \in \mathrm{Gon}(\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle)$.

(iii)⇒(i) By hypothesis, $E' \circ h^2 \in \mathrm{Gon}(\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle)$. So, by Proposition 64, it suffices to show that $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A}') = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}) \circ h'$. First, if $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, then, by hypothesis, $F \circ h' \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$. On the other hand, if $G \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$, then, by Proposition 79, $G \circ h \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ and, moreover,

$$G = (G \circ h) \circ h' \in \mathrm{Fi}_{\mathbb{S}}(\mathbf{A}) \circ h'.$$

Thus, by (iv)⇒(i) of Proposition 64, we get the conclusion.

∎

The next result shows that, if a G-logic is the bilogical image of a G-logic whose closed G-sets consist of all $\mathbb{S}$-filters on the underlying G-algebra, then its own set of closed G-sets must be of the same type. This result will be important when we define and discuss full G-models in Section 3.10. It forms an analog of Proposition 1.22 of [28].

Proposition 82 *Let* $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ *be a sentential* G*-logic and* $\mathcal{A} = \langle \mathbf{A}, E\rangle$ *and* $\mathcal{A}' = \langle \mathbf{A}', E'\rangle$ *be* G*-algebras. If* $h : \langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle \to_b \langle \mathcal{A}', \mathcal{X}\rangle$ *is a bilogical* G*-morphism, then* $\mathcal{X} = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$*. As a consequence, we have:*

- $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^* = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}^*)$*;*
- *If* $\mathbb{L} = \langle \mathcal{A}, \mathcal{X}\rangle$ *and* $\mathbb{L}' = \langle \mathcal{A}', \mathcal{X}'\rangle$ *are* G*-isomorphic, then* $\mathcal{X} = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ *if and only if* $\mathcal{X}' = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$*.*

Proof: If $G \in \mathcal{X}$, then, by hypothesis, $G \circ h \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. Thus, since h is surjective, by Proposition 79, $G \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$. Thus, $\mathcal{X} \subseteq \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$.

Suppose, conversely, that $G \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$. Then, by Proposition 79, $G \circ h \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. Hence, by hypothesis, $G = (G \circ h) \circ h' \in \mathcal{X}$. Hence, we obtain $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A}') \subseteq \mathcal{X}$.

The first consequence is obtained by applying the main statement to the bilogical G-morphism

$$\pi : \langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle \to \langle \mathcal{A}/\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})), \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}) \circ \pi'\rangle,$$

where π' is a section of the quotient G-morphism $\pi : \mathcal{A} \to \mathcal{A}/\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A}))$. It gives $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^* = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}^*)$. Finally, the second consequence results by applying the main statement to both bilogical G-morphisms $h : \mathbb{L} \to_b \mathbb{L}'$ and $h^{-1} : \mathbb{L}' \to_b \mathbb{L}$, where $h : \mathbb{L} \cong \mathbb{L}'$. ■

Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic. Let $\mathfrak{A} = \langle \mathcal{A}, F\rangle$ be an $\mathbb{S}$-matrix. If Θ is a G-congruence on $\mathcal{A}$ compatible with F, then

$$\mathfrak{A}/\Theta = \langle \mathcal{A}/\Theta, F \circ \pi'_{\hat{\Theta}}\rangle,$$

where $\pi'_{\hat{\Theta}}$ is a section of $\pi_{\hat{\Theta}}$, is also an $\mathbb{S}$-matrix. This follows from the equality $(F \circ \pi'_{\hat{\Theta}}) \circ \pi_{\hat{\Theta}} = F$. Therefore, $\mathfrak{A}^* = \langle \mathcal{A}/\Omega_{\mathcal{A}}(F), F \circ \pi'_{\Omega_{\mathcal{A}}(F)}\rangle \in \mathrm{Mat}^*(\mathbb{S})$. So $\mathrm{Mat}^*(\mathbb{S})$ is the class of all reduced $\mathbb{S}$-matrices. Define $\mathrm{Alg}^*(\mathbb{S})$ as the class of all G-algebraic reducts of all G-matrices in $\mathrm{Mat}^*(\mathbb{S})$. We note that, by definition, for every G-algebra $\mathcal{A}$ and any $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, we have $\Omega_{\mathcal{A}}(F) \in \mathrm{Con}_{\mathrm{Alg}^*(\mathbb{S})}(\mathcal{A})$.

By Lemma 78, $\mathbb{S}$-filters on the formula algebra coincide with $\mathbb{S}$-theories. Moreover, in the case of the formula algebra, for all $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$, we have

$$\Omega_{\mathbf{Fm}_{\mathcal{L}}(V)}(X)(\varphi, \psi) = \bigwedge_{\gamma(x) \in \mathrm{Fm}_{\mathcal{L}}(V)} X(\gamma(\varphi)) \leftrightarrow X(\gamma(\psi)).$$

Therefore, we get

$$\widetilde{\Omega}(\mathbb{S})(\varphi, \psi) = \bigwedge_{\substack{X \in \mathrm{Th}(\mathbb{S}) \\ \gamma(x) \in \mathrm{Fm}_{\mathcal{L}}(V)}} X(\gamma(\varphi)) \leftrightarrow X(\gamma(\psi)).$$

The G-logic $\mathbb{S}^* = \mathbb{S}/\widetilde{\Omega}(\mathbb{S})$, obtained from $\mathbb{S}$ by taking the quotient modulo the Tarski G-congruence of $\mathbb{S}$, is called the **Lindenbaum-Tarski quotient of** $\mathbb{S}$. Its G-algebra reduct $\mathcal{F}m^*_{\mathcal{L}}(V) = \mathcal{F}m_{\mathcal{L}}(V)/\widetilde{\Omega}(\mathbb{S})$ is called the **Lindenbaum-Tarski algebra of** $\mathbb{S}$.

We say that a sentential G-logic is **complete with respect to** a class M of G-matrices if

$$\mathrm{Th}(\mathbb{S}) = \{F \circ h : \langle \mathcal{A}, F\rangle \in \mathsf{M}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}\}.$$

In particular, since, by Lemma 77, for all $\langle \mathcal{A}, F\rangle \in \mathsf{M}$, it holds that, for all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$, $F \circ h \in \mathrm{Th}(\mathbb{S})$, we must have $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. Hence, any class of G-matrices with respect to which $\mathbb{S}$ is complete must be a subclass of $\mathrm{Mat}(\mathbb{S})$. Additionally, since, by Lemma 78, $\mathrm{Fi}_{\mathbb{S}}(\mathcal{F}m_{\mathcal{L}}(V)) = \mathrm{Th}(\mathbb{S})$, we have that $\mathbb{S}$ is complete with respect to $\mathrm{Mat}(\mathbb{S})$. Moreover, it can be shown that $\mathbb{S}$ is complete with respect to $\mathrm{Mat}^*(\mathbb{S})$.

3.9 Graded Models

Let $\mathbb{L} = \langle \mathcal{A}, D\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, be a G-logic, that is $D : G^A \to G^A$ is a G-operator on $\mathcal{A}$. Define the sentential G-logic $\mathbb{S}_{\mathbb{L}} = \langle \mathcal{F}m_{\mathcal{L}}(V), C_{\mathbb{L}}\rangle$ by setting, for all $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\begin{aligned} C_{\mathbb{L}}(X) &= \bigwedge\{D(X') \circ h : h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, X' : A \to G, \\ &\qquad \text{such that } X \leq D(X') \circ h\} \\ &= \bigwedge\{X' \circ h : h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, X' \in \mathrm{Cl}(\mathbb{L}), \\ &\qquad \text{such that } X \leq X' \circ h\} \end{aligned}$$

Moreover, given a family $\mathsf{L} = \{\mathbb{L}_i : i \in I\}$ of G-logics $\mathbb{L}_i = \langle \mathcal{A}_i, D_i\rangle$, we set

$$C_{\mathsf{L}} = \bigwedge\{C_{\mathbb{L}} : \mathbb{L} \in \mathsf{L}\}.$$

Proposition 83 *Let $\mathbb{L} = \langle \mathcal{A}, D\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, and $\mathbb{L}_i = \langle \mathcal{A}_i, D_i\rangle$, with $\mathcal{A}_i = \langle \mathbf{A}_i, E_i\rangle$, $i \in I$, be G-logics. Set $\mathsf{L} = \{\mathbb{L}_i : i \in I\}$. Then, $C_{\mathbb{L}}$ and C_{L} are G-operators on $\mathcal{F}m_{\mathcal{L}}(V)$.*

Proof: We show the details for $C_{\mathbb{L}}$. Starting from Inflationarity, we have, for all $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$X \leq \bigwedge_{X',h}\{X' \circ h : X \leq X' \circ h\} = C_{\mathbb{L}}(X).$$

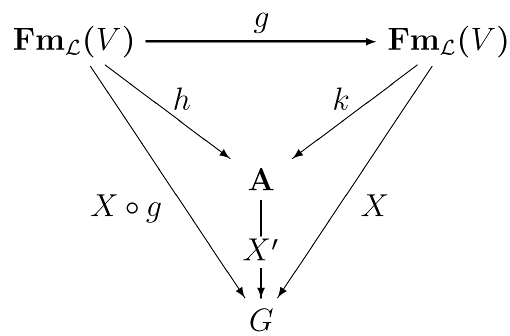
Continuing with Monotonicity, we have, for all $X, Y : \mathrm{Fm}_{\mathcal{L}}(V) \to G$, with $X \leq Y$,

$$C_{\mathbb{L}}(X) = \bigwedge_{X',h}\{X' \circ h : X \leq X' \circ h\} \leq \bigwedge_{X',h}\{X' \circ h : Y \leq X' \circ h\} = C_{\mathbb{L}}(Y).$$

Next, for Idempotency, let $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and compute

$$\begin{array}{rcl} C_{\mathbb{L}}(C_{\mathbb{L}}(X)) & = & \bigwedge_{X',h}\{X' \circ h : C_{\mathbb{L}}(X) \leq X' \circ h\} \\ & = & \bigwedge_{X',h}\{X' \circ h : X \leq X' \circ h\} \quad (X' \circ h \in \mathrm{Cl}(\mathbb{S}_{\mathbb{L}})) \\ & = & C_{\mathbb{L}}(X). \end{array}$$

Finally, for Translation, let $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and $g : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$.



We have

$$\begin{array}{rcl} C_{\mathbb{L}}(X \circ g) & = & \bigwedge_{X',h}\{X' \circ h : X \circ g \leq X' \circ h\} \\ & & \quad \text{(Definition of } C_{\mathbb{L}}) \\ & \leq & \bigwedge_{X',k}\{X' \circ k \circ g : X \circ g \leq X' \circ k \circ g\} \\ & & \quad \text{(Second set smaller)} \\ & \leq & \bigwedge_{X',k}\{X' \circ k : X \leq X' \circ k\} \circ g \\ & & \quad \text{(Second set smaller)} \\ & = & C_{\mathbb{L}}(X) \circ g. \quad \text{(Definition of } C_{\mathbb{L}}) \end{array}$$

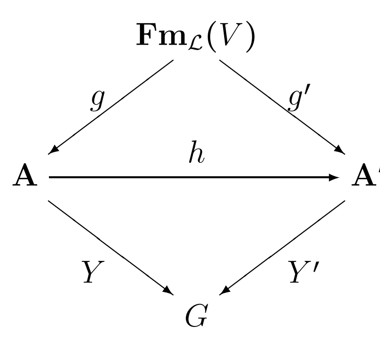
Thus, $C_{\mathbb{L}}$ is a G-operator on $\mathcal{F}m_{\mathcal{L}}(V)$. A similar proof applies to C_{L}. ■

We call $\mathbb{S}_{\mathsf{L}} = \langle \mathcal{F}m_{\mathcal{L}}(V), C_{\mathsf{L}} \rangle$ the (**sentential**) G-**logic induced**, or **generated**, **by** the class L of G-logics (or by the G-logic $\mathbb{L}$ if $\mathsf{L} = \{\mathbb{L}\}$).

In an analog of Proposition 2.3 of [28], we show that if two G-logics are related via a bilogical G-morphism, then the sentential G-logics that they induce on the formula G-algebra are identical.

Proposition 84 *Let $\mathbb{L} = \langle \mathcal{A}, D \rangle$, with $\mathcal{A} = \langle \mathbf{A}, E \rangle$, and $\mathbb{L}' = \langle \mathcal{A}', D' \rangle$, with $\mathcal{A}' = \langle \mathbf{A}', E' \rangle$, be G-logics and $h : \mathbb{L} \to_b \mathbb{L}'$ a bilogical G-morphism. Then $\mathbb{S}_{\mathbb{L}} = \mathbb{S}_{\mathbb{L}'}$.*

Proof: We show that $\mathrm{Cl}(\mathbb{S}_{\mathbb{L}}) = \mathrm{Cl}(\mathbb{S}_{\mathbb{L}'})$. Note that, for all $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,



$$\begin{array}{rcll} C_{\mathbb{L}'}(X) & = & \bigwedge_{\substack{Y' \in \mathrm{Cl}(\mathbb{L}') \\ g' : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}'}} \{Y' \circ g' : X \leq Y' \circ g'\} & (\text{Definition of } C_{\mathbb{L}'}) \\ & = & \bigwedge_{\substack{Y' \in \mathrm{Cl}(\mathbb{L}') \\ g : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}}} \{Y' \circ h \circ g : X \leq Y' \circ h \circ g\} & (h \text{ surjective}) \\ & = & \bigwedge_{\substack{Y \in \mathrm{Cl}(\mathbb{L}) \\ g : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}}} \{Y \circ g : X \leq Y \circ g\} & (h : \mathbb{L} \to_b \mathbb{L}') \\ & = & C_{\mathbb{L}}(X). \quad (\text{Definition of } C_{\mathbb{L}}) & \end{array}$$

Hence $\mathbb{S}_{\mathbb{L}} = \{X : C_{\mathbb{L}}(X) = X\} = \{X : C_{\mathbb{L}'}(X) = X\} = \mathbb{S}_{\mathbb{L}'}$. ∎

Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic and $\mathbb{L} = \langle \mathcal{A}, D\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, a G-logic. We say $\mathbb{L}$ is a G-**model** of $\mathbb{S}$ if

$$C \leq C_{\mathbb{L}}.$$

We let $\mathrm{Mod}(\mathbb{S})$ denote the class of all G-models of $\mathbb{S}$.

We say that a sentential G-logic $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ is **complete with respect to** a class L of G-logics when

$$C = C_{\mathsf{L}}.$$

We now show that modelhood is preserved by bilogical G-morphisms and obtain, as a corollary, that a sentential G-logic is complete with respect to a class of G-logics if and only if it is complete with respect to the class of their reductions. This is an analog of Proposition 2.5 of [28] for G-logics.

Proposition 85 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic and $\mathbb{L} = \langle \mathcal{A}, D\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, and $\mathbb{L} = \langle \mathcal{A}', D'\rangle$, with $\mathcal{A}' = \langle \mathbf{A}', E'\rangle$, be G-logics.*

(a) If there exists a bilogical G-morphism from $\mathbb{L}$ to $\mathbb{L}'$, then $\mathbb{L}$ is a G-model of $\mathbb{S}$ if and only if $\mathbb{L}'$ is a G-model of $\mathbb{S}$. In particular, $\mathbb{L}$ is a G-model of $\mathbb{S}$ if and only if $\mathbb{L}^*$ is.

(b) If $\mathbb{S}$ is complete with respect to a class L of G-logics, then it is also complete with respect to the class L^*.

Proof:

(a) Suppose there exists a bilogical G-morphism from $\mathbb{L}$ to $\mathbb{L}'$. Then, by Proposition 84, $C_{\mathbb{L}} = C_{\mathbb{L}'}$. Now we get

$$\begin{array}{rll} \mathbb{L} \in \mathrm{Mod}(\mathbb{S}) & \text{iff} & C \leq C_{\mathbb{L}} \\ & \text{iff} & C \leq C_{\mathbb{L}'} \\ & \text{iff} & \mathbb{L}' \in \mathrm{Mod}(\mathbb{S}). \end{array}$$

The second statement follows from the fact that, by Lemma 69, the quotient G-morphism $\pi : \mathbb{L} \to_b \mathbb{L}^*$ is a bilogical G-morphism.

(b) This is similar. We get

$$\begin{array}{lll} \mathbb{S} \text{ complete with respect to } \mathsf{L} & \text{iff} & C = C_{\mathsf{L}} \\ & \text{iff} & C = C_{\mathsf{L}^*} \quad \text{(Part (a))} \\ & \text{iff} & \mathbb{S} \text{ complete with respect to } \mathsf{L}^*. \end{array}$$

■

If a class of models L contains the sentential G-logic $\mathbb{S}$ itself or its reduction, then $\mathbb{S}$ is complete with respect to L. This enables us to conclude that a sentential G-logic is complete with respect to both the class of its G-models and the class of its reduced G-models. The statement forms an analog of Proposition 2.6 of [28].

Proposition 86 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic. $\mathbb{S}$ is complete with respect to any class L of its G-models that includes $\mathbb{S}$ or $\mathbb{S}^*$, and also with respect to the corresponding reduced class L^*. In particular, $\mathbb{S}$ is complete with respect to the class of all its G-models and with respect to the class of all its reduced G-models.*

Proof: First, by definition, for any $\mathsf{L} \subseteq \mathrm{Mod}(\mathbb{S})$, we have $C \le C_{\mathsf{L}}$. Next, suppose $\mathbb{S} \in \mathsf{L}$. It suffices to show that $C_{\mathsf{L}} \le C$. This is clear, since, for all $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\begin{array}{rcl} C_{\mathsf{L}}(X) & = & \bigwedge\{X' \circ h : \mathbb{L} = \langle \mathcal{A}, D\rangle \in \mathsf{L}, X' \in \mathrm{Cl}(\mathbb{L}), \\ & & \qquad h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, X \le X' \circ h\} \\ & & \text{(Definition of } C_{\mathsf{L}}) \\ & \le & C(X), \end{array}$$

where the last inequality follows from the fact that $\mathbb{S} \in \mathsf{L}$, $C(X) \in \mathrm{Cl}(\mathbb{S})$ and the identity $i : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$ satisfy the conditions for membership in the set appearing on the left of the inequality. The statement concerning reductions follows from the one proven above and Proposition 85. ■

We close the section by characterizing those G-logics that constitute G-models of a given sentential G-logic $\mathbb{S}$. They are exactly the ones all of whose closed G-sets are $\mathbb{S}$-filters on the underlying G-algebra. This constitutes an analog of Proposition 2.7 of [28].

Proposition 87 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic and $\mathbb{L} = \langle \mathcal{A}, D\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, a G-logic. $\mathbb{L}$ is a G-model of $\mathbb{S}$ if and only if, for all $F \in \mathrm{Cl}(\mathbb{L})$, $\langle \mathbf{A}, F\rangle$ is an $\mathbb{S}$-matrix, i.e., $F \in \mathrm{Fi}_{\mathbb{S}}(\mathbf{A})$.*

Proof: Suppose, first, that, for all $F \in \mathrm{Cl}(\mathbb{L})$, we have $\langle \mathbf{A}, F\rangle \in \mathrm{Mat}(\mathbb{S})$, i.e., that, for all $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$,

$$X \le F \circ h \quad \text{implies} \quad C(X) \le F \circ h.$$

Thus, we get, for all $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$C(X) \leq \bigwedge_{F,h} \{F \circ h : X \leq F \circ h\} = C_{\mathbb{L}}(X).$$

Hence, $\mathbb{L}$ is a G-model of $\mathbb{S}$.

Conversely, suppose that $\mathbb{L}$ is a G-model of $\mathbb{S}$ and let $F \in \mathrm{Cl}(\mathbb{L})$, $X : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$, such that $X \leq F \circ h$. Then

$$\begin{array}{rcll} C(X) & \leq & C(F \circ h) & \text{(Monotonicity)} \\ & \leq & C_{\mathbb{L}}(F \circ h) & (\mathbb{L} \in \mathrm{Mod}(\mathbb{S})) \\ & = & F \circ h. & \text{(Definition of } C_{\mathbb{L}} \text{ and } F \in \mathrm{Cl}(\mathbb{L})) \end{array}$$

Therefore, $F \in \mathrm{Fi}_{\mathbb{S}}(\mathbf{A})$. ■

3.10 Full Graded Models

Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic and $\mathbb{L} = \langle \mathcal{A}, D\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, a G-logic. We say that $\mathbb{L}$ is a **full G-model** of $\mathbb{S}$ if

$$\mathbb{L}^* = \langle \mathcal{A}^*, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}^*)\rangle,$$

that is, if the closed functions of the reduction of $\mathbb{L}$ consist of all $\mathbb{S}$-filters on the quotient G-algebra. The class of all full G-models of $\mathbb{S}$ is denoted $\mathrm{FMod}(\mathbb{S})$. The class of all reduced full G-models of $\mathbb{S}$ is denoted $\mathrm{FMod}^*(\mathbb{S})$. For a given G-algebra $\mathcal{A}$, we write $\mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$ for the class of full G-models of $\mathbb{S}$ on $\mathcal{A}$.

The following result justifies the word "model" in the term "full G-model". It is an analog of Part (1) of Proposition 2.9 of [28] for G-models.

Proposition 88 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic. If a G-logic $\mathbb{L} = \langle \mathcal{A}, D\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, is a full G-model of $\mathbb{S}$, then it is a G-model of $\mathbb{S}$.*

Proof: Suppose $\mathbb{L} = \langle \mathcal{A}, D\rangle$ is a full G-model of $\mathbb{S}$. By definition, $\mathbb{L}^* = \langle \mathcal{A}^*, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}^*)\rangle$. Hence, by Proposition 87, $\mathbb{L}^*$ is a G-model of $\mathbb{S}$. Therefore, by Proposition 85, $\mathbb{L}$ is a G-model of $\mathbb{S}$. ■

It turns out that every model whose collection of closed sets consists of all $\mathbb{S}$-filters on the underlying G-algebra is a full G-model of $\mathbb{S}$. Proposition 89 lifts to G-models Proposition 2.10 of [28].

Proposition 89 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic and $\mathcal{A} = \langle \mathbf{A}, E\rangle$ a G-algebra. The G-logic $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle$ is a full G-model of $\mathbb{S}$. It is the weakest full G-model of $\mathbb{S}$ on the G-algebra $\mathcal{A}$.*

Proof: The first statement follows from Proposition 82, since we have

$$\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^* \cong \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}^*).$$

The second statement is obvious, since, by Proposition 88, $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle$ is a G-model of $\mathbb{S}$ and, by Proposition 87, it is clearly the weakest one. ■

Models of the form $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle$ are called **basic full G-models**.

We show, next, in an analog of Proposition 2.11 of [28], that the class of full G-models of a sentential G-logic is closed under bilogical G-morphisms.

Proposition 90 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic. The class* $\mathrm{FMod}(\mathbb{S})$ *is closed under bilogical G-morphisms. That is, if $\mathbb{L} = \langle \mathcal{A}, D\rangle$ and $\mathbb{L}' = \langle \mathcal{A}', D'\rangle$ are G-logics and $h : \mathbb{L} \to_b \mathbb{L}'$ is a bilogical G-morphism, then $\mathbb{L}$ is a full G-model of $\mathbb{S}$ if and only if $\mathbb{L}'$ is a full G-model of $\mathbb{S}$. In particular, a G-logic $\mathbb{L}$ is a full G-model of $\mathbb{S}$ if and only if its reduction $\mathbb{L}^*$ is a full G-model of $\mathbb{S}$.*

Proof: Let $\mathbb{L} = \langle \mathcal{A}, D\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, $\mathbb{L}' = \langle \mathcal{A}', D'\rangle$, with $\mathcal{A}' = \langle \mathbf{A}', E'\rangle$, be G-logics and $h : \mathbb{L} \to_b \mathbb{L}'$ a bilogical G-morphism. Then, by Proposition 74, $\mathbb{L}^* \cong \mathbb{L}'^*$. Therefore, we have

$$\begin{array}{lll} \mathbb{L} \text{ full } G\text{-model} & \text{iff} & \mathrm{Cl}(\mathbb{L}^*) = \mathrm{Fi}_{\mathbb{S}}(\mathbf{A}^*) \\ & \text{iff} & \mathrm{Cl}(\mathbb{L}'^*) = \mathrm{Fi}_{\mathbb{S}}(\mathbf{A}'^*) \\ & \text{iff} & \mathbb{L}' \text{ full } G\text{-model.} \end{array}$$

The second statement follows immediately, since, by Lemma 69, the quotient G-morphism $\pi : \mathbb{L} \to_b \mathbb{L}^*$ is a bilogical G-morphism. ■

Recall that in Proposition 89, it was shown that basic full G-models of $\mathbb{S}$ are special cases of full G-models of $\mathbb{S}$. It follows from Proposition 90 that any G-model that can be mapped onto a basic full G-model of $\mathbb{S}$ is a full G-model of $\mathbb{S}$. The converse of this also holds.

Corollary 91 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic and $\mathbb{L} = \langle \mathcal{A}, D\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, a G-logic. $\mathbb{L}$ is a full G-model of $\mathbb{S}$ if and only if there is a bilogical G-morphism from $\mathbb{L}$ onto a basic full G-model, i.e., a G-logic of the form $\langle \mathcal{A}', \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')\rangle$.*

Proof: Suppose, first, that $\mathbb{L}$ is a full G-model of $\mathbb{S}$. Then, by Lemma 69, the projection $\pi : \mathbb{L} \to_b \mathbb{L}^*$ is a bilogical G-morphism and, by the definition of a full G-model, $\mathbb{L}^*$ is a basic full G-model.

Assume, conversely, that there exists a bilogical G-morphism $h : \mathbb{L} \to_b \langle \mathcal{A}', \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')\rangle$. By Proposition 89, $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')\rangle$ is a full G-model of $\mathbb{S}$ and, by Proposition 90, $\mathbb{L}$ is a full G-model of $\mathbb{S}$. ■

The preceding results allow us to characterize the class of full G-models as the smallest class containing all basic full G-models and closed under bilogical G-morphisms (in both the forward and backward direction). This forms an analog of Corollary 2.13 of [28].

Corollary 92 *Let* $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ *be a sentential* G*-logic. The class* FMod($\mathbb{S}$) *is the smallest class of* G*-logics containing all basic full* G*-models, i.e.,* G*-logics of the form* $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle$*, and closed under (both images and pre-images of) bilogical* G*-morphisms.*

Proof: Denote by L the smallest class of G-logics containing all basic full G-models and closed under bilogical G-morphisms. We must show that $\mathsf{L} = \mathrm{FMod}(\mathbb{S})$.

First, by Proposition 89, every basic full G-model is in FMod($\mathbb{S}$). Moreover, by Proposition 90, FMod($\mathbb{S}$) is closed under bilogical G-morphisms. Therefore, by the definition of L, $\mathsf{L} \subseteq \mathrm{FMod}(\mathbb{S})$.

Conversely, suppose $\mathbb{L} \in \mathrm{FMod}(\mathbb{S})$. Then, by Lemma 69, the quotient G-morphism $\pi : \mathbb{L} \to_b \mathbb{L}^*$ is a bilogical G-morphism and, by the definition of full G-models, $\mathbb{L}^*$ is a basic full G-model. Therefore, $\mathbb{L} \in \mathsf{L}$. ∎

Full G-models assume their name from the fact that the closed sets of their reductions include all $\mathbb{S}$-filters on the reduced G-algebra. The next theorem gives another sense in which full G-models are "full". The collection of closed sets consists of all $\mathbb{S}$-filters with which the Tarski G-congruence of the G-model is compatible. These are the reasons given by Font and Jansana in [28] for choosing the name for these models in the sentential logic framework. The second justification they provide is the content of Theorem 2.14 of [28] whose analog for G-logics is the next result.

Theorem 93 *Let* $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ *be a sentential* G*-logic and* $\mathbb{L} = \langle \mathcal{A}, D\rangle$*, with* $\mathcal{A} = \langle \mathbf{A}, E\rangle$*, a* G*-logic.* $\mathbb{L}$ *is a full* G*-model of* $\mathbb{S}$ *if and only if*

$$\mathrm{Cl}(\mathbb{L}) = \{F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}) : \widetilde{\Omega}_{\mathcal{A}}(\mathrm{Cl}(\mathbb{L})) \leq \Omega_{\mathcal{A}}(F)\}.$$

Proof: Suppose that $\mathbb{L}$ is a full G-model of $\mathbb{S}$. We must show that the displayed equality holds.

Assume, first, that $F \in \mathrm{Cl}(\mathbb{L})$. By Propositions 89 and 87, $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. Moreover, by the discussion preceding Proposition 62, $\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Cl}(\mathbb{L})) \leq \Omega_{\mathcal{A}}(F)$.

Assume, next, that $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, such that $\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Cl}(\mathbb{L})) \leq \Omega_{\mathcal{A}}(F)$. Then, by Proposition 80, There exists $H \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}^*)$, such that $F = H \circ \pi$, where $\pi : \mathcal{A} \to \mathcal{A}^*$ is the quotient G-morphism. But, by Lemma 69, $\pi : \mathbb{L} \to_b \mathbb{L}^*$ is a bilogical G-morphism. Moreover, since, by hypothesis, $\mathbb{L}$ is a full G-model, $\mathrm{Cl}(\mathbb{L}^*) = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}^*)$. Thus, $F \in \mathrm{Cl}(\mathbb{L})$.

Suppose, conversely, that the displayed equality holds. Using Proposition 80, we can see that $\pi : \mathbb{L} \to_b \langle \mathcal{A}^*, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}^*)\rangle$ is a bilogical G-morphism. Hence, $\mathrm{Cl}(\mathbb{L})^* = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}^*)$. So, by definition, $\mathbb{L}$ is a full G-model of $\mathbb{S}$. ∎

3.11 $\mathbb{S}$-Algebras

Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic. A G-algebra $\mathcal{A} = \langle \mathbf{A}, E\rangle$ is called an $\mathbb{S}$-**algebra** if the G-logic $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle$ is reduced. Equivalently, $\mathcal{A}$ is an $\mathbb{S}$-algebra if and only if it is the G-algebraic reduct of a reduced full G-model of $\mathbb{S}$. We denote the class of all $\mathbb{S}$-algebras by $\mathrm{Alg}(\mathbb{S})$.

Note that the quotient G-algebra $\mathcal{F}m^*_{\mathcal{L}}(V) = \mathcal{F}m_{\mathcal{L}}(V)/\widetilde{\Omega}(\mathbb{S})$ is an $\mathbb{S}$-algebra. This follows from Proposition 82 and Corollary 73. The following is an analog of Proposition 2.17 of [28].

Proposition 94 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic and $\mathbb{L} = \langle \mathcal{A}, D\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, a G-logic. Then the following are equivalent:*

(i) $\mathbb{L}$ *is a reduced full G-model of* $\mathbb{S}$*;*

(ii) $\mathbb{L}$ *is reduced and* $\mathrm{Cl}(\mathbb{L}) = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$*;*

(iii) $\mathcal{A}$ *is an* $\mathbb{S}$*-algebra and* $\mathrm{Cl}(\mathbb{L}) = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$*.*

Proof:

(i)⇒(ii) Suppose that $\mathbb{L} = \langle \mathcal{A}, D\rangle$ is a reduced full G-model of $\mathbb{S}$. The fact that it is reduced shows that its reduction is $\mathbb{L}$ and the fact that it is full shows that $\mathrm{Cl}(\mathbb{L}) = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$.

(ii)⇒(iii) Suppose that $\mathbb{L}$ is reduced and $\mathrm{Cl}(\mathbb{L}) = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. Then $\mathcal{A}$ is an $\mathbb{S}$-algebra by definition. Moreover, by hypothesis, $\mathrm{Cl}(\mathbb{L}) = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$.

(iii)⇒(i) Since $\mathcal{A}$ is an $\mathbb{S}$-algebra, $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle$ is reduced. Since $\mathrm{Cl}(\mathbb{L}) = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, $\mathbb{L}$ is a reduced full G-model of $\mathbb{S}$.

■

Tarski G-congruences of full G-models of a G-logic $\mathbb{S}$ are $\mathrm{Alg}(\mathbb{S})$-congruences, as is the case with sentential logics (see Proposition 2.18 of [28]).

Proposition 95 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic and $\mathbb{L} = \langle \mathcal{A}, D\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, a full G-model of $\mathbb{S}$. Then $\mathcal{A}^*$ is an $\mathbb{S}$-algebra and, hence, $\widetilde{\Omega}(\mathbb{L}) \in \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$.*

Proof: Assume $\mathbb{L} = \langle \mathcal{A}, D\rangle$ is a full G-model of $\mathbb{S}$. Then, by definition, its reduction is $\langle \mathcal{A}^*, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}^*)\rangle$. Thus, again by definition, $\mathcal{A}^*$ is an $\mathbb{S}$-algebra. This shows that $\widetilde{\Omega}(\mathbb{L}) \in \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$. ■

The class $\mathrm{Alg}(\mathbb{S})$ may be characterized as the class of all G-algebraic reducts of all reduced G-models of $\mathbb{S}$. Notice that this characterization, which forms an analog of Proposition 2.19 of [28], does not employ the notion of fullness of models.

Proposition 96 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic. The class of $\mathbb{S}$-algebras is the class of G-algebraic reducts of all reduced G-models of $\mathbb{S}$.*

Proof: Let K be the class of algebraic reducts of all reduced G-models of $\mathbb{S}$. We need to show that $\mathsf{K} = \mathrm{Alg}(\mathbb{S})$. If $\mathcal{A} \in \mathrm{Alg}(\mathbb{S})$, then, by definition, $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle$ is a reduced full G-model of $\mathbb{S}$. Therefore, taking into account Proposition 88, $\mathcal{A} \in \mathsf{K}$. On the other hand, assume $\mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}$. Then, there exists a reduced G-model $\mathbb{L} = \langle \mathcal{A}, D\rangle$ of $\mathbb{S}$. Taking into account Proposition 87, consider the G-model $\mathbb{L}' = \langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle$ of $\mathbb{S}$. We have $\mathbb{L}' \leq \mathbb{L}$, whence, by Proposition 63 and the hypothesis,

$$E \leq \widetilde{\Omega}(\mathbb{L}') \leq \widetilde{\Omega}(\mathbb{L}) = E.$$

Thus, $\mathbb{L}'$ is a reduced full G-model of $\mathbb{S}$, showing that $\mathcal{A} \in \mathrm{Alg}(\mathbb{S})$. ∎

Further, the class $\mathrm{Alg}(\mathbb{S})$ is an abstract class, that is, it is closed under isomorphisms.

Proposition 97 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic. The class of $\mathbb{S}$-algebras is closed under isomorphisms.*

Proof: Suppose that $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic G-algebras. Using Proposition 79, we may show that $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ and $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$ are also isomorphic as lattices. Now, taking into account Propositions 81, 64 and 68, we get that $\mathbb{L} = \langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle$ and $\mathbb{L}' = \langle \mathcal{A}', \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')\rangle$ are G-isomorphic G-logics. Therefore, taking into account Propositions 67 and 68, $\mathbb{L}$ is reduced if and only if $\mathbb{L}'$ is reduced. Thus, $\mathcal{A}$ is an $\mathbb{S}$-algebra if and only if $\mathcal{A}'$ is an $\mathbb{S}$-algebra. ∎

Full G-models of a G-logic $\mathbb{S}$ may be characterized in terms of $\mathbb{S}$-algebras. This is an analog of Proposition 2.21 of [28].

Proposition 98 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic. Then the following conditions are equivalent:*

(i) $\mathbb{L} = \langle \mathcal{A}, D\rangle$*, with* $\mathcal{A} = \langle \mathbf{A}, E\rangle$*, is a full G-model of* $\mathbb{S}$*;*

(ii) $\mathcal{A}^*$ *is an $\mathbb{S}$-algebra and* $\mathrm{Cl}(\mathbb{L})^* = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}^*)$*;*

(iii) There is a bilogical G-morphism between $\mathbb{L}$ and a G-logic $\mathbb{L}' = \langle \mathcal{A}', D'\rangle$, such that $\mathcal{A}'$ is an $\mathbb{S}$-algebra and $\mathrm{Cl}(\mathbb{L}') = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')$.

Proof:

(i)⇒(ii) Suppose $\mathbb{L} = \langle \mathcal{A}, D\rangle$ is a full G-model of $\mathbb{S}$. By definition, $\mathrm{Cl}(\mathbb{L})^* = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}^*)$. Thus, also by definition, $\mathcal{A}^* \in \mathrm{Alg}(\mathbb{S})$.

(ii)⇒(iii) We take $\mathbb{L}' = \mathbb{L}^*$ and consider the quotient G-morphism.

(iii)⇒(i) By Proposition 94, $\mathbb{L}'$ is a reduced full G-model of $\mathbb{S}$. By Corollary 91, $\mathbb{L}$ is a full G-model of $\mathbb{S}$.

■

We have seen in Proposition 86 that a sentential G-logic $\mathbb{S}$ is complete with respect to any class L of its G-models that includes $\mathbb{S}$ or $\mathbb{S}^*$, and also with respect to the corresponding reduced class L^*. We inferred that $\mathbb{S}$ is complete with respect to the class of all its G-models and with respect to the class of all its reduced G-models. We prove next an analog of Theorem 2.22 of [28] which is a "better" or "more advanced" completeness result, in the sense that it asserts completeness with respect to smaller, more targeted, classes of models, which, however, are still large enough to serve the purpose.

Theorem 99 (Completeness Theorem) *Let* $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ *be a sentential G-logic.* $\mathbb{S}$ *is complete with respect to the following classes of G-logics:*

1. *The class of all full G-models of* $\mathbb{S}$*;*
2. *The class of all basic full G-models of* $\mathbb{S}$*, i.e., those of the form* $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle$*, for any G-algebra* $\mathcal{A}$*;*
3. *The class of all reduced full G-models of* $\mathbb{S}$*, that is, the class of all G-logics of the form* $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle$*,* $\mathcal{A} \in \mathrm{Alg}(\mathbb{S})$*.*

Proof: Observe that all three classes described consist of G-models of $\mathbb{S}$. Moreover, $\mathbb{S}^*$ belongs to all three. Therefore, by Proposition 86, $\mathbb{S}$ is complete with respect to each of these classes. ■

Recall the class $\mathrm{Alg}^*(\mathbb{S})$ of all G-algebraic reducts of reduced $\mathbb{S}$-matrices. There is a strong connection between this class and the class of $\mathbb{S}$-algebras. The main thread tying them together is the relation between the Tarski G-congruence of an $\mathbb{S}$-model and the Leibniz G-congruences of the $\mathbb{S}$-matrices associated with its closed functions. Theorem 100 constitutes an analog of Theorem 2.23 of [28] for G-logics and their associated classes of G-algebras.

Theorem 100 *Let* $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ *be a sentential G-logic. Then* $\mathrm{Alg}(\mathbb{S})$ *is the class of all subdirect products of G-algebras in the class* $\mathrm{Alg}^*(\mathbb{S})$*.*

Proof: Suppose, first, that $\mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathrm{Alg}(\mathbb{S})$. Then

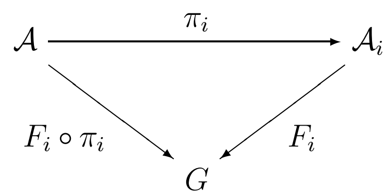
$$E = \widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})) = \bigwedge\{\Omega_{\mathcal{A}}(F) : F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\}.$$

By Proposition 56, $\mathcal{A}$ is a subdirect product of $\{\mathcal{A}/\Omega_{\mathcal{A}}(F) : F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\} \subseteq \mathrm{Alg}^*(\mathbb{S})$.

Assume, conversely, that $\mathcal{A} = \langle \mathbf{A}, E\rangle$ is a subdirect product of

$$\{\mathcal{A}_i = \langle \mathbf{A}_i, E_i\rangle : i \in I\} \subseteq \mathrm{Alg}^*(\mathbb{S}).$$

By definition of $\mathrm{Alg}^*(\mathbb{S})$, for every $i \in I$, there exists $F_i \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}_i)$, such that $\Omega_{\mathcal{A}_i}(F_i) = E_i$.



Let D be the G-operator on $\mathcal{A}$ generated by the family of $\mathbb{S}$-filters $F_i \circ \pi_i$, $i \in I$, where $\pi_i : \mathcal{A} \to \mathcal{A}_i$ denotes the i-th projection morphism of the product restricted to $\mathcal{A}$. Then, by Propositions 79 and 87, $\mathbb{L} = \langle \mathcal{A}, D\rangle$ is a G-model of $\mathbb{S}$. We show it is reduced. We have, for all $a, b \in A$,

$$\begin{aligned}
\widetilde{\Omega}(\mathbb{L}) &= \textstyle\bigwedge_{H \in \mathrm{Cl}(\mathbb{L})} \Omega_{\mathcal{A}}(H) \quad \text{(Tarski \& Leibniz operators)}\\
&\le \textstyle\bigwedge_{i\in I} \Omega_{\mathcal{A}}(F_i \circ \pi_i) \quad (F_i \circ \pi_i \in \mathrm{Cl}(\mathbb{L}))\\
&= \textstyle\bigwedge_{i\in I} \Omega_{\mathcal{A}_i}(F_i) \circ \pi_i^2 \quad \text{(Theorem 58)}\\
&= \textstyle\bigwedge_{i\in I} E_i \circ \pi_i^2 \quad (\mathcal{A}_i \in \mathrm{Alg}^*(\mathbb{S}))\\
&= E. \quad (\mathcal{A} \subseteq_{\mathsf{sd}} \textstyle\prod_{i\in I} \mathcal{A}_i)
\end{aligned}$$

This shows that $\mathbb{L}$ is reduced. Therefore, by Proposition 96, $\mathcal{A} \in \mathrm{Alg}(\mathbb{S})$. ■

Corollary 101 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic. Then*

$$\mathrm{Alg}^*(\mathbb{S}) \subseteq \mathrm{Alg}(\mathbb{S})$$

and the two classes coincide iff $\mathrm{Alg}^*(\mathbb{S})$ is closed under subdirect products.

Proof: Directly from Theorem 100. ■

Finally, we can state a relationship between the corresponding associated classes of G-algebras of two sentential G-logics one of which is an extension of the other. This forms an analog of Proposition 2.27 of [28].

Proposition 102 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ and $\mathbb{S}' = \langle \mathcal{F}m_{\mathcal{L}}(V), C'\rangle$ be sentential G-logics, such that $\mathbb{S} \le \mathbb{S}'$. Then $\mathrm{Alg}(\mathbb{S}') \subseteq \mathrm{Alg}(\mathbb{S})$ and $\mathrm{Alg}^*(\mathbb{S}') \subseteq \mathrm{Alg}^*(\mathbb{S})$.*

Proof: Assume $\mathbb{S} \le \mathbb{S}'$. Then, by Lemma 77, for every G-algebra $\mathcal{A}$, $\mathrm{Fi}_{\mathbb{S}'}(\mathcal{A}) \subseteq \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. Therefore, directly by the respective definitions, $\mathrm{Alg}^*(\mathbb{S}') \subseteq \mathrm{Alg}^*(\mathbb{S})$. By Theorem 100, we obtain, also, that $\mathrm{Alg}(\mathbb{S}') \subseteq \mathrm{Alg}(\mathbb{S})$. ■

3.12 The Lattice of Full G-Models

In this section, we present an analog for G-logics, G-models and G-congruences of the celebrated Isomorphism Theorem 2.30 of [28], which asserts that $\widetilde{\Omega}_{\mathbf{A}}$ is an order-isomorphism between the ordered set of full models of a sentential logic on an algebra $\mathbf{A}$ and that of $\mathrm{Alg}(\mathcal{S})$-congruences on $\mathbf{A}$.

Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic, $\mathcal{A} = \langle \mathbf{A}, E\rangle$ a G-algebra and $\Theta \in \mathrm{Gon}(\mathcal{A})$. We denote by

$$\widetilde{H}_{\mathcal{A}}(\Theta) = \langle \mathcal{A}, C_{\Theta}\rangle$$

the G-logic on $\mathcal{A}$ projectively generated from $\langle \mathcal{A}/\Theta, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)\rangle$ by the quotient G-morphism $\pi_{\hat{\Theta}} : \mathcal{A} \to \mathcal{A}/\Theta$.

The definition implies that

$$\pi_{\hat{\Theta}} : \widetilde{H}_{\mathcal{A}}(\Theta) \to_b \langle \mathcal{A}/\Theta, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)\rangle$$

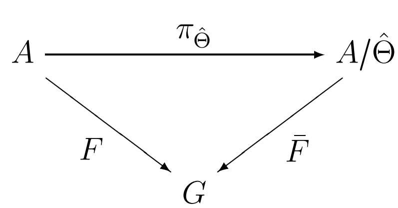
is a bilogical G-morphism.

Lemma 103 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic, $\mathcal{A} = \langle \mathbf{A}, E\rangle$ a G-algebra and $\Theta \in \mathrm{Gon}(\mathcal{A})$. Then:*

- $\Theta \in \mathrm{Gon}(\widetilde{H}_{\mathcal{A}}(\Theta))$;
- $\widetilde{H}_{\mathcal{A}}(\Theta)/\Theta = \langle \mathcal{A}/\Theta, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)\rangle$;
- $\widetilde{H}_{\mathcal{A}}(\Theta) \in \mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$.

Moreover, the mapping $\Theta \mapsto \widetilde{H}_{\mathcal{A}}(\Theta)$ is order preserving, that is, if $\Theta, \Theta' \in \mathrm{Gon}(\mathcal{A})$, such that $\Theta \le \Theta'$, then $\widetilde{H}_{\mathcal{A}}(\Theta) \le \widetilde{H}_{\mathcal{A}}(\Theta')$.

Proof: Fist, we show that $\Theta \in \mathrm{Gon}(\widetilde{H}_{\mathcal{A}}(\Theta))$. Let $a, b \in A$ and consider $F \in \mathrm{Fi}_{\mathbb{S}}(\widetilde{H}_{\mathcal{A}}(\Theta))$. Then, since $\widetilde{H}_{\mathcal{A}}(\Theta)$ is projectively generated from $\langle \mathcal{A}/\Theta, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)\rangle$, there exists $\bar{F} \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)$, such that $F = \bar{F} \circ \pi_{\hat{\Theta}}$.

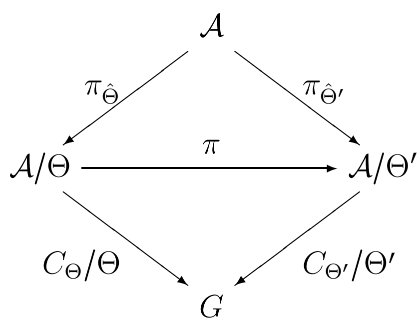


Now we get

$$\begin{aligned} \Theta(a,b) \wedge F(a) &= \bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) \wedge \bar{F}(a/\hat{\Theta}) \quad (\text{Definitions of } \bar{\Theta} \text{ and } \bar{F}) \\ &\le \bar{F}(b/\hat{\Theta}) \quad (\bar{F} \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)) \\ &= F(b). \quad (\text{Definition of } \bar{F}) \end{aligned}$$

Since $\Theta \in \mathrm{Gon}(\widetilde{H}_{\mathcal{A}}(\Theta))$, it makes sense to consider the quotient $\widetilde{H}_{\mathcal{A}}(\Theta)/\Theta$ and, by the definition of $\widetilde{H}_{\mathcal{A}}(\Theta)$, $\widetilde{H}_{\mathcal{A}}(\Theta)/\Theta = \langle \mathcal{A}/\Theta, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)\rangle$. By Proposition 89, $\langle \mathcal{A}/\Theta, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)\rangle$ is a full G-model of $\mathbb{S}$, whence, by Corollary 91, $\widetilde{H}_{\mathcal{A}}(\Theta) \in \mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$.

Finally, let $\Theta, \Theta' \in \mathrm{Gon}(\mathcal{A})$, with associated quotient morphisms $\pi_{\hat{\Theta}} : \mathcal{A} \to \mathcal{A}/\Theta$ and $\pi_{\hat{\Theta}'} : \mathcal{A} \to \mathcal{A}/\Theta'$. Assume that $\Theta \le \Theta'$ and let $\pi : \mathcal{A}/\Theta \to \mathcal{A}/\Theta'$ be the homomorphism $a/\hat{\Theta} \mapsto a/\hat{\Theta}'$, which is well-defined because of the inequality $\Theta \le \Theta'$ and, moreover, makes the following diagram commute.



Now we have

$$\begin{array}{rcll}
\mathrm{Cl}(\widetilde{H}_{\mathcal{A}}(\Theta')) & = & \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta') \circ \pi_{\hat{\Theta}'} & (\pi_{\hat{\Theta}'} : \widetilde{H}_{\mathcal{A}}(\Theta') \to_b \langle \mathcal{A}/\Theta', \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta')\rangle) \\
& = & \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta') \circ \pi \circ \pi_{\hat{\Theta}} & (\pi_{\hat{\Theta}'} = \pi \circ \pi_{\hat{\Theta}}) \\
& \subseteq & \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta) \circ \pi_{\hat{\Theta}} & (\text{Proposition } 79) \\
& = & \mathrm{Cl}(\widetilde{H}_{\mathcal{A}}(\Theta)). & (\pi_{\hat{\Theta}} : \widetilde{H}_{\mathcal{A}}(\Theta) \to_b \langle \mathcal{A}/\Theta, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)\rangle)
\end{array}$$

Therefore, by Proposition 61, we conclude that $\widetilde{H}_{\mathcal{A}}(\Theta) \le \widetilde{H}_{\mathcal{A}}(\Theta')$. ∎

The following Isomorphism Theorem is a version in the current setting of the well known Isomorphism Theorem (Theorem 2.30 of [28]) devised by Font and Jansana in the framework of sentential logics.

Theorem 104 (The Isomorphism Theorem) *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic and $\mathcal{A} = \langle \mathbf{A}, E\rangle$ a G-algebra. The Tarski operator $\widetilde{\Omega}_{\mathcal{A}}$ is an order isomorphism between $\langle \mathrm{FMod}_{\mathbb{S}}(\mathcal{A}), \le\rangle$ and $\langle \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A}), \le\rangle$, with $\widetilde{H}_{\mathcal{A}}$ as its inverse,*

$$\widetilde{\Omega}_{\mathcal{A}} : \quad \langle \mathrm{FMod}_{\mathbb{S}}(\mathcal{A}), \le\rangle \rightleftarrows \langle \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A}), \le\rangle \; : \widetilde{H}_{\mathcal{A}}$$

Proof: By Proposition 95, if $\mathbb{L} = \langle \mathcal{A}, D\rangle \in \mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$, then we have $\widetilde{\Omega}(\mathbb{L}) \in \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$. Conversely, by Lemma 103, if $\Theta \in \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$, then $\widetilde{H}_{\mathcal{A}}(\Theta) \in \mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$. Thus, both mappings are well defined. We show that they are inverse bijections.

Suppose, first, that $\mathbb{L} = \langle \mathcal{A}, D\rangle \in \mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$. By Proposition 95, $\mathcal{A}^* \in \mathrm{Alg}(\mathbb{S})$ and $\widetilde{\Omega}(\mathbb{L}) \in \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$. Moreover, $\mathbb{L}$ is projectively generated from $\langle \mathcal{A}^*, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}^*)\rangle$ by the quotient G-morphism. Thus, by definition, $\mathbb{L} = \widetilde{H}_{\mathcal{A}}(\widetilde{\Omega}_{\mathcal{A}}(\mathbb{L}))$.

Suppose, on the other hand, that $\Theta \in \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$. By definition, $\langle \mathcal{A}/\Theta, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)\rangle$ is reduced, i.e.,

$$\widetilde{\Omega}_{\mathcal{A}/\Theta}(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)) = \bar{\Theta}.$$

Then, we have

$$\begin{array}{rcll}\widetilde{\Omega}_{\mathcal{A}}(\widetilde{H}_{\mathcal{A}}(\Theta)) & = & \widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta) \circ \pi_{\hat{\Theta}}) & (\text{Definition of } \widetilde{H}_{\mathcal{A}}(\Theta)) \\ & = & \widetilde{\Omega}_{\mathcal{A}/\Theta}(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)) \circ \pi^2_{\hat{\Theta}} & (\text{Proposition 67}) \\ & = & \bar{\Theta} \circ \pi^2_{\hat{\Theta}} & (\text{Display above}) \\ & = & \Theta. & (\text{Definition of } \bar{\Theta})\end{array}$$

We conclude that $\widetilde{\Omega}_{\mathcal{A}}$ and $\widetilde{H}_{\mathcal{A}}$ are inverse bijections. Finally, note that, by Proposition 63, $\widetilde{\Omega}_{\mathcal{A}}$ is order preserving and, by Lemma 103, $\widetilde{H}_{\mathcal{A}}$ is also order preserving, whence they are order isomorphisms between $\langle \mathrm{FMod}_{\mathbb{S}}(\mathcal{A}), \le\rangle$ and $\langle \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A}), \le\rangle$. ■

It can be shown that one of the two ordered sets proven isomorphic in the Isomorphism Theorem has the structure of a complete lattice and this allows us, via the Isomorphism Theorem, to conclude that the other does also. See Theorem 2.31 and Corollary 2.32 of [28] for the corresponding statements in the context of sentential logics.

Theorem 105 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic and $\mathcal{A} = \langle \mathbf{A}, E\rangle$ a G-algebra. The ordered set $\langle \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A}), \le\rangle$ is a complete lattice, with the meet operation $\wedge$ in $\mathbf{Gon}(\mathcal{A})$ as its meet.*

Proof: Let $\{\Theta_i : i \in I\} \subseteq \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$ be nonempty and consider $\Theta := \bigwedge_{i\in I} \Theta_i$. We must show that $\Theta \in \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$. Define, for every $i \in I$, $h_i : \mathcal{A}/\Theta \to \mathcal{A}/\Theta_i$ by

$$h_i(a/\hat{\Theta}) = a/\hat{\Theta}_i, \quad a \in A.$$

This is well defined. To see this, note that, for all $a, b \in A$,

$$\begin{array}{rcll}\langle a,b\rangle \in \hat{\Theta} & \text{iff} & \Theta(a,b) = \top & (\text{Definition of } \hat{\Theta}) \\ & \text{iff} & \bigwedge_{i\in I}\Theta_i(a,b) = \top & (\text{Definition of } \Theta) \\ & \text{iff} & \Theta_i(a,b) = \top,\ i \in I, & (\text{Property of } \bigwedge) \\ & \text{iff} & \langle a,b\rangle \in \hat{\Theta}_i,\ i \in I. & (\text{Definition of } \hat{\Theta}_i)\end{array}$$

Thus, we obtain

$$\begin{array}{rcll}\bar{\Theta}(a/\hat{\Theta}, b/\hat{\Theta}) & = & \Theta(a,b) & (\text{Definition of } \bar{\Theta}) \\ & = & \bigwedge_{i\in I}\Theta_i(a,b) & (\text{Definition of } \Theta) \\ & = & \bigwedge_{i\in I}\bar{\Theta}_i(a/\hat{\Theta}_i, b/\hat{\Theta}_i) & (\text{Definition of } \bar{\Theta}_i) \\ & = & \bigwedge_{i\in I}\bar{\Theta}_i(h_i(a/\hat{\Theta}), h_i(b/\hat{\Theta})). & (\text{Definition of } h_i)\end{array}$$

Since, for every $i \in I$, $\Theta_i \in \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$, the G-logic $\mathbb{L}_i = \langle \mathcal{A}/\Theta_i, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta_i)\rangle$ is reduced, that is, we have, for all $i \in I$,

$$\widetilde{\Omega}_{\mathcal{A}/\Theta_i}(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta_i)) = \bar{\Theta}_i.$$

Our goal is to show that $\mathbb{L} = \langle \mathcal{A}/\Theta, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)\rangle$ is reduced. We calculate

$$\begin{array}{lll}\widetilde{\Omega}(\mathbb{L}) & \leq & \bigwedge_{i\in I}\widetilde{\Omega}_{\mathcal{A}/\Theta}(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta_i)\circ h_i) \quad \text{(Proposition 79)}\\ & = & \bigwedge_{i\in I}\widetilde{\Omega}_{\mathcal{A}/\Theta_i}(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta_i))\circ h_i^2 \quad \text{(Theorem 58)}\\ & = & \bigwedge_{i\in I}\bar{\Theta}_i\circ h_i^2 \quad \text{(Displayed above)}\\ & = & \bar{\Theta}. \quad \text{(Shown above)}\end{array}$$

We conclude that $\mathbb{L}$ is reduced and $\Theta \in \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$.

Finally, observe that any one-element algebra belongs to $\mathrm{Alg}(\mathbb{S})$, whence $\nabla^{\mathcal{A}} \in \mathrm{Con}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$ and, hence, it is the largest element in this lattice. It corresponds to the G-logic whose only closed G-set is the function mapping every sentence to $\top$ in G, which is clearly a full G-model of $\mathbb{S}$. ■

Corollary 106 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic and $\mathcal{A} = \langle \mathbf{A}, E\rangle$ a G-algebra. Then $\langle \mathrm{FMod}_{\mathbb{S}}(\mathcal{A}), \leq\rangle$ is a complete lattice and the Tarski operator is a lattice isomorphism from $\langle \mathrm{FMod}_{\mathbb{S}}(\mathcal{A}), \leq\rangle$ to $\langle \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A}), \leq\rangle$.*

Proof: This follows from Theorems 104 and 105. ■

$\mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$ is a subset of the complete lattice of all G-logics over $\mathcal{A}$. However, it is not a sublattice. As a consequence of the preceding results, we can see that, given a collection $\{\mathbb{L}_i : i \in I\} \subseteq \mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$, its meet in $\mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$ can be obtained as the G-logic projectively generated from $\langle \mathcal{A}/\Theta, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Theta)\rangle$ by the projection homomorphism $\pi_{\hat{\Theta}} : \mathcal{A} \to \mathcal{A}/\Theta$, where $\Theta = \bigwedge_{i\in I}\widetilde{\Omega}(\mathbb{L}_i)$.

Some extensions of the isomorphisms detailed above may be established via the use of bilogical morphisms. The following proposition and corollary are analogs in the G-logic framework of Proposition 2.33 and of Corollary 2.34, respectively, of [28].

Proposition 107 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic, $\mathbb{L} = \langle \mathcal{A}, D\rangle$, $\mathbb{L}' = \langle \mathcal{A}', D'\rangle$ two full G-models of $\mathbb{S}$ and $h : \mathbb{L} \to_b \mathbb{L}'$ a bilogical G-morphism, with h' a section of h. Then the mapping*

$$\mathcal{X} \mapsto \{X \circ h' : X \in \mathcal{X}\}$$

establishes an isomorphism between the lattice of all full G-models of $\mathbb{S}$ on $\mathcal{A}$ extending $\mathbb{L}$ and the lattice of all full G-models of $\mathbb{S}$ on $\mathcal{A}'$ extending $\mathbb{L}'$. Moreover, the principal ideals of $\mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$ and of $\mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A}')$ determined by $\widetilde{\Omega}(\mathbb{L})$ and $\widetilde{\Omega}(\mathbb{L}')$, respectively, are isomorphic.

Proof: By Corollary 66, the given mapping is an isomorphism between the lattice of all G-logics extending $\mathbb{L}$ and the lattice of all G-logics extending $\mathbb{L}'$. By Proposition 90, this restricts to an isomorphism between the lattice of all full G-models of $\mathbb{S}$ on $\mathcal{A}$ extending $\mathbb{L}$ and the lattice of all full G-models of $\mathbb{S}$ on $\mathcal{A}'$ extending $\mathbb{L}'$. The last statement is a direct consequence of the Isomorphism Theorem 104. ■

Corollary 108 *Let* $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ *be a sentential G-logic,* $\mathcal{A} = \langle \mathbf{A}, E\rangle$*,* $\mathcal{A}' = \langle \mathbf{A}', E'\rangle$ *G-algebras and* $h : \mathcal{A} \to \mathcal{A}'$ *an epimorphism, such that* $h : \langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle \to_b \langle \mathcal{A}', \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')\rangle$ *is a bilogical G-morphism. Then h induces an isomorphism between the complete lattices* $\mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$ *and* $\mathrm{FMod}_{\mathbb{S}}(\mathcal{A}')$*. In addition, the lattices* $\mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$ *and of* $\mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A}')$ *are isomorphic.*

Proof: By Proposition 89, $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})\rangle$ is the weakest full G-model of $\mathbb{S}$ on $\mathcal{A}$ and $\langle \mathcal{A}', \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}')\rangle$ is the weakest full G-model of $\mathbb{S}$ on $\mathcal{A}'$. Therefore, by Proposition 107, h induces an isomorphism between the complete lattices $\mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$ and $\mathrm{FMod}_{\mathbb{S}}(\mathcal{A}')$. ■

3.13 Protoalgebraic G-Logics

Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic. We say that $\mathbb{S}$ is **protoalgebraic** if, for all $X \in \mathrm{Th}(\mathbb{S})$, $\Omega_{\mathcal{F}m_{\mathcal{L}}(V)}(X)$ is compatible with every $X' \in \mathrm{Th}(\mathbb{S})^X$. Assuming that G has an implication $\to$, this can be expressed by the condition that, for all $X \in \mathrm{Th}(\mathbb{S})$ and all all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Omega_{\mathcal{F}m_{\mathcal{L}}(V)}(X)(\varphi, \psi) \leq \bigwedge_{X \leq X' \in \mathrm{Th}(\mathbb{S})} (X'(\varphi) \leftrightarrow X'(\psi)).$$

This definition, which in the traditional framework is encapsulated in the motto "indistinguishability implies interderivability", turns out to be equivalent to the, more semantic in flavor, property of the Leibniz operator being monotone on $\mathrm{Th}(\mathbb{S})$.

Proposition 109 *Let* $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ *be a sentential G-logic. $\mathbb{S}$ is protoalgebraic if and only if* $\Omega_{\mathcal{F}m_{\mathcal{L}}(V)}$ *is monotone on* $\mathrm{Th}(\mathbb{S})$*.*

Proof: Suppose, first, that $\mathbb{S}$ is protoalgebraic. To show monotonicity, let $X, X' \in \mathrm{Th}(\mathbb{S})$, such that $X \leq X'$. By definition of protoalgebraicity, $\Omega_{\mathcal{F}m_{\mathcal{L}}(V)}(X)$ is compatible with X'. Hence, by the maximality property of $\Omega_{\mathcal{F}m_{\mathcal{L}}(V)}(X')$, we get $\Omega_{\mathcal{F}m_{\mathcal{L}}(V)}(X) \leq \Omega_{\mathcal{F}m_{\mathcal{L}}(V)}(X')$. So $\Omega_{\mathcal{F}m_{\mathcal{L}}(V)}$ is monotone on $\mathrm{Th}(\mathbb{S})$.

Suppose, conversely, that $\Omega_{\mathcal{F}m_{\mathcal{L}}(V)}$ is monotone on $\mathrm{Th}(\mathbb{S})$. Let $X, X' \in \mathrm{Th}(\mathbb{S})$, with $X \leq X'$. Then, by monotonicity, $\Omega_{\mathcal{F}m_{\mathcal{L}}(V)}(X') \leq \Omega_{\mathcal{F}m_{\mathcal{L}}(V)}(X')$.

This shows that $\Omega_{\mathcal{F}m_{\mathcal{L}}(V)}(X)$ is compatible with X'. Thus, $\mathbb{S}$ is protoalgebraic. ∎

Further, monotonicity of the Leibniz operator on $\mathrm{Th}(\mathbb{S})$ is equivalent to the monotonicity of $\Omega_{\mathcal{A}}$ on the $\mathbb{S}$-filters of any G-algebra $\mathcal{A}$. The fact that the property of monotonicity of the Leibniz operator "transfers" from $\mathrm{Th}(\mathbb{S})$ to $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, for every G-algebra $\mathcal{A}$, is an example of a class of theorems of a similar type which are collectively termed **transfer theorems**.

Proposition 110 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic. $\mathbb{S}$ is protoalgebraic if and only if, for every G-algebra $\mathcal{A}$, admitting a surjective G-morphism $h : \mathcal{F}m_{\mathcal{L}}(V) \to \mathcal{A}$, $\Omega_{\mathcal{A}}$ is monotone on $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$.*

Proof: First, suppose the given condition holds. Then take $\mathcal{A} = \mathcal{F}m_{\mathcal{L}}(V)$ and recall that, by Lemma 78, $\mathrm{Fi}_{\mathbb{S}}(\mathcal{F}m_{\mathcal{L}}(V)) = \mathrm{Th}(\mathbb{S})$. Hence, $\Omega_{\mathcal{F}m_{\mathcal{L}}(V)}$ is monotone on $\mathrm{Th}(\mathbb{S})$. Therefore, by Proposition 109, $\mathbb{S}$ is protoalgebraic.

For the converse, suppose $\mathbb{S}$ is protoalgebraic. Thus, by Proposition 109, $\Omega_{\mathcal{F}m_{\mathcal{L}}(V)}$ is monotone on $\mathrm{Th}(\mathbb{S})$. Now assume $F, F' \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, such that $F \leq F'$. Let $h : \mathcal{F}m_{\mathcal{L}}(V) \to \mathcal{A}$ be a surjective G-morphism. Recall, by Lemma 77, that if $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, then $F \circ h \in \mathrm{Th}(\mathbb{S})$. So we have

$$\begin{array}{rcll} \Omega_{\mathcal{A}}(F) \circ h^2 & = & \Omega_{\mathcal{F}m_{\mathcal{L}}(V)}(F \circ h) & \text{(Lemma 58)} \\ & \leq & \Omega_{\mathcal{F}m_{\mathcal{L}}(V)}(F' \circ h) & \text{(Hypothesis)} \\ & = & \Omega_{\mathcal{A}}(F') \circ h^2. & \text{(Lemma 58)} \end{array}$$

Hence, by the surjectivity of h, $\Omega_{\mathcal{A}}(F) \leq \Omega_{\mathcal{A}}(F')$. Thus, $\Omega_{\mathcal{A}}$ is monotone on $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. ∎

Another characterization involves the commutativity of the Leibniz operator with arbitrary meets.

Corollary 111 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic. $\mathbb{S}$ is protoalgebraic if and only if, for every G-algebra $\mathcal{A}$, admitting a surjective homomorphism $h : \mathcal{F}m_{\mathcal{L}}(V) \to \mathcal{A}$, $\Omega_{\mathcal{A}}$ commutes with arbitrary meets, i.e., for all $F_i \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$,*

$$\Omega_{\mathcal{A}}\left(\bigwedge_{i\in I} F_i\right) = \bigwedge_{i\in I} \Omega_{\mathcal{A}}(F_i).$$

Proof: First, assume that $\mathbb{S}$ is protoalgebraic. Since, by definition, $\bigwedge_{i\in I} F_i \leq F_i$, for all $i \in I$, we get, by Proposition 109, $\Omega_{\mathcal{A}}(\bigwedge_{i\in I} F_i) \leq \Omega_{\mathcal{A}}(F_i)$. Since this holds for all $i \in I$, $\Omega_{\mathcal{A}}(\bigwedge_{i\in I} F_i) \leq \bigwedge_{i\in I} \Omega_{\mathcal{A}}(F_i)$. Conversely, suppose $a, b \in A$. Then

$$\begin{array}{rcl} \bigwedge_{i\in I} \Omega_{\mathcal{A}}(F_i)(a,b) \wedge \bigwedge_{i\in I} F_i(a) & \leq & \Omega_{\mathcal{A}}(F_i)(a,b) \wedge F_i(a) \\ & \leq & F_i(b). \end{array}$$

So we get

$$\bigwedge_{i\in I} \Omega_{\mathcal{A}}(F_i)(a) \wedge \bigwedge_{i\in I} F_i(a) \leq \bigwedge_{i\in I} F_i(b).$$

This shows that $\bigwedge_{i\in I}\Omega_{\mathcal{A}}(F_i)$ is compatible with $\bigwedge_{i\in I}F_i$. Thus, by the maximality property of the Leibniz G-congruence, $\bigwedge_{i\in I}\Omega_{\mathcal{A}}(F_i)\le\Omega_{\mathcal{A}}(\bigwedge_{i\in I}F_i)$.

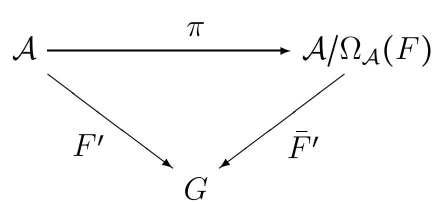
Assume, conversely, that $\Omega_{\mathcal{A}}$ is meet continuous and let $F,F'\in\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, such that $F\le F'$. Then, we have

$$\Omega_{\mathcal{A}}(F)=\Omega_{\mathcal{A}}(F\wedge F')=\Omega_{\mathcal{A}}(F)\wedge\Omega_{\mathcal{A}}(F').$$

Thus, $\Omega_{\mathcal{A}}(F)\le\Omega_{\mathcal{A}}(F')$. We conclude that $\Omega_{\mathcal{A}}$ is monotone on $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ and, hence, by Proposition 109, $\mathbb{S}$ is protoalgebraic. ■

Note that the preceding results hold for all G-algebras such that they hold for all their subalgebras generated by sets of generators equipotent with the cardinality of the free algebra. This ensures the existence of surjective homomorphisms from the formula algebra onto those subalgebras. *In the sequel we assume that such homomorphisms exist.*

We dwell briefly on the *compatibility property*, since it gives protoalgebraic logics their distinctive flavor and, as Blok and Pigozzi noted, makes them amenable to the methods of Universal Algebra. Let $\mathcal{A}=\langle\mathbf{A},E\rangle$ be a G-algebra. Let $F,F'\in\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, such that $F\le F'$. Then protoalgebraicity implies that $\Omega_{\mathcal{A}}(F)\le\Omega_{\mathcal{A}}(F')$. Hence, the G-congruence $\Omega_{\mathcal{A}}(F)$ is compatible with the $\mathbb{S}$-filter F'. That is, F' is constant on the equivalence classes of $\Omega_{\mathcal{A}}(F)$. Another way to express this is that, for the quotient G-morphism $\pi:\mathcal{A}\to\mathcal{A}/\Omega_{\mathcal{A}}(F)$, we have that, for all $F'\in\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, with $F\le F'$, there exists an $\bar{F}'\in\mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$, such that the following triangle commutes.



The correspondence $F'\mapsto\bar{F}'$ establishes a lattice isomorphism between the lattice of all $\mathbb{S}$-filters on $\mathcal{A}$ greater than F and the lattice of all $\mathbb{S}$-filters on $\mathcal{A}/\Omega_{\mathcal{A}}(F)$ greater than $\bar{F}$.

The next proposition, an analog of Proposition 3.1 of [28], characterizes protoalgebraicity via the behavior of the Tarski operator. It shows that protoalgebraicity of the G-logic is tantamount to the Tarski G-congruence of any G-model of the logic having the same value as the Leibniz G-congruence of the least closed G-set of the G-model on the underlying G-algebra.

Proposition 112 *Let* $\mathbb{S}=\langle\mathcal{F}m_{\mathcal{L}}(V),C\rangle$ *be a sentential G-logic. The following conditions are equivalent.*

(i) $\mathbb{S}$ *is protoalgebraic;*

(ii) For any G-model $\mathbb{L}=\langle\mathcal{A},D\rangle$ *of* $\mathbb{S}$, $\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Cl}(\mathbb{L}))=\Omega_{\mathcal{A}}(D(\bot))$;

(iii) *For any* $\mathbb{S}$*-matrix* $\mathbb{L} = \langle \mathcal{A}, F\rangle$, $\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F) = \Omega_{\mathcal{A}}(F)$;

(iv) *For any* $X \in \mathrm{Th}(\mathbb{S})$, $\widetilde{\Omega}_{\mathcal{F}m_{\mathcal{L}}(V)}(\mathrm{Th}(\mathbb{S})^X) = \Omega_{\mathcal{F}m_{\mathcal{L}}(V)}(X)$.

Proof:

(i)⇒(ii) Suppose $\mathbb{S}$ is protoalgebraic and let $\mathbb{L} = \langle \mathcal{A}, D\rangle$ be a G-model of $\mathbb{S}$. Then, by Proposition 87, $\mathrm{Cl}(\mathbb{L}) \subseteq \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. Therefore, by Proposition 109, $\Omega_{\mathcal{A}}$ is monotone on $\mathrm{Cl}(\mathbb{L})$. Now we get

$$\begin{array}{rcl}\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Cl}(\mathbb{L})) & = & \bigwedge\{\Omega_{\mathcal{A}}(F) : F \in \mathrm{Cl}(\mathbb{L})\} \quad \text{(Before Proposition 62)}\\ & = & \Omega_{\mathcal{A}}(D(\perp)). \quad \text{(Monotonicity of } \Omega_{\mathcal{A}})\end{array}$$

(ii)⇒(iii) Take $\mathbb{L} = \langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F\rangle$ in Part (ii).

(iii)⇒(iv) Take $\mathcal{A} = \mathcal{F}m_{\mathcal{L}}(V)$ and $F = X$ in Part (iii).

(iv)⇒(i) Let $X, X' \in \mathrm{Th}(\mathbb{S})$, such that $X \leq X'$. Then $X' \in \mathrm{Th}(\mathbb{S})^X$. Hence,

$$\begin{array}{rcl}\Omega_{\mathcal{F}m_{\mathcal{L}}(V)}(X) & = & \widetilde{\Omega}_{\mathcal{F}m_{\mathcal{L}}(V)}(\mathrm{Th}(\mathbb{S})^X) \quad \text{(Hypothesis)}\\ & \leq & \Omega_{\mathcal{F}m_{\mathcal{L}}(V)}(X'). \quad (X' \in \mathrm{Th}(\mathbb{S})^X)\end{array}$$

So $\Omega_{\mathcal{F}m_{\mathcal{L}}(V)}$ is monotone on $\mathrm{Th}(\mathbb{S})$. By Proposition 109, $\mathbb{S}$ is protoalgebraic.

■

Proposition 101 asserted that, for any G-logic $\mathbb{S}$, the class $\mathrm{Alg}^*(\mathbb{S})$ is, in general, included in the class $\mathrm{Alg}(\mathbb{S})$ of $\mathbb{S}$-algebras. It is now shown that for protoalgebraic G-logics the two classes coincide. This is an analog of Proposition 3.2 of [28].

Proposition 113 *Let* $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ *be a protoalgebraic sentential G-logic. Then* $\mathrm{Alg}(\mathbb{S}) = \mathrm{Alg}^*(\mathbb{S})$.

Proof: By Proposition 101, $\mathrm{Alg}^*(\mathbb{S}) \subseteq \mathrm{Alg}(\mathbb{S})$. Suppose, conversely, that $\mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathrm{Alg}(\mathbb{S})$. Then, there exists $\mathbb{L} = \langle \mathcal{A}, D\rangle$, such that $\widetilde{\Omega}(\mathbb{L}) = E$. But then, if F is the least function in $\mathrm{Cl}(\mathbb{L})$, we get, by Proposition 112, $\Omega_{\mathcal{A}}(F) = \widetilde{\Omega}(\mathbb{L}) = E$ and, hence, $\mathcal{A} \in \mathrm{Alg}^*(\mathbb{S})$. ■

Further, protoalgebraicity of a G-logic $\mathbb{S}$ entails that a full G-model of the logic on a given G-algebra is fully determined by its least closed function, in the sense that any two full G-models on the same G-algebra which happen to have the same least closed functions are identical. See Lemma 3.3 of [28] for the corresponding result in the context of sentential logics.

Lemma 114 *Let* $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ *be a protoalgebraic sentential G-logic and* $\mathbb{L} = \langle \mathcal{A}, D\rangle$, $\mathbb{L}' = \langle \mathcal{A}, D'\rangle$ *be full G-models of* $\mathbb{S}$ *over the same G-algebra* $\mathcal{A} = \langle \mathbf{A}, E\rangle$. *If the least elements* F *and* F' *of* $\mathrm{Cl}(\mathbb{L})$ *and* $\mathrm{Cl}(\mathbb{L}')$, *respectively, coincide, then* $\mathbb{L} = \mathbb{L}'$.

Proof: By Proposition 112 and the hypothesis,

$$\widetilde{\Omega}(\mathbb{L}) = \Omega_{\mathcal{A}}(F) = \Omega_{\mathcal{A}}(F') = \widetilde{\Omega}(\mathbb{L}').$$

Hence, by Theorem 104, $\mathbb{L} = \mathbb{L}'$. ∎

We turn, now, to the question of when all full G-models of a G-logic $\mathbb{S}$ over a G-algebra $\mathcal{A}$ have the form $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F$, for some $\mathbb{S}$-filter F on $\mathcal{A}$. The answer, expressed in an analog of Theorem 3.4 of [28], provides an additional view and characterization of protoalgebraicity in terms of the structure of full G-models.

Theorem 115 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic. $\mathbb{S}$ is protoalgebraic iff, for any G-algebra $\mathcal{A} = \langle \mathbf{A}, E\rangle$, all full G-models of $\mathbb{S}$ over $\mathcal{A}$ have the form $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F\rangle$, for some $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$.*

Proof: Suppose, first, that $\mathbb{S}$ is protoalgebraic. Let $\mathbb{L} = \langle \mathcal{A}, D\rangle$, with $\mathcal{A} = \langle \mathbf{A}, E\rangle$, be a full G-model of $\mathbb{S}$. Consider the least closed set $F \in \mathrm{Cl}(\mathbb{L})$. Clearly, $\mathrm{Cl}(\mathbb{L}) \subseteq \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F$. We must show the reverse inclusion. By protoalgebraicity and Proposition 112, $\widetilde{\Omega}(\mathbb{L}) = \Omega_{\mathcal{A}}(F)$. Thus, the quotient G-morphism $\pi : \mathcal{A} \to \mathcal{A}/\Omega_{\mathcal{A}}(F)$ is a bilogical morphism

$$\pi : \mathbb{L} \to_b \langle \mathcal{A}/\Omega_{\mathcal{A}}(F), \mathrm{Cl}(\mathbb{L})/\Omega_{\mathcal{A}}(F)\rangle.$$

Since $\mathbb{L}$ is a full model, we get $\mathrm{Cl}(\mathbb{L})/\Omega_{\mathcal{A}}(F) = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. Now let $F' \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F$. Since $\mathbb{S}$ is protoalgebraic, $\Omega_{\mathcal{A}}(F)$ is compatible with F'. Therefore, if $\pi' : \mathcal{A}/\Omega_{\mathcal{A}}(F) \to \mathcal{A}$ is a section of π, we get $F' = (F' \circ \pi') \circ \pi$. By the surjectivity of π and Proposition 79, $F' \circ \pi' \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. Thus, by the bilogical G-morphism property, $F' = (F' \circ \pi') \circ \pi \in \mathrm{Cl}(\mathbb{L})$. Therefore, $\mathrm{Cl}(\mathbb{L}) = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F$.

Assume, conversely, that every full G-model of $\mathbb{S}$ over $\mathcal{A}$ is of the form $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^H\rangle$, for some $H \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. Let $F, F' \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, such that $F \leq F'$. By Corollary 101, $\mathrm{Alg}^*(\mathbb{S}) \subseteq \mathrm{Alg}(\mathbb{S})$. Therefore, $\Omega_{\mathcal{A}}(F) \in \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$. By the Isomorphism Theorem 104, there exists a full G-model $\mathbb{L}$ of $\mathbb{S}$, such that $\Omega_{\mathcal{A}}(F) = \widetilde{\Omega}(\mathbb{L})$. By fullness, the quotient morphism $\pi : \mathcal{A} \to \mathcal{A}/\Omega_{\mathcal{A}}(F)$ is a bilogical G-morphism

$$\pi : \mathbb{L} \to_b \langle \mathcal{A}/\Omega_{\mathcal{A}}(F), \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))\rangle.$$

Since $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, we get $F \circ \pi' \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$, whence, by the bilogical G-morphism property, $F = (F \circ \pi') \circ \pi \in \mathrm{Cl}(\mathbb{L})$. Now assume, according to the hypothesis, that $\mathrm{Cl}(\mathbb{L}) = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^H$. Then we get $H \leq F \leq F'$. So $F' \in \mathrm{Cl}(\mathbb{L})$, whence

$$\Omega_{\mathcal{A}}(F) = \widetilde{\Omega}(\mathbb{L}) \leq \Omega_{\mathcal{A}}(F').$$

Thus, $\Omega_{\mathcal{A}}$ is monotone and, hence, by Proposition 110, $\mathbb{S}$ is protoalgebraic. ∎

3.14 Leibniz G-Filters

The characterization of protoalgebraicity in terms of the form of full G-models in Theorem 115 raises another natural question. To single out, if possible, among all $\mathbb{S}$-filters F of a protoalgebraic G-logic $\mathbb{S}$ those for which $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F$ is the set of closed functions of a full G-model.

Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic. Given an algebra $\mathcal{A}$, define

$$\mathrm{Fi}^{\bigstar}_{\mathbb{S}}(\mathcal{A}) = \left\{F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}) : \left\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F\right\rangle \in \mathrm{FMod}(\mathbb{S})\right\}.$$

The $\mathbb{S}$-filters in $\mathrm{Fi}^{\bigstar}_{\mathbb{S}}(\mathcal{A})$ are called **Leibniz $\mathbb{S}$-filters**.

Based on the Isomorphism Theorem 104 and the form of full G-models of a protoalgebraic G-logic, given by Theorem 115, we may establish another isomorphism, akin to the one established in Proposition 3.5 of [28], between Leibniz $\mathbb{S}$-filters on a G-algebra and $\mathrm{Alg}^*(\mathbb{S})$-congruences on the same G-algebra.

Theorem 116 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a protoalgebraic sentential G-logic. Then, for every G-algebra $\mathcal{A} = \langle \mathbf{A}, E\rangle$, the Leibniz operator $\Omega_{\mathcal{A}}$ is a lattice isomorphism between $\mathrm{Fi}^{\bigstar}_{\mathbb{S}}(\mathcal{A})$ and $\mathrm{Gon}_{\mathrm{Alg}^*(\mathbb{S})}(\mathcal{A}) = \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$.*

Proof: For the proof we compose two isomorphisms.

$$\begin{array}{ccccc} \mathrm{Fi}^{\bigstar}_{\mathbb{S}}(\mathcal{A}) & \longrightarrow & \mathrm{FMod}_{\mathbb{S}}(\mathcal{A}) & \longrightarrow & \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A}) \\ F & \longmapsto & \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F & \longmapsto & \widetilde{\Omega}_{\mathcal{A}}\left(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F\right) \end{array}$$

By the definition of $\mathrm{Fi}^{\bigstar}_{\mathbb{S}}(\mathcal{A})$, the first one is well defined. It is one-to-one, and it is both order preserving and order reflecting. Finally, since $\mathbb{S}$ is protoalgebraic, by Theorem 115, it is surjective. Therefore, it is an order isomorphism.

By Theorem 104, $\mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$ is isomorphic to $\mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$ via the Tarski operator. Thus, $F \mapsto \widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F)$ is an isomorphism from $\mathrm{Fi}^{\bigstar}_{\mathbb{S}}(\mathcal{A})$ to $\mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$. However, by Proposition 112, $\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F) = \Omega_{\mathcal{A}}(F)$. Moreover, by Proposition 113, $\mathrm{Alg}(\mathbb{S}) = \mathrm{Alg}^*(\mathbb{S})$ and, therefore, $\mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A}) = \mathrm{Gon}_{\mathrm{Alg}^*(\mathbb{S})}(\mathcal{A})$. Combining, we get the result. ■

The Leibniz $\mathbb{S}$-filters on a G-algebra $\mathcal{A}$ have a characterization that does not rely on the notion of full G-model. To formalize this characterization, we define, for every G-algebra $\mathcal{A}$, a binary relation $\sim_\Omega$ on $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ by setting, for all $F, F' \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$,

$$F \sim_\Omega F' \quad \text{iff} \quad \Omega_{\mathcal{A}}(F) = \Omega_{\mathcal{A}}(F').$$

That is, $\sim_\Omega$ is the kernel of the Leibniz operator acting on the $\mathbb{S}$-filters on the G-algebra $\mathcal{A}$. By Theorem 116, when $\mathbb{S}$ is protoalgebraic, at most one

$\mathbb{S}$-filter in each $\sim_\Omega$-equivalence class can belong to $\mathrm{Fi}^{\bigstar}_{\mathbb{S}}(\mathcal{A})$. We obtain a characterization of this filter.

Suppose $\mathbb{S}$ is protoalgebraic. Let $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ and denote by $[F]_\Omega$ its $\sim_\Omega$-equivalence class. Then $\bigwedge[F]_\Omega \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ and, moreover,

$$\begin{array}{rcll}\Omega_{\mathcal{A}}(\bigwedge[F]_\Omega) & = & \bigwedge_{H\in[F]_\Omega}\Omega_{\mathcal{A}}(H) & \text{(Corollary 111)}\\ & = & \Omega_{\mathcal{A}}(F). & \text{(Intersection over } [F]_\Omega)\end{array}$$

Thus, $\bigwedge[F]_\Omega \in [F]_\Omega$. This shows that $[F]_\Omega$ has a minimum element.

The following result characterizes Leibniz $\mathbb{S}$-filters on any G-algebra $\mathcal{A}$ as being the minimum elements in their own $\sim_\Omega$-equivalence class and, also, as those that induce the least $\mathbb{S}$-filter on the quotient G-algebra formed by their Leibniz G-congruence. It constitutes an analog of Proposition 3.6 of [28] for protoalgebraic G-logics,

Proposition 117 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a protoalgebraic sentential G-logic. Then, for all G-algebras $\mathcal{A} = \langle \mathbf{A}, E\rangle$ and all $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, the following conditions are equivalent:*

(i) $F \in \mathrm{Fi}^{\bigstar}_{\mathbb{S}}(\mathcal{A})$, *i.e.*, $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F\rangle \in \mathrm{FMod}(\mathbb{S})$;

(ii) F *is the minimum element in* $[F]_\Omega$;

(iii) $F \circ \pi'$ *is the least $\mathbb{S}$-filter on* $\mathcal{A}/\Omega_{\mathcal{A}}(F)$, *where* π' *is a section of the quotient morphism* $\pi : \mathcal{A} \to \mathcal{A}/\Omega_{\mathcal{A}}(F)$.

Proof:

(ii)⇒(iii) Suppose F is the minimum element in $[F]_\Omega$. Consider a filter $H \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))$. Construct $F' = (H \circ \pi) \wedge F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. We have

$$\begin{array}{rcll}F' & = & (H\circ\pi)\wedge((F\circ\pi')\circ\pi) & (\Omega_{\mathcal{A}}(F) \text{ compatible with } F)\\ & = & (H\wedge(F\circ\pi'))\circ\pi. & \end{array}$$

Thus, for all $a, b \in A$,

$$\begin{array}{l}\Omega_{\mathcal{A}}(F)(a,b)\wedge F'(a)\\ \quad = \overline{\Omega_{\mathcal{A}}(F)}(a/\widehat{\Omega_{\mathcal{A}}(F)}, a/\widehat{\Omega_{\mathcal{A}}(F)})\wedge(H\wedge(F\circ\pi'))(a/\widehat{\Omega_{\mathcal{A}}(F)})\\ \qquad \text{(Definition of } \overline{\Omega_{\mathcal{A}}(F)} \text{ and } F' = (H\wedge(F\circ\pi'))\circ\pi)\\ \quad \le (H\wedge(F\circ\pi'))(b/\widehat{\Omega_{\mathcal{A}}(F)})\\ \qquad (H\wedge(F\circ\pi')\in\mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Omega_{\mathcal{A}}(F)))\\ \quad = F'(b). \quad (F' = (H\wedge(F\circ\pi'))\circ\pi)\end{array}$$

Thus, $\Omega_{\mathcal{A}}(F)$ is compatible with F'. By the maximality property of the Leibniz G-congruence, $\Omega_{\mathcal{A}}(F) \le \Omega_{\mathcal{A}}(F')$. However, since, by definition, $F' \le F$ and $\mathbb{S}$ is protoalgebraic, we get $\Omega_{\mathcal{A}}(F') \le \Omega_{\mathcal{A}}(F)$. Hence,

$\Omega_{\mathcal{A}}(F) = \Omega_{\mathcal{A}}(F')$. So $F \sim_\Omega F'$ and, by hypothesis, $F \leq F'$. We conclude that $F = F'$ and, hence, $F \leq H \circ \pi$. This yields

$$F \circ \pi' \leq (H \circ \pi) \circ \pi' = H.$$

Thus, $F \circ \pi'$ is the least $\mathbb{S}$-filter on $\mathcal{A}/\Omega_{\mathcal{A}}(F)$.

(iii)⇒(i) Suppose $F \circ \pi'$ is the least $\mathbb{S}$-filter on $\mathcal{A}/\Omega_{\mathcal{A}}(F)$. Taking into account that, by protoalgebraicity, $\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F) = \Omega_{\mathcal{A}}(F)$, our goal is to prove that

$$\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F \circ \pi' \cong \mathrm{Fi}_{\mathbb{S}}\left(\mathcal{A}/\widetilde{\Omega}_{\mathcal{A}}\left(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F\right)\right).$$

We have

$$\begin{array}{lcll} \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F \circ \pi' & \cong & \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Omega_{\mathcal{A}}(F))^{F\circ\pi'} & \text{(Protoalgebraicity)} \\ & = & \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\Omega_{\mathcal{A}}(F)) & \text{(Hypothesis)} \\ & = & \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}/\widetilde{\Omega}_{\mathcal{A}}(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F)). & \text{(Protoalgebraicity)} \end{array}$$

(i)⇒(ii) Suppose $F \in \mathrm{Fi}_{\mathbb{S}}^{\bigstar}(\mathcal{A})$. By protoalgebraicity, its class $[F]_\Omega$ has a minimum element, say H. Applying the implications (ii)⇒(iii)⇒(i) for H, we get that $\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^H \rangle$ is a full G-model of $\mathbb{S}$. Thus, we have

$$\widetilde{\Omega}_{\mathcal{A}}\left(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^H\right) = \Omega_{\mathcal{A}}(H) = \Omega_{\mathcal{A}}(F) = \widetilde{\Omega}_{\mathcal{A}}\left(\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F\right).$$

By the Isomorphism Theorem 104, $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^H = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F$, whence $F = H$. So F is the minimum element in its $\sim_\Omega$-class $[F]_\Omega$.

■

An important consequence for what follows is the fact that injectivity of the Leibniz operator on all $\mathbb{S}$-filters on a G-algebra is equivalent to the condition that every $\mathbb{S}$-filter on the G-algebra is actually a Leibniz $\mathbb{S}$-filter. This forms an analog of Proposition 3.7 of [28].

Proposition 118 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a protoalgebraic sentential G-logic. Then, for all G-algebras $\mathcal{A} = \langle \mathbf{A}, E\rangle$,* $\mathrm{Fi}_{\mathbb{S}}^{\bigstar}(\mathcal{A}) = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ *iff the Leibniz operator $\Omega_{\mathcal{A}}$ is injective on* $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$.

Proof: We have $\mathrm{Fi}_{\mathbb{S}}^{\bigstar}(\mathcal{A}) = \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ if and only if, by Proposition 117, for all $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, F is the least element in $[F]_\Omega$ if and only if, for all $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, $[F]_\Omega = \{F\}$ if and only if $\Omega_{\mathcal{A}}$ is injective on $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. ■

3.15 Weakly Algebraizable G-Logics

We have seen that protoalgebraicity of a G-logic $\mathbb{S}$ is characterized by the monotonicity of the Leibniz operator on the theories of the G-logic and, also, by the monotonicity of the Leibniz operator on the $\mathbb{S}$-filters of the G-logic on any G-algebra. Another important class of logics in the traditional algebraic hierarchy is that of *weakly algebraizable logics*. It is obtained from the class of protoalgebraic logics if one insists that, apart from being monotone, the Leibniz operator also be injective. So we add, and study the effects of, injectivity of the Leibniz operator in order to introduce the class of weakly algebraizable G-logics.

Theorem 119, an analog of Theorem 3.8 of [28], characterizes those sentential G-logics $\mathbb{S}$ for which the Leibniz operator is both monotone and injective on the $\mathbb{S}$-filters of any G-algebra.

Theorem 119 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a sentential G-logic. The following conditions are equivalent:*

(i) $\mathbb{S}$ is protoalgebraic and, for all G-algebras $\mathcal{A} = \langle \mathbf{A}, E\rangle$ and all $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$, $F \circ \pi'$ is the least $\mathbb{S}$-filter on $\mathcal{A}/\Omega_{\mathcal{A}}(F)$, where π' is a section of the quotient G-morphism $\pi : \mathcal{A} \to \mathcal{A}/\Omega_{\mathcal{A}}(F)$;

(ii) For all G-algebras $\mathcal{A} = \langle \mathbf{A}, E\rangle$, the Leibniz operator $\Omega_{\mathcal{A}}$ is monotone and injective on $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$;

(iii) For all G-algebras $\mathcal{A} = \langle \mathbf{A}, E\rangle$, the mapping $F \mapsto \langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F\rangle$ is a bijection (hence, a lattice isomorphism) from $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ onto $\mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$;

(iv) For all G-algebras $\mathcal{A} = \langle \mathbf{A}, E\rangle$, $\Omega_{\mathcal{A}} : \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}) \to \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$ is a lattice isomorphism;

(v) For all G-algebras $\mathcal{A} = \langle \mathbf{A}, E\rangle$, $\Omega_{\mathcal{A}} : \mathrm{Fi}_{\mathbb{S}}(\mathcal{A}) \to \mathrm{Gon}_{\mathrm{Alg}^*(\mathbb{S})}(\mathcal{A})$ is a lattice isomorphism.

Proof:

(i)⇔(ii) Since $\mathbb{S}$ is protoalgebraic, by Proposition 110, $\Omega_{\mathcal{A}}$ is monotone on $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. Moreover, by Proposition 117, for every G-algebra $\mathcal{A}$, $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A}) = \mathrm{Fi}^{\bigstar}_{\mathbb{S}}(\mathcal{A})$. Hence, by Proposition 118, $\Omega_{\mathcal{A}}$ is injective on $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$.

(i)⇒(iii) The mapping $F \mapsto \langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F\rangle$ is injective. By Proposition 117, it is well defined and, by Theorem 115, it is surjective.

(iii)⇒(iv) Since $F \mapsto \langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F\rangle$ is surjective, by Theorem 115, $\mathbb{S}$ is protoalgebraic. Composing the postulated isomorphism with the one given by the Isomorphism Theorem 104, we get an isomorphism from $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ onto $\mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$, given by

$$F \mapsto \widetilde{\Omega}\left(\left\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F \right\rangle\right) \stackrel{\text{Prop. 112}}{=} \Omega_{\mathcal{A}}(F),$$

i.e., it is the Leibniz operator on $\mathcal{A}$.

(iv)⇒(v) By Corollary 101, the inclusion $\mathrm{Alg}^*(\mathbb{S}) \subseteq \mathrm{Alg}(\mathbb{S})$ holds in general and gives

$$\mathrm{Gon}_{\mathrm{Alg}^*(\mathbb{S})}(\mathcal{A}) \subseteq \mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A}).$$

On the other hand, by hypothesis, every congruence in $\mathrm{Gon}_{\mathrm{Alg}(\mathbb{S})}(\mathcal{A})$ is of the form $\Omega_{\mathcal{A}}(F)$, for some $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. This yields the reverse of the displayed inclusion, whence we get the isomorphism of Part (v).

(v)⇒(ii) This is straightforward.

■

A sentential G-logic $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ is called **weakly algebraizable** if, for every G-algebra $\mathcal{A} = \langle \mathbf{A}, E\rangle$, the Leibniz operator $\Omega_{\mathcal{A}}$ is monotone and injective on $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$.

Given a G-algebra $\mathcal{A} = \langle \mathbf{A}, E\rangle$, we say that the Leibniz operator on $\mathcal{A}$ is **continuous** if, for every directed family $\{F_i : i \in I\}$ of $\mathbb{S}$-filters on $\mathcal{A}$, we have $\bigvee_{i\in I} F_i \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ and

$$\Omega_{\mathcal{A}}\left(\bigvee_{i\in I} F_i\right) = \bigvee_{i\in I} \Omega_{\mathcal{A}}(F_i).$$

If, in addition to $\Omega_{\mathcal{A}}$ being monotone and injective on every G-algebra $\mathcal{A}$, $\Omega_{\mathcal{A}}$ is also continuous, then we call the sentential G-logic $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ **algebraizable**.

Given a G-algebra $\mathcal{A} = \langle \mathbf{A}, E\rangle$, we say that the Tarski operator on $\mathcal{A}$ is **continuous** if, for every directed family $\{\mathbb{L}_i : i \in I\}$ of full G-models of $\mathbb{S}$ on $\mathcal{A}$, $\bigvee_{i\in I} \mathbb{L}_i$ is also a full G-model of $\mathbb{S}$ and

$$\widetilde{\Omega}_{\mathcal{A}}\left(\bigvee_{i\in I} \mathbb{L}_i\right) = \bigvee_{i\in I} \widetilde{\Omega}_{\mathcal{A}}(\mathbb{L}_i).$$

Then we have the following partial analog of Theorem 3.10 of [28] in the context of weakly algebraizable sentential G-logics.

Theorem 120 *Let $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ be a weakly algebraizable sentential G-logic. Then the following conditions are equivalent:*

(i) $\mathbb{S}$ *is algebraizable;*

(ii) For every G-algebra $\mathcal{A} = \langle \mathbf{A}, E\rangle$, $\Omega_{\mathcal{A}}$ is continuous on $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$*;*

(iii) For every G-algebra $\mathcal{A} = \langle \mathbf{A}, E\rangle$, $\widetilde{\Omega}_{\mathcal{A}}$ is continuous on $\mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$*.*

Proof: By the definition of algebraizability and Theorem 119, Statements (i) and (ii) are equivalent. So it suffices to prove the equivalence of (ii) and (iii).

Suppose, first, that $F \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. Define

$$\Phi_{\mathcal{A}}(F) = \left\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F \right\rangle.$$

By protoalgebraicity and Proposition 112, $\Omega_{\mathcal{A}} = \widetilde{\Omega}_{\mathcal{A}} \circ \Phi_{\mathcal{A}}$. By Theorem 119, $\Phi_{\mathcal{A}}$ is a bijection. Consequently, $\widetilde{\Omega}_{\mathcal{A}} = \Omega_{\mathcal{A}} \circ \Phi_{\mathcal{A}}^{-1}$. Suppose $\Omega_{\mathcal{A}}$ is continuous on $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. Let $\{\mathbb{L}_i : i \in I\} \subseteq \mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$ be directed. Consider $F_i = \Phi_{\mathcal{A}}^{-1}(\mathbb{L}_i)$. Then $\{F_i : i \in I\} \subseteq \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ is also directed, whence, by hypothesis, $\bigvee_{i\in I} F_i \in \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. Thus,

$$\Phi_{\mathcal{A}}\left(\bigvee_{i\in I} F_i\right) = \left\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^{\bigvee F_i} \right\rangle \in \mathrm{FMod}_{\mathbb{S}}(\mathcal{A}).$$

Moreover,

$$\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^{\bigvee F_i} = \bigcap_{i\in I} \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^{F_i}.$$

So $\Phi_{\mathcal{A}}(\bigvee_{i\in I} F_i) = \bigvee_{i\in I} \mathbb{L}_i$. Now we get

$$\begin{array}{rcl}
\widetilde{\Omega}_{\mathcal{A}}(\bigvee_{i\in I} \mathbb{L}_i) & = & (\widetilde{\Omega}_{\mathcal{A}} \circ \Phi_{\mathcal{A}})(\Phi_{\mathcal{A}}^{-1}(\bigvee_{i\in I} \mathbb{L}_i)) \quad (\Phi_{\mathcal{A}} \circ \Phi_{\mathcal{A}}^{-1} = Id) \\
& = & \Omega_{\mathcal{A}}(\bigvee_{i\in I} F_i) \quad (\widetilde{\Omega}_{\mathcal{A}} \circ \Phi_{\mathcal{A}} = \Omega_{\mathcal{A}}) \\
& = & \bigvee_{i\in I} \Omega_{\mathcal{A}}(F_i) \quad (\text{Hypothesis}) \\
& = & \bigvee_{i\in I} \widetilde{\Omega}_{\mathcal{A}}(\Phi_{\mathcal{A}}(F_i)) \quad (\widetilde{\Omega}_{\mathcal{A}} \circ \Phi_{\mathcal{A}} = \Omega_{\mathcal{A}}) \\
& = & \bigvee_{i\in I} \widetilde{\Omega}_{\mathcal{A}}(\mathbb{L}_i). \quad (\Phi_{\mathcal{A}}(F_i) = \mathbb{L}_i)
\end{array}$$

This proves that $\widetilde{\Omega}_{\mathcal{A}}$ is continuous on $\mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$.

Suppose, conversely, that $\widetilde{\Omega}_{\mathcal{A}}$ is continuous on $\mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$. Consider a directed $\{F_i : i \in I\} \subseteq \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. Clearly, $\{\Phi_{\mathcal{A}}(F_i) : i \in I\}$ is also directed. Moreover,

$$\begin{array}{rcl}
\Omega_{\mathcal{A}}(\bigvee_{i\in I} F_i) & = & \widetilde{\Omega}_{\mathcal{A}}(\Phi_{\mathcal{A}}(\bigvee_{i\in I} F_i)) \quad (\widetilde{\Omega}_{\mathcal{A}} \circ \Phi_{\mathcal{A}} = \Omega_{\mathcal{A}}) \\
& = & \widetilde{\Omega}_{\mathcal{A}}(\bigvee_{i\in I} \mathbb{L}_i) \quad (\Phi_{\mathcal{A}}(\bigvee_{i\in I} F_i) = \bigvee_{i\in I} \mathbb{L}_i) \\
& = & \bigvee_{i\in I} \widetilde{\Omega}_{\mathcal{A}}(\mathbb{L}_i) \quad (\text{Hypothesis}) \\
& = & \bigvee_{i\in I} \Omega_{\mathcal{A}}(\Phi_{\mathcal{A}}^{-1}(\mathbb{L}_i)) \quad (\widetilde{\Omega}_{\mathcal{A}} = \Omega_{\mathcal{A}} \circ \Phi_{\mathcal{A}}^{-1}) \\
& = & \bigvee_{i\in I} \Omega_{\mathcal{A}}(F_i). \quad (\Phi_{\mathcal{A}}^{-1}(\mathbb{L}_i) = F_i)
\end{array}$$

Therefore, $\Omega_{\mathcal{A}}$ is continuous on $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$. ■

Corollary 121 *Let* $\mathbb{S} = \langle \mathcal{F}m_{\mathcal{L}}(V), C\rangle$ *be a sentential G-logic. Then* $\mathbb{S}$ *is algebraizable if and only if, for every G-algebra* $\mathcal{A} = \langle \mathbf{A}, E\rangle$*, the mapping*

$$F \mapsto \left\langle \mathcal{A}, \mathrm{Fi}_{\mathbb{S}}(\mathcal{A})^F \right\rangle$$

is a bijection between $\mathrm{Fi}_{\mathbb{S}}(\mathcal{A})$ *and* $\mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$ *and the Tarski operator* $\widetilde{\Omega}_{\mathcal{A}}$ *is continuous on* $\mathrm{FMod}_{\mathbb{S}}(\mathcal{A})$.

Proof: By definition, $\mathcal{S}$ is algebraizable if and only if the Leibniz operator is monotone, injective and continuous on $\mathrm{Fi}_{\mathcal{S}}(\mathcal{A})$ if and only if, by Theorems 119 and 120, the mapping $F \mapsto \langle \mathcal{A}, \mathrm{Fi}_{\mathcal{S}}(\mathcal{A})^F \rangle$ is a bijection between $\mathrm{Fi}_{\mathcal{S}}(\mathcal{A})$ and $\mathrm{FMod}_{\mathcal{S}}(\mathcal{A})$ and the Tarski operator $\widetilde{\Omega}_{\mathcal{A}}$ is continuous on $\mathrm{FMod}_{\mathcal{S}}(\mathcal{A})$. ■