Chapter 2

Algebraizable Graded Logics

2.1 Introduction

Our goal in this chapter is to generalize the framework of Blok and Pigozzi outlined in Chapter 1, to the extent possible, to cover consequences that deal with sentences whose truth values are drawn from a complete lattice. On some contexts, the lattice may be assumed to have additional properties, e.g., being completely distributive or a Boolean algebra. In any case, all results are abstractions of those dealing with the classical framework, where a formula assumes two possible truth values.

We outline the contents of the chapter by section.

In Section 2.2, we introduce the central structure of *G-logic*. Roughly speaking, it is a structural closure operator on *G-sets*, which are mappings of the set of formulas $\mathrm{Fm}_{\mathcal{L}}(V)$ into a fixed complete lattice $\mathbf{G} = \langle G, \leq \rangle$ to be thought of as the lattice of truth values. Under a natural definition of an ordering $\leq$ on G-logics, induced in a "pointwise" manner by the $\leq$ ordering of $\mathbf{G}$, we show that the class $\mathrm{Log}_{\mathbf{G}}(\mathcal{L})$ of G-logics forms a complete lattice.

In Section 2.3, we look at *G-theories*. These are G-sets that are closed sets of the closure operator of the G-logic. They form a complete lattice under $\leq$. We also define a notion of *finitarity* for G-logics. We finally establish some properties of G-theories. We show that, due to structurality, for every G-theory T and all substitutions σ, $T \circ \sigma$ is also a G-theory. Moreover, meets and joins of G-theories commute with substitutions, in the sense that, for every collection $\{T_i : i \in I\}$ of G-theories and all substitutions σ, $\bigwedge_{i \in I}(T_i \circ \sigma) = (\bigwedge_{i \in I} T_i) \circ \sigma$ and similarly for joins.

In Section 2.4, we look at *G-matrices*. These are pairs $\mathfrak{A} = \langle \mathbf{A}, F \rangle$, where $\mathbf{A}$ is an $\mathcal{L}$-algebra and F is a G-set on A, i.e., $F : A \to G$. Any given class of G-matrices M induces a G-logic $\mathcal{S}_{\mathsf{M}} = \langle \mathcal{L}, C_{\mathsf{M}} \rangle$ on $\mathrm{Fm}_{\mathcal{L}}(V)$. We say that $\mathfrak{A}$ is a *matrix of* $\mathcal{S}$ or an *$\mathcal{S}$-matrix* is $\mathcal{S} \leq \mathcal{S}_{\mathsf{M}}$, that is, if $C \leq C_{\mathsf{M}}$. In this case F is called an *$\mathcal{S}$-filter*. The collection of all $\mathcal{S}$-filters on an algebra $\mathbf{A}$ is denoted by $\mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$. Ordered by $\leq$ it forms a complete lattice. Furthermore, $\mathcal{S}$-filters on the formula algebra $\mathbf{Fm}_{\mathcal{L}}(V)$ coincide with $\mathcal{S}$-theories. Finally, a class M of G-matrices constitutes a *G-matrix semantics* for a G-logic $\mathcal{S}$ if $\mathcal{S} = \mathcal{S}_{\mathsf{M}}$ and, in that case, M is said to be *strongly adequate for $\mathcal{S}$*.

In Section 2.5, we define *G-congruences*. These are G-sets of equations on $\mathbf{A}$, that is, mappings $\Theta : A^2 \to G$, that satisfy reflexivity, symmetry, transitivity and congruence. Reflexivity means that $\Theta(a,a) = \top$, for all $a \in A$, symmetry means that $\Theta(a,b) = \Theta(b,a)$, for all $a, b \in A$, transitivity that $\Theta(a,b) \wedge \Theta(b,c) \leq \Theta(a,c)$, for all $a,b,c \in A$ and, similarly, congruence means that, for all n-ary $\lambda \in \mathcal{L}$ and all $\bar{a}, \bar{b} \in A^n$, $\bigwedge_{i=1}^n \Theta(a_i, b_i) \leq \Theta(\lambda^{\mathbf{A}}(\bar{a}), \lambda^{\mathbf{A}}(\bar{b}))$. We show that G-congruences, ordered by $\leq$, form a complete lattice. We also introduce *stratified congruences*. These are families $\theta = \{\theta_g : g \in G\}$ indexed by the elements of the complete lattice $\mathbf{G} = \langle G, \leq \rangle$ that, in addition, satisfy $\theta_{g_2} \subseteq \theta_{g_1}$, for all $g_1, g_2 \in G$, such that $g_1 \leq g_2$. We show that, under certain conditions, the two mappings

$$\begin{array}{ll}\Theta \mapsto \hat{\Theta}; & \hat{\Theta}_g = \{\langle a,b\rangle : \Theta(a,b) \geq g\},\ g \in G,\\ \theta \mapsto \check{\theta}; & \check{\theta}(a,b) = \bigvee\{g : \langle a,b\rangle \in \theta_g\},\ a,b \in A,\end{array}$$

are inverse mappings from G-congruences to stratified congruences on $\mathbf{A}$.

Section 2.6 discusses *compatibility* of G-congruences with G-filters and the existence of the *Leibniz G-congruence*. We say that a G-congruence Θ is *compatible with* a G-filter F, or that Θ is a *G-congruence of the G-matrix* $\mathfrak{A} = \langle \mathbf{A}, F\rangle$, if, for all $a,b \in A$, $\Theta(a,b) \wedge F(a) \leq F(b)$. In case $\mathbf{G}$ has an implication $\to$, such that, for all $g,g',g'' \in G$, $g \wedge g' \leq g''$ if and only if $g \leq g' \to g''$, then the condition above becomes equivalent to $\Theta(a,b) \leq F(a) \leftrightarrow F(b)$, where $\leftrightarrow$ is the biconditional corresponding to $\to$. We denote by $\mathrm{Gon}(\mathfrak{A})$ the collection of all G-congruences on $\mathbf{A}$ that are compatible with F. In some special cases, it can be shown that, regardless of $\mathfrak{A}$, $\mathrm{Gon}(\mathfrak{A})$ is a principal ideal of the complete lattice $\mathbf{Gon(A)}$ of all G-congruences on $\mathbf{A}$. For us, this is a firm desideratum. So we call a complete lattice $\mathbf{G}$ *Leibniz permitting* if it is such that, for all $\mathfrak{A}$, $\mathrm{Gon}(\mathfrak{A})$ is a principal ideal in $\mathbf{Gon(A)}$ and we restrict attention, throughout our work, to such complete lattices. The generator of this principal ideal, i.e., the largest G-congruence on $\mathbf{A}$ that is compatible with the G-filter F, is called the *Leibniz G-congruence of* $\mathfrak{A}$, or the *Leibniz G-congruence of F on* $\mathbf{A}$. We provide a Blok-Pigozzi style characterization of Leibniz G-congruences. Namely, we show that, for all $a,b \in A$,

$$\Omega_{\mathbf{A}}(F)(a,b) = \bigwedge\{F(\varphi^{\mathbf{A}}(a,\overline{c})) \leftrightarrow F(\varphi^{\mathbf{A}}(b,\overline{c})) : \varphi \in \mathrm{Fm}_{\mathcal{L}}(V), \overline{c} \in A\}.$$

The section concludes with a result of a rather technical nature. We say that a G-set Θ of equations is *definable* (*with parameters*) *in* $\mathfrak{A} = \langle \mathbf{A}, F\rangle$ if there exist a formula $\varphi(x,y,\bar{z})$ and parameters $\bar{c} \in A$, such that, for all $a,b \in A$, $\Theta(a,b) = F(\varphi^{\mathbf{A}}(a,b,\bar{c}))$. We show that, if Θ is a G-congruence on $\mathbf{A}$, definable in $\mathfrak{A}$, and compatible with F, then $\Theta = \Omega_{\mathbf{A}}(F)$.

Section 2.7 is parenthetical, briefly introducing and giving a characterization of *protoalgebraic G-logics*. This topic will be discussed more extensively in Chapter 3, together with other classes of G-logics in the algebraic hierarchy, when additional machinery will be at our disposal.

In Section 2.8, we introduce and study *G-2-logics*. These are analogs of G-logics, but act on G-sets of equations rather than on G-sets of formulas. They are needed, in the same way that 2-deductive systems are needed, to formalize a Blok-Pigozzi style theory of algebraizability of G-logics. A *G-2-logic* is a mapping $C : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$ that satisfies appropriate versions of inflationarity, monotonicity, idempotency and structurality. Perhaps the most distinguishing and important feature of G-2-logics is that they can be used to express equational logics of specific classes of G-algebras, i.e., pairs $\mathcal{A} = \langle \mathbf{A}, \Theta\rangle$, where $\Theta : A^2 \to G$ is a G-congruence on $\mathbf{A}$. Each such class K gives rise to a G-2-logic $\mathcal{S}_{\mathsf{K}}$. The theories of this logic are G-congruences on $\mathbf{Fm}_{\mathcal{L}}(V)$ and, conversely, each G-congruence on $\mathbf{Fm}_{\mathcal{L}}(V)$, such that $\langle \mathbf{Fm}_{\mathcal{L}}(V), \Theta\rangle \in \mathsf{K}$ is an $\mathcal{S}_{\mathsf{K}}$-theory. The collection $\mathrm{Th}(\mathcal{S}_{\mathsf{K}})$, ordered by $\le$, forms a complete lattice $\mathbf{Th}(\mathcal{S}_{\mathsf{K}}) = \langle \mathrm{Th}(\mathcal{S}_{\mathsf{K}}), \le\rangle$.

In Section 2.9, we define *translations* between G-logics and G-2-logics and vice-versa. These form analogs of the ordinary translations that form the building blocks of the theory of algebraizable logics of Blok and Pigozzi [6]. A *translation from* G-formulas *to* G-equations is a join preserving mapping $\mathcal{E} : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$. Dually, a *translation from* G-equations *to* G-formulas is a join preserving mapping $\mathcal{F}$ in the opposite direction. Consider, now, a G-logic $\mathcal{S} = \langle \mathcal{L}, C\rangle$, a class K of $\mathcal{L}$-algebras and the G-2-logic $\mathcal{S}_{\mathsf{K}} = \langle \mathcal{L}, C_{\mathsf{K}}\rangle$ associated with K. A translation $\mathcal{E}$ from G-formulas to G-equations is an *interpretation* $\mathcal{E} : \mathcal{S} \to \mathcal{S}_{\mathsf{K}}$ if, for all $\Gamma, \Phi \in G^{\mathrm{Fm}_{\mathcal{L}}(V)}$,

$$\Phi \le C(\Gamma) \quad \text{iff} \quad \mathcal{E}(\Phi) \le C_{\mathsf{K}}(\mathcal{E}(\Gamma)).$$

Dually, an interpretation from $\mathcal{S}_{\mathsf{K}}$ to $\mathcal{S}$ is a translation $\mathcal{F}$ from G-equations to G-formulas, such that, for all $\Theta, E \in G^{\mathrm{Eq}_{\mathcal{L}}(V)}$,

$$E \le C_{\mathsf{K}}(\Theta) \quad \text{iff} \quad \mathcal{F}(E) \le C(\mathcal{F}(\Theta)).$$

We provide characterizations of when a given translation $\mathcal{E}$ or $\mathcal{F}$ is an interpretation. E.g., $\mathcal{E}$ is an interpretation $\mathcal{E} : \mathcal{S} \to \mathcal{S}_{\mathsf{K}}$ if and only if, for all $\Gamma \in G^{\mathrm{Fm}_{\mathcal{L}}(V)}$, $C_{\mathsf{K}}(\mathcal{E}(\Gamma)) = C_{\mathsf{K}}(\mathcal{E}(C(\Gamma)))$ and $C(\Gamma) = \bigwedge\{C(\Phi) : \mathcal{E}(\Gamma) \le C_{\mathsf{K}}(\mathcal{E}(\Phi))\}$, and, dually for $\mathcal{F}$.

In Section 2.10, we give an alternative, but equivalent, representation of translations. A *hybrid translation* from formulas to equations is a mapping $E : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \times \mathrm{Fm}_{\mathcal{L}}(V) \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$, such that, for all G-sets of formulas $\{\Gamma_i : i \in I\} \cup \{\Gamma\}$ and all formulas φ, $E(\bot, \varphi) = \bot$, $E(\Gamma, \varphi) = E(\Gamma^\varphi, \varphi)$ and $E(\bigvee_i \Gamma_i, \varphi) = \bigvee_i E(\Gamma_i, \varphi)$, where Γ^φ is the G-set of formulas that picks the value $\Gamma(\varphi)$ at φ and assigns $\bot$ to all other formulas. The definition of *hybrid translations* from equations to formulas is defined dually. It turns out that translations and hybrid translations are two faces of the same coin. Given a translation $\mathcal{E}$ from G-formulas to G-equations, we define the hybrid translation $\mathcal{E}^h$ by setting, for all G-sets of equations Γ and all formulas φ, $\mathcal{E}^h(\Gamma, \varphi) = \mathcal{E}(\Gamma^\varphi)$. Conversely, given a hybrid translation E, we define E^t by setting, for all G-sets Γ of formulas, $E^t(\Gamma) = \bigvee_\varphi E(\Gamma^\varphi, \varphi)$. Then $\mathcal{E}^h$ is a hybrid translation, E^t is a translation and, further, $\mathcal{E}^{ht} = \mathcal{E}$ and $E^{th} = E$. The same situation occurs, in a completely dual fashion, when translations and hybrid translations from equations to formulas are treated.

Section 2.11 continues the work started in Section 2.10. Here, it is shown that the equivalence between translations and hybrid translations can be restricted to *structural translations* and *structural hybrid translations*. To give the main idea, a translation $\mathcal{E}$ from G-formulas to G-equations is *structural* if, for all G-sets of formulas Γ, all formulas φ and all substitutions σ, $\mathcal{E}(\Gamma^{\sigma(\varphi)}) = \mathcal{E}(\Gamma^\varphi) \circ \sigma$. Similarly, a hybrid translation E from formulas to equations is *structural* if, for all G-sets of formulas Γ, all formulas φ and all substitutions σ, $E(\Gamma^{\sigma(\varphi)}, \sigma(\varphi)) = E(\Gamma^{\varphi}, \varphi) \circ \sigma$. We show that the mappings $\mathcal{E} \mapsto \mathcal{E}^h$ and $E \mapsto E^t$ of Section 2.10 are inverse mappings between structural translations and structural hybrid translations. The dual equivalence between translations from G-equations to G-formulas and hybrid translations from equations to formulas yields an equivalence between structural translations and structural hybrid translations as well.

In Section 2.12, we introduce the dual notions of *G-algebraic semantics* for a G-logic $\mathcal{S}$ and of *G-logical semantics* for a class K of G-algebras. We say that a class K of G-algebras is a *G-algebraic semantics* for a G-logic $\mathcal{S}$ if there exists an interpretation $\mathcal{E} : \mathcal{S} \to \mathcal{S}_{\mathsf{K}}$. Dually, a G-logic $\mathcal{S}$ is a *G-logical semantics* for a class K of G-algebras if there exists an interpretation $\mathcal{F} : \mathcal{S}_{\mathsf{K}} \to \mathcal{S}$. Under certain conditions on the witnessing interpretations, the notions of G-algebraic semantics and of G-matrix semantics are related. The same happens, dually, with the notions of G-logical semantics and of G-2-matrix semantics, a G-matrix semantics applicable to G-2-logics. We say that a translation $\mathcal{E}$ from G-formulas to G-equations is *order reflecting* if, for all G-sets of formulas Γ, Γ',

$$\mathcal{E}(\Gamma) \le \mathcal{E}(\Gamma') \quad \text{implies} \quad \Gamma \le \Gamma'.$$

Dually for a translation $\mathcal{F}$ from G-equations to G-formulas. Furthermore, we say that $\mathcal{E}$ is *reflectively structural* if, for every algebra $\mathbf{A}$ and all G-congruences Θ on $\mathbf{A}$, there exists a G-filter F on $\mathbf{A}$, such that, for every $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$,

$$\mathcal{E}(F \circ h) = \Theta \circ h^2.$$

Dually for a translation $\mathcal{F}$ from G-equations to G-formulas. Reflective structurality of $\mathcal{E}$ allows one to construct, given a class K of G-algebras, a corresponding class $\mathsf{K}^{\mathcal{E}}$ of G-matrices. Dually, reflective structurality of $\mathcal{F}$ allows one to construct, given a class M of G-matrices, a corresponding class $\mathsf{M}^{\mathcal{F}}$ of G-algebras. We show that, if $\mathcal{E}$ is order reflecting and reflectively structural, then K is a G-algebraic semantics for $\mathcal{S}$ via $\mathcal{E}$ if and only if $\mathsf{K}^{\mathcal{E}}$ is a G-matrix semantics for $\mathcal{S}$ and, dually, if $\mathcal{F}$ is order reflecting and reflectively structural, then, for a class M of G-matrices, $\mathcal{S}_{\mathsf{M}}$ is a G-logical semantics for K via $\mathcal{F}$ if and only if $\mathsf{M}^{\mathcal{F}}$ is a G-2-matrix semantics for $\mathcal{S}_{\mathsf{K}}$.

In Section 2.13, starting from interpretations, we define the concept of an *equivalent G-algebraic semantics* for a G-logic $\mathcal{S}$. A class K of G-algebras is an *equivalent G-algebraic semantics* for $\mathcal{S} = \langle \mathcal{L}, C \rangle$ if there exist two translations $\mathcal{E}$ from G-formulas to G-equations and $\mathcal{F}$ from G-equations to G-formulas, such that, for all G-sets of formulas Γ, Φ and all G-sets of equations Θ,

(i) $\Phi \le C(\Gamma)$ iff $\mathcal{E}(\Phi) \le C_{\mathsf{K}}(\mathcal{E}(\Gamma))$, i.e., $\mathcal{E}$ is an interpretation $\mathcal{E} : \mathcal{S} \to \mathcal{S}_{\mathsf{K}}$;

(ii) $C_{\mathsf{K}}(\Theta) = C_{\mathsf{K}}(\mathcal{E}(\mathcal{F}(\Theta)))$.

As in the theory of Blok and Pigozzi (see Corollary 2.9 of [6]), it turns out that Conditions (i) and (ii) together are equivalent to their dual statements, namely, that, for all G-sets of equations E, Θ and for all G-sets of formulas Γ,

(iii) $E \leq C_{\mathsf{K}}(\Theta)$ iff $\mathcal{F}(E) \leq C(\mathcal{F}(\Theta))$, i.e., $\mathcal{F}$ is an interpretation $\mathcal{F} : \mathcal{S}_{\mathsf{K}} \to \mathcal{S}$;

(iv) $C(\Gamma) = C(\mathcal{F}(\mathcal{E}(\Gamma)))$.

Thus, the roles played by the interpretations $\mathcal{E}$ and $\mathcal{F}$ are completely symmetric. We say that $\mathcal{E}$ and $\mathcal{F}$ are *inverse interpretations* when Conditions (i)-(iv) hold. In the context of G-logics, we are not able to replicate the Uniqueness Theorem 2.15 of [6]. However, we are able to show uniqueness in case the interpretations are of a special type, which allows us to emulate very closely the framework of sentential logics. We describe this partial result briefly. If we assume that $\mathcal{S}$ has two equivalent G-algebraic semantics K via $\mathcal{E}, \mathcal{F}$ and K' via $\mathcal{E}', \mathcal{F}'$ and that, for every G-set of equations Θ, it holds that $C(\mathcal{F}(\Theta)) = C(\mathcal{F}'(\Theta))$, then we can show that $\mathcal{S}_{\mathsf{K}} = \mathcal{S}_{\mathsf{K}'}$ and that, for every G-set of formulas Γ, $C_{\mathsf{K}}(\mathcal{E}(\Gamma)) = C_{\mathsf{K}}(\mathcal{E}'(\Gamma))$. But the hypothesis of this implication does not seem to be valid in the context of G-logics. We call a translation $\mathcal{E}$ from G-formulas to G-equations *standard* if, roughly speaking, it is induced by a set $\delta \approx \varepsilon$ of equations in a single variable. Dually, we call $\mathcal{F}$ *standard* if it is induced by a set Δ of formulas in two variables. We can now show that if K via standard $\mathcal{E}, \mathcal{F}$ and K' via standard $\mathcal{E}', \mathcal{F}'$ are two equivalent G-algebraic semantics for $\mathcal{S}$, then the equality $C(\mathcal{F}(\Theta)) = C(\mathcal{F}'(\Theta))$ is guaranteed and, thus, we recover the Uniqueness Theorem of an equational G-algebraic semantics and of the associated interpretations up to G-consequence.

Section 2.14 is the main (and longest) section of Chapter 2. It corresponds to Chapter 3 of [6], culminating in a Characterization Theorem, Theorem 42, of algebraizability of G-logics in terms of isomorphisms between lattices of theories, satisfying additional conditions. It forms an analog of the well known Theorem 3.7 of Blok and Pigozzi [6]. The section starts with given a G-logic $\mathcal{S} = \langle \mathcal{L}, C \rangle$ which has a G-algebraic semantics K via an interpretation $\mathcal{E} : \mathcal{S} \to \mathcal{S}_{\mathsf{K}}$. Based on these data, two mappings $H_{\mathsf{K}} : \mathrm{Th}(\mathcal{S}_{\mathsf{K}}) \to \mathrm{Th}(\mathcal{S})$ and $\Omega_{\mathsf{K}} : \mathrm{Th}(\mathcal{S}) \to \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$ are defined. More concretely, for all $\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$ and all $T \in \mathrm{Th}(\mathcal{S})$, we set

$$\begin{array}{rcl} H_{\mathsf{K}}(\Theta) & = & \bigvee\{\Gamma \in G^{\mathrm{Fm}_{\mathcal{L}}(V)} : \mathcal{E}(\Gamma) \leq \Theta\}; \\ \Omega_{\mathsf{K}}(T) & = & C_{\mathsf{K}}(\mathcal{E}(T)). \end{array}$$

We show that $\Omega_{\mathsf{K}} : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$ is a join continuous mapping and, for all $T \in \mathrm{Th}(\mathcal{S})$, $H_{\mathsf{K}}(\Omega_{\mathsf{K}}(T)) = T$, whereas, for all $\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$, $\Omega_{\mathsf{K}}(H_{\mathsf{K}}(\Theta)) \leq \Theta$, with equality holding if and only if $\Theta \in \Omega_{\mathsf{K}}(\mathrm{Th}(\mathcal{S}))$. These observations allow us to show that, if $\mathcal{E} : \mathcal{S} \to \mathcal{S}_{\mathsf{K}}$, then Ω_{K} maps $\mathbf{Th}(\mathcal{S})$ isomorphically onto a join complete subsemilattice of $\mathbf{Th}(\mathcal{S}_\mathsf{K})$ and that, moreover, if K is equivalent to $\mathcal{S}$ via $\mathcal{E}$, then $\Omega_\mathsf{K} : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_\mathsf{K})$ is an isomorphism, with $\mathcal{E}$ *invertible*, in the sense that, for some translation $\mathcal{F}$ from G-equations to G-formulas, $H_\mathsf{K}(C_\mathsf{K}(\Theta)) = C(\mathcal{F}(\Theta))$, for every G-set of equations Θ.

To abstract the framework, we look at an arbitrary join complete embedding $\Xi : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_\mathsf{K})$. We say that Ξ is *$\mathcal{E}$-regular* if it is induced by a some translation $\mathcal{E}$, in the sense that, for all G-sets of formulas Γ, $\Xi(C(\Gamma)) = C_\mathsf{K}(\mathcal{E}(\Gamma))$. Dually, for $Z : \mathbf{Th}(\mathcal{S}_\mathsf{K}) \to \mathbf{Th}(\mathcal{S})$, Z is *$\mathcal{F}$-regular* if it is induced by some translation $\mathcal{F}$ in the opposite direction. It is shown that, if $\Xi : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_\mathsf{K})$ is an $\mathcal{E}$-regular order isomorphism, then $\mathcal{E}$ is invertible via $\mathcal{F}$ if and only if Ξ^{-1} is $\mathcal{F}$-regular. This enables us to prove that a class K of G-algebras is a G-algebraic semantics for $\mathcal{S}$ if and only if, there exists an $\mathcal{E}$-regular isomorphism $\Xi : \mathbf{Th}(\mathcal{S}) \to \Xi(\mathbf{Th}(\mathcal{S}))$, where $\Xi(\mathbf{Th}(\mathcal{S}))$ is a join complete subsemilattice of $\mathbf{Th}(\mathcal{S}_\mathsf{K})$. Further, K is equivalent to $\mathcal{S}$ if and only if there exists an $\mathcal{E}$-regular isomorphism $\Xi : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_\mathsf{K})$, with $\mathcal{E}$ invertible.

In Section 2.15, the final section of Chapter 2, we deal exclusively with algebraization via standard interpretations. The developments here are rather technical, but, in a nutshell, this allows us to recover many of the distinctive features of Blok and Pigozzi's theory [6], perhaps the most striking among them the fact that Ω_K must coincide with the Leibniz operator Ω. As a result, one may provide in this setting an intrinsic characterization of standard algebraizability. We show that $\mathcal{S}$ is algebraizable via standard interpretations if and only if the Leibniz operator Ω is $\mathcal{E}$-regular, with $\mathcal{E}$ standard and standardly invertible, injective and join continuous on $\mathbf{Th}(\mathcal{S})$.

2.2 Graded Logics

Let $\mathcal{L}$ be a **logical language**, i.e., a set of logical connectives, or an algebraic signature, i.e., a set of operation symbols, depending on the point of view taken. Let V be a countably infinite set of variables. Denote by $\mathrm{Fm}_\mathcal{L}(V)$ the set of **$\mathcal{L}$-formulas** or **$\mathcal{L}$-terms** with variables in V and by $\mathbf{Fm}_\mathcal{L}(V) = \langle \mathrm{Fm}_\mathcal{L}(V), \mathcal{L}\rangle$ the corresponding absolutely free algebra generated by V.
A **substitution** $\sigma : \mathbf{Fm}_\mathcal{L}(V) \to \mathbf{Fm}_\mathcal{L}(V)$ is an endomorphism of $\mathbf{Fm}_\mathcal{L}(V)$, which is completely determined by the values it assigns to the variables in V.

Let $\mathbf{G} = \langle G, \leq\rangle$ be a poset, which, often, will be assumed to have additional structure, e.g., be a complete lattice. Given any set X and functions $f, g : X \to G$, also written $f, g \in G^X$, we define

$$f \leq g \quad \text{iff} \quad f(x) \leq g(x), \text{ for all } x \in X.$$

A **G-set of formulas** is a function

$$\Gamma : \mathrm{Fm}_\mathcal{L}(V) \to G.$$

A G-**logic** is a pair $\mathcal{S} = \langle \mathcal{L}, C \rangle$, where

$$C : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$$

satisfies the following axioms, for all $\Gamma, \Delta : \mathrm{Fm}_{\mathcal{L}}(V) \to G$.

(**Inflationarity**) $\Gamma \leq C(\Gamma)$;

(**Monotonicity**) $\Gamma \leq \Delta$ implies $C(\Gamma) \leq C(\Delta)$;

(**Idempotency**) $C(C(\Gamma)) = C(\Gamma)$;

(**Structurality**) $C(\Gamma \circ \sigma) \leq C(\Gamma) \circ \sigma$, for all $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$.

Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ and $\mathcal{S}' = \langle \mathcal{L}, C' \rangle$ be two G-logics over the same signature $\mathcal{L}$. We say $\mathcal{S}'$ is an **extension** of $\mathcal{S}$ and that $\mathcal{S}$ is a **sublogic** of $\mathcal{S}'$, written $\mathcal{S} \leq \mathcal{S}'$, if, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$C(\Gamma) \leq C'(\Gamma).$$

We denote by $\mathsf{Log}_{\mathbf{G}}(\mathcal{L})$ be the collection of all G-logics over $\mathcal{L}$.

Proposition 1 *If* $\mathbf{G} = \langle G, \leq \rangle$ *is a complete lattice, then* $\mathsf{Log}_{\mathbf{G}}(\mathcal{L})$*, ordered by* $\leq$*, becomes a complete lattice.*

Proof: Let $\top$ be the top element in $\mathbf{G}$. Define $C_\top : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$ by setting, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$C_\top(\Gamma)(\varphi) = \top, \quad \varphi \in \mathrm{Fm}_{\mathcal{L}}(V).$$

Note that this defines a G-logic $\mathcal{S}_\top = \langle \mathcal{L}, C_\top \rangle$, which is obviously a top element in $\mathsf{Log}_{\mathbf{G}}(\mathcal{L})$ under $\leq$.

Next consider a collection $\mathcal{S}_i = \langle \mathcal{L}, C_i \rangle$, $i \in I$. Define $\bigwedge_i \mathcal{S}_i = \langle \mathcal{L}, \bigwedge_i C_i \rangle$ by setting, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\left(\bigwedge_i C_i\right)(\Gamma)(\varphi) = \bigwedge_i C_i(\Gamma)(\varphi), \quad \varphi \in \mathrm{Fm}_{\mathcal{L}}(V).$$

We show that $\bigwedge_i \mathcal{S}_i$ is a G-logic.

- First, by Inflationarity, $\Gamma \leq C_i(\Gamma)$, for all $i \in I$ and all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$. Thus, $\Gamma \leq \bigwedge_i C_i(\Gamma)$. Hence, by definition, $\Gamma \leq (\bigwedge_i C_i)(\Gamma)$.

- Next, for Monotonicity, suppose $\Gamma \leq \Delta$. By Monotonicity, $C_i(\Gamma) \leq C_i(\Delta)$, for all $i \in I$. Thus, $\bigwedge_{i \in I} C_i(\Gamma) \leq \bigwedge_{i \in I} C_i(\Delta)$. Hence, by definition, $(\bigwedge_i C_i)(\Gamma) \leq (\bigwedge_i C_i)(\Delta)$.

- For Idempotency, note, first, that, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\left(\bigwedge_i C_i\right)\left(\left(\bigwedge_i C_i\right)(\Gamma)\right) \leq C_i(C_i(\Gamma)) = C_i(\Gamma).$$

  This gives $(\bigwedge_i C_i)((\bigwedge_i C_i)(\Gamma)) \leq (\bigwedge_i C_i)(\Gamma)$. The reverse inequality is assured by Inflationarity.

- Finally, for Structurality, let $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and σ be a substitution. Then we have

$$\begin{array}{rcl} (\bigwedge_i C_i)(\Gamma \circ \sigma) & = & \bigwedge_i C_i(\Gamma \circ \sigma) \\ & \leq & \bigwedge_i (C_i(\Gamma) \circ \sigma) \\ & = & (\bigwedge_i C_i(\Gamma)) \circ \sigma \\ & = & ((\bigwedge_i C_i)(\Gamma)) \circ \sigma. \end{array}$$

$\bigwedge_i \mathcal{S}_i$ is clearly a lower bound of $\{\mathcal{S}_i\}_{i\in I}$ under $\leq$. Finally, it follows directly from the definition that $\bigwedge_i \mathcal{S}_i$ is the greatest lower bound of $\{\mathcal{S}_i\}_{i\in I}$.

A dual reasoning shows that $\bigvee_i \mathcal{S}_i = \langle \mathcal{L}, \bigvee_i C_i\rangle$, defined dually, forms a least upper bound of $\{\mathcal{S}_i\}_{i\in I}$ under $\leq$. Thus, $\langle \mathsf{Log}_{\mathbf{G}}(\mathcal{L}), \leq\rangle$ is a complete lattice, as asserted. ∎

The complete lattice of Proposition 1 is denoted by

$$\mathbf{Log}_{\mathbf{G}}(\mathcal{L}) = \langle \mathsf{Log}_{\mathbf{G}}(\mathcal{L}), \leq\rangle.$$

2.3 Graded Theories

Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic. A G-set of formulas T is called a G-**theory** of $\mathcal{S}$ if

$$C(T) = T.$$

For any $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$, $C(\Gamma)$ is the smallest G-theory of $\mathcal{S}$, such that $\Gamma \leq C(\Gamma)$. We say that Γ **generates** the G-theory $C(\Gamma)$. The collection of all G-theories of $\mathcal{S}$ is denoted by $\mathrm{Th}(\mathcal{S})$.

Proposition 2 *Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic, with $\mathbf{G} = \langle G, \leq\rangle$ a complete lattice. Then $\mathrm{Th}(\mathcal{S})$, ordered by $\leq$, forms a complete lattice.*

Proof: Consider an arbitrary collection $\{T_i : i \in I\} \subseteq \mathrm{Th}(\mathcal{S})$. We must show that $\bigwedge_{i\in I} T_i \in \mathrm{Th}(\mathcal{S})$. Note that

$$\begin{array}{rcll} C(\bigwedge_{i\in I} T_i) & \leq & C(T_i) & \text{(Monotonicity)} \\ & = & T_i. & (T_i \text{ a } G\text{-theory}) \end{array}$$

Thus, we get $C(\bigwedge_{i\in I} T_i) \le \bigwedge_{i\in I} T_i$. Since the reverse inequality holds by Inflationarity, we get that $C(\bigwedge_{i\in I} T_i) = \bigwedge_{i\in I} T_i$, i.e., $\bigwedge_{i\in I} T_i \in \mathrm{Th}(\mathcal{S})$. Hence, $\mathrm{Th}(\mathcal{S})$ forms a complete lattice under $\le$. ■

We denote the complete lattice of G-theories of a G-logic $\mathcal{S}$ by $\mathbf{Th}(\mathcal{S}) = \langle \mathrm{Th}(\mathcal{S}), \le \rangle$. Its largest element is the constant function $\top : \mathrm{Fm}_{\mathcal{L}}(V) \to G$, with

$$\top(\varphi) = \top, \quad \varphi \in \mathrm{Fm}_{\mathcal{L}}(V).$$

Its smallest element is $C(\bot)$, where $\bot : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ is given by

$$\bot(\varphi) = \bot, \quad \varphi \in \mathrm{Fm}_{\mathcal{L}}(V).$$

Let $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ be a G-set of formulas. We say that Γ is **finite** if, for all but finitely many formulas φ,

$$\Gamma(\varphi) = \bot.$$

Given G-sets of formulas Γ, Δ, we write

$$\Gamma \le_f \Delta$$

to signify that $\Gamma \le \Delta$ and Γ is finite.

We say a G-logic $\mathcal{S} = \langle \mathcal{L}, C \rangle$ is **finitary** if, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$C(\Gamma) = \bigvee_{\Gamma_0 \le_f \Gamma} C(\Gamma_0).$$

The following lemmas give some properties of the G-theories of $\mathcal{S}$.

Lemma 3 *Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic. Then, for all $T \in \mathrm{Th}(\mathcal{S})$ and all $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$,*

$$T \circ \sigma \in \mathrm{Th}(\mathcal{S}).$$

Proof: Using Structurality and the fact that T is a G-theory, we get

$$C(T \circ \sigma) \le C(T) \circ \sigma = T \circ \sigma.$$

Since the reverse inclusion always holds, $C(T \circ \sigma) = T \circ \sigma$. Therefore, $T \circ \sigma$ is a G-theory. ■

Lemma 4 *Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic. Then, for all $\{T_i : i \in I\} \subseteq \mathrm{Th}(\mathcal{S})$ and all $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$,*

$$\bigwedge_{i\in I}(T_i \circ \sigma) = \left(\bigwedge_{i\in I} T_i\right) \circ \sigma \quad \text{and} \quad \bigvee_{i\in I}(T_i \circ \sigma) = \left(\bigvee_{i\in I} T_i\right) \circ \sigma.$$

Proof: This follows directly from the definitions involved. E.g., we have, for all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\begin{array}{rcl} \bigvee_{i\in I}(T_i \circ \sigma)(\varphi) & = & \bigvee_{i\in I} T_i(\sigma(\varphi)) \\ & = & (\bigvee_{i\in I} T_i)(\sigma(\varphi)) \\ & = & ((\bigvee_{i\in I} T_i) \circ \sigma)(\varphi). \end{array}$$

Therefore, $\bigvee_{i\in I}(T_i \circ \sigma) = (\bigvee_{i\in I} T_i) \circ \sigma$. Similarly for meet. ■

2.4 Graded Matrix Semantics

A **graded $\mathcal{L}$-matrix** or simply G-**matrix** is a pair $\mathfrak{A} = \langle \mathbf{A}, F\rangle$, where:

- $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}}\rangle$ is an $\mathcal{L}$-algebra;
- $F : A \to G$ is a function, called the **graded set of designated elements**, the **graded filter** or, simply, the G-**filter** of the G-matrix.

Given a G-matrix $\mathfrak{A} = \langle \mathbf{A}, F\rangle$, we define a mapping

$$C_{\mathfrak{A}} : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$$

by, setting, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$C_{\mathfrak{A}}(\Gamma) = \bigwedge\{F \circ h : h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Gamma \leq F \circ h\}.$$

More generally, given a class M of G-matrices, we define a mapping

$$C_{\mathsf{M}} : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$$

by, setting, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$C_{\mathsf{M}}(\Gamma) = \bigwedge\{C_{\mathfrak{A}} : \mathfrak{A} \in \mathsf{M}\}.$$

Proposition 5 *Let M be a class of G-matrices, with $\mathbf{G} = \langle G, \leq\rangle$ a complete lattice. Then $\mathcal{S}_{\mathsf{M}} = \langle \mathcal{L}, C_{\mathsf{M}}\rangle$ is a G-logic.*

Proof: We fist check Inflationarity. Let $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$. Then, for all $\mathfrak{A} = \langle \mathbf{A}, F\rangle \in \mathsf{M}$, we get

$$\Gamma \leq \bigwedge\{F \circ h : h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Gamma \leq F \circ h\} = C_{\mathfrak{A}}(\Gamma).$$

Therefore, $\Gamma \leq \bigwedge_{\mathfrak{A}\in\mathsf{M}} C_{\mathfrak{A}}(\Gamma) = C_{\mathsf{M}}(\Gamma)$.

We turn, next, to Monotonicity. Suppose $\Gamma, \Delta : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ are such that $\Gamma \leq \Delta$. Then we have

$$\begin{array}{rcl} C_{\mathsf{M}}(\Gamma) & = & \bigwedge\{F \circ h : \mathfrak{A} = \langle \mathbf{A}, F\rangle \in \mathsf{M}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Gamma \leq F \circ h\} \\ & \leq & \bigwedge\{F \circ h : \mathfrak{A} = \langle \mathbf{A}, F\rangle \in \mathsf{M}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Delta \leq F \circ h\} \\ & = & C_{\mathsf{M}}(\Delta). \end{array}$$

Next, we look at Idempotency. For all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$, we get

$$\begin{array}{rcl} C_{\mathsf{M}}(C_{\mathsf{M}}(\Gamma)) & = & \bigwedge\{F \circ h : \mathfrak{A} = \langle \mathbf{A}, F\rangle \in \mathsf{M}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \\ & & \qquad C_{\mathsf{M}}(\Gamma) \leq F \circ h\} \\ & = & \bigwedge\{F \circ h : \mathfrak{A} = \langle \mathbf{A}, F\rangle \in \mathsf{M}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Gamma \leq F \circ h\} \\ & = & C_{\mathsf{M}}(\Gamma). \end{array}$$

Finally, for Structurality, let $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and σ be a substitution. Then we get

$$\begin{array}{rcl} C_{\mathsf{M}}(\Gamma \circ \sigma) & = & \bigwedge\{F \circ h : \mathfrak{A} = \langle \mathbf{A}, F \rangle \in \mathsf{M}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Gamma \circ \sigma \leq F \circ h\} \\ & \leq & \bigwedge\{F \circ h \circ \sigma : \mathfrak{A} = \langle \mathbf{A}, F \rangle \in \mathsf{M}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \\ & & \qquad \Gamma \circ \sigma \leq F \circ h \circ \sigma\} \\ & \leq & \bigwedge\{F \circ h : \mathfrak{A} = \langle \mathbf{A}, F \rangle \in \mathsf{M}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Gamma \leq F \circ h\} \circ \sigma \\ & = & C_{\mathsf{M}}(\Gamma) \circ \sigma. \end{array}$$

Thus, $\mathcal{S}_{\mathsf{M}} = \langle \mathcal{L}, C_{\mathsf{M}} \rangle$ is a G-logic. ∎

$\mathcal{S}_{\mathsf{M}}$ is called the ***G*-logic determined by** or **induced by** the class M of G-matrices. If $\mathsf{M} = \{\mathfrak{A}\}$, then we write $C_{\mathfrak{A}}$ instead of $C_{\{\mathfrak{A}\}}$.

Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic. A G-matrix $\mathfrak{A} = \langle \mathbf{A}, F \rangle$ is called a ***G*-matrix of $\mathcal{S}$** or an **$\mathcal{S}$-matrix** if

$$C \leq C_{\mathfrak{A}}.$$

In this case, the G-filter F of $\mathfrak{A}$ is called an **$\mathcal{S}$-filter** on $\mathbf{A}$. The collection of all $\mathcal{S}$-filters on $\mathbf{A}$ is denoted by $\mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$. We now look at the structure of this set. We have the following technical lemma.

Lemma 6 *Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic, with $\mathbf{G} = \langle G, \leq \rangle$ a complete lattice, $\mathbf{A}$ an $\mathcal{L}$-algebra and $\{F_i : i \in I\}$ a collection of G-filters on $\mathbf{A}$. Then*

$$\bigwedge_{i \in I} C_{\langle \mathbf{A}, F_i \rangle} = C_{\langle \mathbf{A}, \bigwedge_{i \in I} F_i \rangle}.$$

Proof: We have, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\begin{array}{rcl} (\bigwedge_{i \in I} C_{\langle \mathbf{A}, F_i \rangle})(\Gamma) & = & \bigwedge_i \bigwedge_h \{F_i \circ h : \Gamma \leq F_i \circ h\} \\ & = & \bigwedge_h \bigwedge_i \{F_i \circ h : \Gamma \leq F_i \circ h\} \\ & = & \bigwedge_h \{(\bigwedge_i F_i) \circ h : \Gamma \leq (\bigwedge_i F_i) \circ h\} \\ & = & C_{\langle \mathbf{A}, \bigwedge_i F_i \rangle}(\Gamma). \end{array}$$

∎

Lemma 7 *Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic, with $\mathbf{G} = \langle G, \leq \rangle$ a complete lattice, and $\mathbf{A}$ an $\mathcal{L}$-algebra. Then*

$$\mathbf{Fi}_{\mathcal{S}}(\mathbf{A}) = \langle \mathrm{Fi}_{\mathcal{S}}(\mathbf{A}), \leq \rangle$$

is a complete lattice.

Proof: It is clear that the constant function $\top : A \to G$, with

$$\top(a) = \top, \quad a \in A,$$

is a maximum element in $\mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$. Moreover, given $F_i \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$, $i \in I$, we have, by definition, $C \leq C_{\langle \mathbf{A}, F_i\rangle}$, for all $i \in I$, whence, by Lemma 6,

$$C \leq \bigwedge_{i \in I} C_{\langle \mathbf{A}, F_i\rangle} = C_{\langle \mathbf{A}, \bigwedge_{i\in I} F_i\rangle}.$$

Thus, $\bigwedge F_i$ is also an $\mathcal{S}$-filter on $\mathbf{A}$. ∎

In closing, we show that the collection of $\mathcal{S}$-filters on the formula algebra $\mathbf{Fm}_{\mathcal{L}}(V)$ coincides with the collection of G-theories of $\mathcal{S}$.

Lemma 8 *Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic. Then*

$$\mathrm{Fi}_{\mathcal{S}}(\mathbf{Fm}_{\mathcal{L}}(V)) = \mathrm{Th}(\mathcal{S}).$$

Proof: Suppose, first, that $T \in \mathrm{Th}(\mathcal{S})$. Let $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$, such that $\Gamma \leq T \circ \sigma$. Then, we have

$$C(\Gamma) \leq C(T \circ \sigma) \leq C(T) \circ \sigma = T \circ \sigma.$$

Therefore, $T \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{Fm}_{\mathcal{L}}(V))$.

Suppose, conversely, that $T \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{Fm}_{\mathcal{L}}(V))$. Clearly, $T \leq T \circ i$, where i is the identity homomorphism on $\mathbf{Fm}_{\mathcal{L}}(V)$. Thus, by hypothesis,

$$C(T) \leq T \circ i = T.$$

This shows that $T = C(T)$ and, hence, $T \in \mathrm{Th}(\mathcal{S})$. ∎

Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic and M a class of G-matrices. M is called a G-**matrix semantics** of (or for) $\mathcal{S}$ if

$$C = C_{\mathsf{M}}.$$

In this case it is said that M is **strongly adequate for** $\mathcal{S}$. Both the class of all $\mathcal{S}$-models and the class of all $\mathcal{S}$-models on the formula algebra are strongly adequate for $\mathcal{S}$.

2.5 G-Congruences and Congruences

Since most of our results require that G be a complete lattice, we make this assumption from now own, even if it is not explicitly mentioned.

We denote by $\mathrm{Eq}_{\mathcal{L}}(V)$ the set of all $\mathcal{L}$-**equations**, i.e.,

$$\mathrm{Eq}_{\mathcal{L}}(V) = \mathrm{Fm}^2_{\mathcal{L}}(V).$$

For $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$, such an equation may be denoted by $\langle \varphi, \psi\rangle$ or $\varphi \approx \psi$.

Let $\mathbf{A}$ be an algebra. A G-**congruence** on $\mathbf{A}$ is a mapping $\Theta : A^2 \to G$, such that, for all $\lambda \in \mathcal{L}$ and all $a, b, c, \overline{a}, \overline{b} \in A$, (**Reflexivity**) $\Theta(a,a) = \top$;

(**Symmetry**) $\Theta(a,b) = \Theta(b,a)$;

(**Transitivity**) $\Theta(a,b) \wedge \Theta(b,c) \leq \Theta(a,c)$;

(**Congruence**) $\bigwedge_i \Theta(a_i,b_i) \leq \Theta(\lambda^{\mathbf{A}}(\overline{a}),\lambda^{\mathbf{A}}(\overline{b}))$.

Let $\mathrm{Gon}(\mathbf{A})$ denote the collection of all G-congruences on $\mathbf{A}$.

Proposition 9 *Let* $\mathbf{A}$ *be an* $\mathcal{L}$*-algebra and* $\mathbf{G} = \langle G, \leq\rangle$ *a complete lattice. The collection* $\mathrm{Gon}(\mathbf{A})$ *of* G*-congruences on* $\mathbf{A}$*, ordered by* $\leq$*, forms a complete lattice.*

Proof: First, note that $\top : A^2 \to G$, with

$$\top(a,b) = \top, \text{ for all } a,b \in A,$$

is a G-congruence on $\mathbf{A}$ and is clearly the largest G-congruence under $\leq$.

Next consider the collection Θ_i, $i \in I$, of G-congruences on $\mathbf{A}$. Define $\bigwedge_i \Theta_i : A^2 \to G$ by setting, for all $a,b \in A$,

$$\left(\bigwedge_i \Theta_i\right)(a,b) = \bigwedge_i \Theta_i(a,b).$$

We show that $\bigwedge_i \Theta_i$ is a G-congruence.

- First, for all $a \in A$, $(\bigwedge_i \Theta_i)(a,a) = \bigwedge_i \Theta_i(a,a) = \bigwedge_i \top = \top$.
- Next, for all $a,b \in A$,

$$\left(\bigwedge_i \Theta_i\right)(a,b) = \bigwedge_i \Theta_i(a,b) = \bigwedge_i \Theta_i(b,a) = \left(\bigwedge_i \Theta_i\right)(b,a).$$

- Next, for all $a,b,c \in A$,

$$\begin{array}{rcl}(\bigwedge_i \Theta_i)(a,b) \wedge (\bigwedge_i \Theta_i)(b,c) & = & \bigwedge_i \Theta_i(a,b) \wedge \bigwedge_i \Theta_i(b,c)\\ & = & \bigwedge_i(\Theta_i(a,b) \wedge \Theta_i(b,c))\\ & \leq & \bigwedge_i \Theta_i(a,c)\\ & = & (\bigwedge_i \Theta_i)(a,c).\end{array}$$

- Finally, for every n-ary $\lambda \in \mathcal{L}$ and all $a_1,\ldots,a_n,b_1,\ldots,b_n \in A$, we have

$$\begin{array}{rcl}\bigwedge_{j=1}^n(\bigwedge_i \Theta_i)(a_j,b_j) & = & \bigwedge_{j=1}^n \bigwedge_i \Theta_i(a_j,b_j)\\ & = & \bigwedge_i(\bigwedge_{j=1}^n \Theta_i(a_j,b_j))\\ & \leq & \bigwedge_i \Theta_i(\lambda^{\mathbf{A}}(\overline{a}),\lambda^{\mathbf{A}}(\overline{b}))\\ & = & (\bigwedge_i \Theta_i)(\lambda^{\mathbf{A}}(\overline{a}),\lambda^{\mathbf{A}}(\overline{b})).\end{array}$$

$\bigwedge_i \Theta_i$ is clearly a lower bound of $\{\Theta_i\}_{i\in I}$ under $\leq$. Finally, it follows directly from the definition that $\bigwedge_i \Theta_i$ is the greatest lower bound of $\{\Theta_i\}_{i\in I}$.

This proves that $\langle \mathrm{Gon}(\mathcal{L}), \leq\rangle$ is a complete lattice, as asserted. ∎

The lattice of G-congruences on $\mathbf{A}$, given by Proposition 9, is denoted by

$$\mathbf{Gon}(\mathbf{A}) = \langle \mathrm{Gon}(\mathbf{A}), \leq\rangle.$$

G-congruences are closely associated with *families of ordinary congruences*. We give this correspondence in detail because, on the one hand, it illuminates the concept of G-congruence by relating it to a more familiar concept and, on the other, passing back and forth between G-congruences and congruences is very useful in applying the concepts, especially in reductions.

Let $\mathbf{A}$ be an $\mathcal{L}$-algebra and $\Theta : A^2 \to G$ be a G-congruence. Let $g \in G$. Define a binary relation

$$\hat{\Theta}_g \subseteq A \times A$$

by setting, for all $a, b \in A$,

$$\langle a, b\rangle \in \hat{\Theta}_g \quad \text{iff} \quad \Theta(a, b) \geq g.$$

Proposition 10 *Let* $\mathbf{A}$ *be an* $\mathcal{L}$*-algebra,* $\Theta : A^2 \to G$ *be a* G*-congruence on* $\mathbf{A}$ *and* $g \in G$*. Then* $\hat{\Theta}_g$ *is a congruence on* $\mathbf{A}$*. Moreover, for all* $g_1, g_2 \in G$*,*

$$g_1 \leq g_2 \quad \textit{implies} \quad \hat{\Theta}_{g_2} \subseteq \hat{\Theta}_{g_1}.$$

Proof: For all $a \in A$, we have $\Theta(a, a) = \top \geq g$. Hence, $\langle a, a\rangle \in \hat{\Theta}_g$ and $\hat{\Theta}_g$ is reflexive. For symmetry, let $a, b \in A$. Then we have

$$\langle a, b\rangle \in \hat{\Theta}_g \;\Rightarrow\; \Theta(a, b) \geq g \;\Rightarrow\; \Theta(b, a) \geq g \;\Rightarrow\; \langle b, a\rangle \in \hat{\Theta}_g.$$

For transitivity, let $a, b, c \in A$. Then

$$\begin{aligned} \langle a, b\rangle \in \hat{\Theta}_g \text{ and } \langle b, c\rangle \in \hat{\Theta}_g &\Rightarrow \Theta(a, b) \geq g \text{ and } \Theta(b, c) \geq g \\ &\Rightarrow \Theta(a, b) \wedge \Theta(b, c) \geq g \\ &\Rightarrow \Theta(a, c) \geq g \\ &\Rightarrow \langle a, c\rangle \in \hat{\Theta}_g. \end{aligned}$$

Thus, $\hat{\Theta}_g$ is an equivalence relation. To finish the demonstration that it is a congruence, let $\lambda \in \mathcal{L}$ be n-ary and suppose $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$. Then

$$\begin{aligned} \langle a_i, b_i\rangle \in \hat{\Theta}_g,\ i \in I, &\Rightarrow \Theta(a_i, b_i) \geq g,\ i \in I, \\ &\Rightarrow \textstyle\bigwedge_i \Theta(a_i, b_i) \geq g \\ &\Rightarrow \Theta(\lambda^{\mathbf{A}}(\overline{a}), \lambda^{\mathbf{A}}(\overline{b})) \geq g \\ &\Rightarrow \langle \lambda^{\mathbf{A}}(\overline{a}), \lambda^{\mathbf{A}}(\overline{b})\rangle \in \hat{\Theta}_g. \end{aligned}$$

Finally, note that, if $g_1, g_2 \in G$, such that $g_1 \le g_2$, we have, for all $a, b \in A$,

$$\langle a,b\rangle \in \hat{\Theta}_{g_2} \;\Rightarrow\; \Theta(a,b) \ge g_2 \;\Rightarrow\; \Theta(a,b) \ge g_1 \;\Rightarrow\; \langle a,b\rangle \in \hat{\Theta}_{g_1}.$$

This shows that the displayed antimonotonicity property holds. ■

We set

$$\hat{\Theta} = \{\hat{\Theta}_g : g \in G\}$$

and call $\hat{\Theta}$ the **stratified congruence associated with** the G-congruence Θ. Further, for a fixed $g \in G$, we call $\hat{\Theta}_g$ the g-**stratum** of Θ.

Conversely, let us call a family $\theta = \{\theta_g : g \in G\}$ of congruences on $\mathbf{A}$ a **stratified congruence** if it satisfies

$$\theta_{g_2} \subseteq \theta_{g_1}, \text{ for all } g_1 \le g_2.$$

Define the mapping $\check{\theta} : A^2 \to G$ by setting, for all $a, b \in A$,

$$\check{\theta}(a,b) = \bigvee\{g \in G : \langle a,b\rangle \in \theta_g\}.$$

Proposition 11 *Suppose* $\mathbf{G} = \langle G, \le\rangle$ *satisfies complete distributivity. Let* $\mathbf{A}$ *be an* $\mathcal{L}$*-algebra and* $\theta = \{\theta_g : g \in G\}$ *a stratified congruence on* $\mathbf{A}$*. Then* $\check{\theta}$ *is a* G*-congruence on* $\mathbf{A}$*.*

Proof: For all $a \in A$, we have

$$\check{\theta}(a,a) = \bigvee\{g : \langle a,a\rangle \in \theta_g\} = \bigvee G = \top.$$

For all $a, b \in A$,

$$\check{\theta}(a,b) = \bigvee\{g : \langle a,b\rangle \in \theta_g\} = \bigvee\{g : \langle b,a\rangle \in \theta_g\} = \check{\theta}(b,a).$$

For all $a, b, c \in A$,

$$\begin{array}{rcl}\check{\theta}(a,b) \wedge \check{\theta}(b,c) & = & \bigvee\{g : \langle a,b\rangle \in \theta_g\} \wedge \bigvee\{h : \langle b,c\rangle \in \theta_h\}\\ & = & \bigvee\{g \wedge h : \langle a,b\rangle \in \theta_g \text{ and } \langle b,c\rangle \in \theta_h\}\\ & \le & \bigvee\{g \wedge h : \langle a,b\rangle \in \theta_{g\wedge h} \text{ and } \langle b,c\rangle \in \theta_{g\wedge h}\}\\ & \le & \bigvee\{g \wedge h : \langle a,c\rangle \in \theta_{g\wedge h}\}\\ & \le & \check{\theta}(a,c).\end{array}$$

Similarly, for all $\lambda \in \mathcal{L}$ n-ary and all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$,

$$\begin{array}{rcl}\bigwedge_{i=1}^n \check{\theta}(a_i,b_i) & = & \bigwedge_{i=1}^n \bigvee\{g_i : \langle a_i,b_i\rangle \in \theta_{g_i}\}\\ & = & \bigvee\{\bigwedge_{i=1}^n g_i : \langle a_i,b_i\rangle \in \theta_{g_i}, i \in I\}\\ & \le & \bigvee\{\bigwedge_{i=1}^n g_i : \langle a_i,b_i\rangle \in \theta_{\wedge_i g_i}\}\\ & \le & \bigvee\{g : \langle\lambda^{\mathbf{A}}(\overline{a}), \lambda^{\mathbf{A}}(\overline{b})\rangle \in \theta_g\}\\ & = & \check{\theta}(\lambda^{\mathbf{A}}(\overline{a}), \lambda^{\mathbf{A}}(\overline{b})).\end{array}$$

This proves that $\check{\theta}$ is a G-congruence on $\mathbf{A}$. ∎

We call $\check{\theta}$ the G-**congruence associated with** the stratified congruence $\theta = \{\theta_g : g \in G\}$.

Finally, we show that, under special circumstances, e.g., when G is completely distributive and every element in G is a (possibly infinite) join of completely join-irreducibles, then the correspondences established via $\hat{}$ and $\check{}$ are inverses of one another. In such cases, therefore, the point of view taken, G-congruence versus stratified congruence, is a matter of preference and/or convenience.

Proposition 12 *Suppose* $\mathbf{G} = \langle G, \leq \rangle$ *satisfies complete distributivity. Let* $\mathbf{A}$ *be an* $\mathcal{L}$*-algebra. Then the following statements hold.*

(a) $\check{\hat{\Theta}} = \Theta$, *for every* G*-congruence* Θ *on* $\mathbf{A}$*;*

(b) $\hat{\check{\theta}}_g = \theta_g$, *for every stratified congruence* $\theta = \{\theta_g : g \in G\}$ *on* $\mathbf{A}$, *provided* g *is completely join irreducible in* $\mathbf{G}$*.*

Proof: For Part (a), suppose $a, b \in A$. Then we have

$$\check{\hat{\Theta}}(a,b) = \bigvee\{g : \langle a,b\rangle \in \hat{\Theta}_g\} = \bigvee\{g : \Theta(a,b) \geq g\} = \Theta(a,b).$$

For Part (b), we have, for all $a, b \in A$,

$$\begin{array}{rll} \langle a,b\rangle \in \hat{\check{\theta}}_g & \text{iff} & \check{\theta}(a,b) \geq g \\ & \text{iff} & \bigvee\{g' : \langle a,b\rangle \in \theta_{g'}\} \geq g \\ & \text{iff} & \langle a,b\rangle \in \theta_{g'}, \text{ for some } g' \geq g, \\ & \text{iff} & \langle a,b\rangle \in \theta_g. \end{array}$$

The one equivalence before the last is a consequence of complete distributivity and complete join irreducibility. ∎

Corollary 13 *Let* $\mathbf{G} = \langle G, \leq \rangle$ *be such that, for every algebra* $\mathbf{A}$, *the mappings* $\Theta \mapsto \hat{\Theta}$ *and* $\theta \mapsto \check{\theta}$ *are inverses of one another. Then, for all algebras* $\mathbf{A}$, *all* G*-congruences* $\Theta', \Theta'' \in \mathrm{Gon}(\mathbf{A})$, *and all* $a, b \in A$,

$$(\Theta' \vee^{\mathbf{Gon(A)}} \Theta'')(a,b) = \bigvee\{g : \langle a,b\rangle \in \hat{\Theta}'_g \vee^{\mathbf{Con(A)}} \hat{\Theta}''_g\}.$$

Proof: We have

$$\begin{array}{rcl} (\Theta' \vee^{\mathbf{Gon(A)}} \Theta'')(a,b) & = & \bigvee\{g : \langle a,b\rangle \in \widehat{\Theta' \vee^{\mathbf{Gon(A)}} \Theta''}_g\} \\ & = & \bigvee\{g : \langle a,b\rangle \in \hat{\Theta}'_g \vee^{\mathbf{Con(A)}} \hat{\Theta}''_g\}. \end{array}$$

This proves the statement. ∎

2.6 Compatibility and Leibniz Congruences

Let $\mathbf{A}$ be an algebra and $F : A \to G$ a G-filter. A G-congruence Θ on $\mathbf{A}$ is said to be **compatible with** F if, for all $a, b \in A$,

$$\Theta(a,b) \wedge F(a) \leq F(b).$$

Suppose $\mathbf{G}$ has an implication $\to$, satisfying

$$g_1 \wedge g_2 \leq g_3 \quad \text{iff} \quad g_2 \leq g_1 \to g_3,$$

and denote

$$g_1 \leftrightarrow g_2 := (g_1 \to g_2) \wedge (g_2 \to g_1).$$

Then Θ is compatible with F if and only if, for all $a, b \in A$,

$$\Theta(a,b) \leq F(a) \leftrightarrow F(b).$$

If Θ is compatible with F, we also say that Θ is a G-**congruence of** the G-matrix $\mathfrak{A} = \langle \mathbf{A}, F \rangle$. By $\mathrm{Gon}(\mathfrak{A})$ we denote the collection of all G-congruences of $\mathfrak{A}$.

We show that, under some hypotheses, $\mathrm{Gon}(\mathfrak{A})$, ordered by $\leq$, forms a principal ideal of the complete lattice $\mathbf{Gon(A)} = \langle \mathrm{Gon}(\mathbf{A}), \leq \rangle$ of all G-congruences on $\mathbf{A}$. From then on, we shall assume that G satisfies those hypotheses and take the conclusion for granted. We start with a lemma.

Lemma 14 *Suppose $\mathbf{G} = \langle G, \leq \rangle$ is a complete lattice with an implication $\to$. Let $\mathfrak{A} = \langle \mathbf{A}, F \rangle$ be a G-matrix. The G-relation $R_{\mathfrak{A}} : A^2 \to G$, defined, for all $a, b \in A$, by*

$$R_{\mathfrak{A}}(a,b) = \bigwedge \{F(\varphi^{\mathbf{A}}(a,\overline{c})) \leftrightarrow F(\varphi^{\mathbf{A}}(b,\overline{c})) : \varphi \in \mathrm{Fm}_{\mathcal{L}}(V), \overline{c} \in A\}$$

is a G-congruence on $\mathbf{A}$ compatible with F, i.e., $R_{\mathfrak{A}} \in \mathrm{Gon}(\mathfrak{A})$.

Proof: It is clear that $R_{\mathfrak{A}}$ satisfies Reflexivity and Symmetry. For Transitivity, let $a, b, c \in A$. Then we have

$$\begin{aligned} & R_{\mathfrak{A}}(a,b) \wedge R_{\mathfrak{A}}(b,c) \\ & \quad = [\textstyle\bigwedge_{\varphi,\overline{d}} F(\varphi^{\mathbf{A}}(a,\overline{d})) \leftrightarrow F(\varphi^{\mathbf{A}}(b,\overline{d}))] \\ & \qquad\qquad \wedge [\textstyle\bigwedge_{\psi,\overline{e}} F(\psi^{\mathbf{A}}(b,\overline{e})) \leftrightarrow F(\psi^{\mathbf{A}}(c,\overline{e}))] \\ & \quad \leq \textstyle\bigwedge_{\varphi,\overline{d}} [(F(\varphi^{\mathbf{A}}(a,\overline{d})) \leftrightarrow F(\varphi^{\mathbf{A}}(b,\overline{d}))) \\ & \qquad\qquad \wedge (F(\varphi^{\mathbf{A}}(b,\overline{d})) \leftrightarrow F(\varphi^{\mathbf{A}}(c,\overline{d})))] \\ & \quad \leq \textstyle\bigwedge_{\varphi,\overline{d}} F(\varphi^{\mathbf{A}}(a,\overline{d})) \leftrightarrow F(\varphi^{\mathbf{A}}(c,\overline{d})) \\ & \quad = R_{\mathfrak{A}}(a,c). \end{aligned}$$

For the Congruence property, let $\lambda \in \mathcal{L}$ be n-ary and $a_i, b_i \in A$, $i < n$. Then, for all $\varphi(x, \overline{z}) \in \mathrm{Fm}_{\mathcal{L}}(V)$ and all $\overline{c} \in A$,

$$\begin{aligned} R_{\mathfrak{A}}(a_i,b_i) \leq {} & F(\varphi^{\mathbf{A}}(\lambda^{\mathbf{A}}(a_0,\ldots,a_{i-1},a_i,b_{i+1},\ldots,b_{n-1}),\overline{c})) \\ & \leftrightarrow F(\varphi^{\mathbf{A}}(\lambda^{\mathbf{A}}(a_0,\ldots,a_{i-1},b_i,b_{i+1},\ldots,b_{n-1}),\overline{c})). \end{aligned}$$

Thus, by Transitivity,

$$\bigwedge_i R_{\mathfrak{A}}(a_i,b_i) \leq F(\varphi^{\mathbf{A}}(\lambda^{\mathbf{A}}(\overline{a}),\overline{c})) \leftrightarrow F(\varphi^{\mathbf{A}}(\lambda^{\mathbf{A}}(\overline{a}),\overline{c})),$$

i.e.,

$$\bigwedge_i R_{\mathfrak{A}}(a_i,b_i) \leq R_{\mathfrak{A}}(\lambda^{\mathbf{A}}(\overline{a}),\lambda^{\mathbf{A}}(\overline{b})).$$

So $R_{\mathfrak{A}}$ is a G-congruence on $\mathbf{A}$. Compatibility with F is straightforward by taking $\varphi = x$ in the defining property of $R_{\mathfrak{A}}$. ∎

Next, we prove that $\mathrm{Gon}(\mathfrak{A})$ forms a principal ideal of the complete lattice $\mathbf{Gon}(\mathbf{A}) = \langle \mathrm{Gon}(\mathbf{A}), \leq \rangle$ of all G-congruences on $\mathbf{A}$.

Proposition 15 *Suppose* $\mathbf{G} = \langle G, \leq \rangle$ *is a complete lattice with an implication* $\rightarrow$. *Let* $\mathfrak{A} = \langle \mathbf{A}, F \rangle$ *be a* G*-matrix. Then* $\mathrm{Gon}(\mathfrak{A})$ *is a principal ideal in* $\mathbf{Gon}(\mathbf{A})$.

Proof: First, suppose $\Theta, \Theta' \in \mathrm{Gon}(\mathbf{A})$, such that $\Theta \leq \Theta' \in \mathrm{Gon}(\mathfrak{A})$. Then, for all $a, b \in A$,

$$\Theta(a,b) \wedge F(a) \leq \Theta'(a,b) \wedge F(a) \leq F(b).$$

Thus, $\Theta \in \mathrm{Gon}(\mathfrak{A})$. So $\mathrm{Gon}(\mathfrak{A})$ is a downset in $\mathbf{Gon}(\mathbf{A})$.

Suppose, next, that $\Theta, \Theta' \in \mathrm{Gon}(\mathfrak{A})$. One may show by induction on the structure of an $\mathcal{L}$-term φ that, for all $a, b, \bar{c} \in A$,

$$\Theta(a,b) \leq \Theta(\varphi^{\mathbf{A}}(a,\bar{c}),\varphi^{\mathbf{A}}(b,\bar{c})).$$

By compatibility of Θ with F, it follows that

$$\Theta(a,b) \leq F(\varphi^{\mathbf{A}}(a,\bar{c})) \leftrightarrow F(\varphi^{\mathbf{A}}(b,\bar{c})).$$

Therefore, $\Theta(a,b) \leq R_{\mathfrak{A}}(a,b)$. Similarly, $\Theta'(a,b) \leq R_{\mathfrak{A}}(a,b)$. This yields

$$(\Theta \bigvee^{\mathbf{Gon(A)}} \Theta')(a,b) \wedge F(a) \leq R_{\mathfrak{A}}(a,b) \wedge F(a) \leq F(b).$$

Therefore, $\Theta \bigvee^{\mathbf{Gon(A)}} \Theta' \in \mathrm{Gon}(\mathfrak{A})$. This shows that $\mathrm{Gon}(\mathfrak{A})$ is an ideal in $\mathbf{Gon}(\mathbf{A})$. Finally, to see that it is principal, it suffices to show that it has a maximal element. This can be shown using Zorn's Lemma. Indeed every chain in $\mathrm{Gon}(\mathfrak{A})$ is upper bounded by its join, which is also compatible with F. ∎

We shall restrict attention to lattices $\mathbf{G} = \langle G, \leq \rangle$ for which the conclusion of Proposition 15 holds. This is due to the fact that our theory, attempting to emulate the main points of the theory of Blok and Pigozzi [6] and of Font and Jansana [28], requires the existence of a maximum element in $\mathbf{Gon}(\mathfrak{A})$. Let us call such lattices **Leibniz permitting**. Some of the techniques and results presented, however, may be transferrable to more relaxed settings.

The generator of Gon($\mathfrak{A}$), i.e., the largest G-congruence on $\mathbf{A}$ compatible with F, is called the **Leibniz G-congruence of F on A** or the **Leibniz G-congruence of $\mathfrak{A}$**. It is denoted by $\Omega_{\mathbf{A}}(F)$ or $\Omega(\mathcal{A})$. The operator

$$\Omega_{\mathbf{A}} : G^A \to \mathrm{Gon}(\mathbf{A})$$

is called the **Leibniz operator** on $\mathbf{A}$. In the special case in which $\mathbf{A} = \mathbf{Fm}_{\mathcal{L}}(V)$, we write Ω for $\Omega_{\mathbf{Fm}_{\mathcal{L}}(V)}$ to simplify notation.

Based on bits and pieces of preceding work, we can provide a characterization of the Leibniz G-congruence of a G-filter on an algebra in terms of indistinguishability, paralleling the one given by Blok and Pigozzi to explain the name "Leibniz congruence" for the original notion they introduced (Page 11 of [6]).

Theorem 16 *Suppose* $\mathbf{G} = \langle G, \leq\rangle$ *is a complete lattice with an implication* $\to$*. Let* $\mathfrak{A} = \langle \mathbf{A}, F\rangle$ *be a G-matrix. Then* $\Omega_{\mathbf{A}}(F) = R_{\mathfrak{A}}$.

Proof: Suppose, first, that $a, b \in A$. Then, since $\Omega_{\mathbf{A}}(F)$ is a G-congruence, we have, for all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$ and all $\overline{c} \in A$,

$$\Omega_{\mathbf{A}}(F)(a,b) \leq \Omega_{\mathbf{A}}(F)(\varphi^{\mathbf{A}}(a,\overline{c}), \varphi^{\mathbf{A}}(b,\overline{c})).$$

Finally, by compatibility of $\Omega_{\mathbf{A}}(F)$ with F, we get

$$\begin{aligned} \Omega_{\mathbf{A}}(F)(a,b) \wedge F(\varphi^{\mathbf{A}}(a,\overline{c})) &\leq \Omega_{\mathbf{A}}(F)(\varphi^{\mathbf{A}}(a,\overline{c}), \varphi^{\mathbf{A}}(b,\overline{c})) \wedge F(\varphi^{\mathbf{A}}(a,\overline{c})) \\ &\leq F(\varphi^{\mathbf{A}}(b,\overline{c})). \end{aligned}$$

Thus, we conclude that

$$\Omega_{\mathbf{A}}(F)(a,b) \leq \bigwedge_{\varphi,\overline{c}} F(\varphi^{\mathbf{A}}(a,\overline{c})) \leftrightarrow F(\varphi^{\mathbf{A}}(b,\overline{c})),$$

i.e., $\Omega_{\mathbf{A}}(F) \leq R_{\mathfrak{A}}$. For the reverse inequality, because of the definition of $\Omega_{\mathbf{A}}(F)$, it suffices to show that $R_{\mathfrak{A}}$ is a G-congruence on $\mathbf{A}$ compatible with F. This, however, holds by Lemma 14. Since $\Omega_{\mathbf{A}}(F)$ is, by definition, the largest congruence on $\mathbf{A}$ compatible with F, we get $R_{\mathfrak{A}} \leq \Omega_{\mathbf{A}}(F)$. ■

We close the section with a result of a slightly more technical nature pertaining to the identification of the Leibniz G-congruence of a G-filter. Let $\mathfrak{A} = \langle \mathbf{A}, F\rangle$ be a G-matrix. We say that $\Theta : A^2 \to G$ is **definable (with parameters) in $\mathfrak{A}$** if there exist $\varphi(x,y,\overline{z}) \in \mathrm{Fm}_{\mathcal{L}}(V)$ and $\overline{c} \in A$, such that, for all $a, b \in A$,

$$\Theta(a,b) = F(\varphi^{\mathbf{A}}(a,b,\overline{c})).$$

Theorem 17 *Suppose* $\mathbf{G} = \langle G, \leq\rangle$ *is a complete lattice with* $\to$*. Let* $\mathfrak{A} = \langle \mathbf{A}, F\rangle$ *be a G-matrix and* $\Theta : A^2 \to G$ *definable in* $\mathfrak{A}$.

(a) If $\Theta(a,a) = \top$, *for all* $a \in A$, *then* $\Omega_{\mathbf{A}}(F) \leq \Theta$.

(b) If, in addition, Θ *is a* G*-congruence on* $\mathbf{A}$ *compatible with* F, *then* $\Omega_{\mathbf{A}}(F) = \Theta$.

Proof:

(a) Let $a, b \in A$. Suppose Θ is definable by $\psi(x, y, \overline{z})$, with parameters $\overline{d}$. Then we have

$$\begin{array}{lll} \Omega_{\mathbf{A}}(F)(a,b) & = & \bigwedge_{\varphi,\overline{c}} F(\varphi^{\mathbf{A}}(a,\overline{c})) \leftrightarrow F(\varphi^{\mathbf{A}}(b,\overline{c})) \quad \text{(Theorem 16)} \\ & \leq & F(\psi^{\mathbf{A}}(a,b,\overline{d})) \leftrightarrow F(\psi^{\mathbf{A}}(b,b,\overline{d})) \quad \text{(Instantiation)} \\ & = & \Theta(a,b) \leftrightarrow \Theta(b,b) \quad \text{(Definability of } \Theta \text{ in } \mathfrak{A}) \\ & = & \Theta(a,b) \leftrightarrow \top \quad (\Theta(b,b) = \top) \\ & = & \Theta(a,b). \quad \text{(Property of } \to) \end{array}$$

Hence, $\Omega_{\mathbf{A}}(F) \leq \Theta$.

(b) By hypothesis, $\Theta \in \mathrm{Gon}(\mathfrak{A})$. Since $\Omega_{\mathbf{A}}(F)$ is the largest G-congruence in $\mathrm{Gon}(\mathfrak{A})$, we get $\Theta \leq \Omega_{\mathbf{A}}(F)$. Therefore, by Part (a), $\Omega_{\mathbf{A}}(F) = \Theta$.

■

2.7 Protoalgebraic Graded Logics

Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic. We say that $\mathcal{S}$ is **protoalgebraic** if, for all $T \in \mathrm{Th}(\mathcal{S})$ and all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Omega(T)(\varphi,\psi) \wedge T'(\varphi) \leq T'(\psi), \quad \text{for all } T \leq T' \in \mathrm{Th}(\mathcal{S}).$$

In implication form, should such a connective exist, we can reformulate the defining condition equivalently as

$$\Omega(T)(\varphi,\psi) \leq T'(\varphi) \leftrightarrow T'(\psi), \quad \text{for all } T \leq T' \in \mathrm{Th}(\mathcal{S}).$$

Protoalgebraic G-logics are characterized by the monotonicity of the Leibniz operator on the complete lattice of their G-theories.

Theorem 18 *Let* $\mathcal{S} = \langle \mathcal{L}, C \rangle$ *be a* G*-logic.* $\mathcal{S}$ *is protoalgebraic if and only if, for all* $T, T' \in \mathrm{Th}(\mathcal{S})$,

$$T \leq T' \quad \textit{implies} \quad \Omega(T) \leq \Omega(T').$$

Proof: Suppose, first, that $\mathcal{S}$ is protoalgebraic. Let $T, T' \in \mathrm{Th}(\mathcal{S})$, such that $T \leq T'$. To see that $\Omega(T) \leq \Omega(T')$, it suffices to show, by the maximality property of the Leibniz G-congruence, that $\Omega(T)$ is compatible with T'. Let $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$. Then, by G-protoalgebraicity,

$$\Omega(T)(\varphi,\psi) \wedge T'(\varphi) \leq T'(\psi).$$

Hence $\Omega(T)$ is compatible with T' and $\Omega(T) \leq \Omega(T')$.

Assume, conversely, that, for all $T, T' \in \mathrm{Th}(\mathcal{S})$, with $T \leq T'$, we have $\Omega(T) \leq \Omega(T')$. Thus, $\Omega(T)$ is a G-congruence that is compatible with T'. By compatibility, for all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Omega(T)(\varphi,\psi) \wedge T'(\varphi) \leq T'(\psi).$$

This proves that $\mathcal{S}$ is protoalgebraic. ■

2.8 Graded 2-Logics

Let $\mathrm{Eq}_{\mathcal{L}}(V)$ be the set of all $\mathcal{L}$-equations and $\mathbf{G} = \langle G, \leq\rangle$ a complete lattice. A G-**set of equations** is a mapping

$$\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G.$$

A G-2-**logic** is a pair $\mathcal{S} = \langle \mathcal{L}, C\rangle$, where

$$C : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$$

is a mapping that satisfies the following axioms, for all $\Theta, \Theta' : \mathrm{Eq}_{\mathcal{L}}(V) \to G$.

(**Inflationarity**) $\Theta \leq C(\Theta)$;

(**Monotonicity**) $\Theta \leq \Theta'$ implies $C(\Theta) \leq C(\Theta')$;

(**Idempotency**) $C(C(\Theta)) = C(\Theta)$;

(**Structurality**) $C(\Theta \circ \sigma) \leq C(\Theta) \circ \sigma$, for all $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$.

Structurality would be more accurately defined by the inequality

$$C(\Theta \circ \langle\sigma,\sigma\rangle) \leq C(\Theta) \circ \langle\sigma,\sigma\rangle,$$

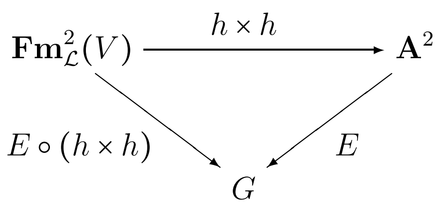
but it is very common to overload notation and apply a substitution σ on equations by applying the substitution to each side of the equation. A G-**algebra** is a pair $\mathcal{A} = \langle \mathbf{A}, E\rangle$, where $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}}\rangle$ is an $\mathcal{L}$-algebra and $E : A^2 \to G$ is a G-congruence on $\mathbf{A}$. In the context of G-2-logics, G-algebras play the role that G-matrices play in the context of G-logics.

A key concept from the point of view of algebraic logic is that of the G-2-logic determined by a given class of G-algebras. This parallels, for equations, the concept of a G-logic determined by a G-matrix, defined at the beginning of Section 2.4.

Let $\mathcal{A} = \langle \mathbf{A}, E\rangle$ be a G-algebra. Define the mapping

$$C_{\mathcal{A}} : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$$

by setting, for all $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$,



$$C_{\mathcal{A}}(\Theta) = \bigwedge\{E \circ h : h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta \le E \circ h\}.$$

Extending this definition, for a class K of G-algebras, we define

$$C_{\mathsf{K}} = \bigwedge\{C_{\mathcal{A}} : \mathcal{A} \in \mathsf{K}\}.$$

Proposition 19 *Let K be a class of G-algebras. Then $\mathcal{S}_{\mathsf{K}} = \langle \mathcal{L}, C_{\mathsf{K}}\rangle$ is a G-2-logic.*

Proof: The proof is similar to the proof of Proposition 5. We must show that C_{K} satisfies Inflationarity, Monotonicity, Idempotency and Structurality. For Inflationarity, let $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$. Then we have

$$\begin{aligned} \Theta &\le \wedge\{E \circ h : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta \le E \circ h\} \\ &= C_{\mathsf{K}}(\Theta). \end{aligned}$$

For Monotonicity, suppose $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$ and $\Theta' : \mathrm{Eq}_{\mathcal{L}}(V) \to G$, such that $\Theta \le \Theta'$. Then we have

$$\begin{aligned} C_{\mathsf{K}}(\Theta) &= \wedge\{E \circ h : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta \le E \circ h\} \\ &\le \wedge\{E \circ h : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta' \le E \circ h\} \\ &= C_{\mathsf{K}}(\Theta'). \end{aligned}$$

For Idempotency, let $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$. Then we have

$$\begin{aligned} &C_{\mathsf{K}}(C_{\mathsf{K}}(\Theta)) \\ &\quad = \wedge\{E \circ h : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, C_{\mathsf{K}}(\Theta) \le E \circ h\} \\ &\quad = \wedge\{E \circ h : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta \le E \circ h\} \\ &\quad = C_{\mathsf{K}}(\Theta). \end{aligned}$$

Finally, for Structurality, suppose $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$ and let σ be a substitution. Then

$$\begin{aligned}
&C_{\mathsf{K}}(\Theta \circ \sigma)\\
&\quad = \bigwedge\{E \circ h : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta \circ \sigma \leq E \circ h\}\\
&\quad \leq \bigwedge\{E \circ h \circ \sigma : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta \circ \sigma \leq E \circ h \circ \sigma\}\\
&\quad \leq \bigwedge\{E \circ h : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta \leq E \circ h\} \circ \sigma\\
&\quad = C_{\mathsf{K}}(\Theta) \circ \sigma.
\end{aligned}$$

We have now shown that $\mathcal{S}_{\mathsf{K}} = \langle \mathcal{L}, C_{\mathsf{K}}\rangle$ is a G-2-logic. ■

$\mathcal{S}_{\mathsf{K}} = \langle \mathcal{L}, C_{\mathsf{K}}\rangle$ is called the G-2-**logic determined by** or **induced by** the class K of G-algebras.

A G-set of equations $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$ is said to be a G-2-**theory** or a G-**congruence** of $\mathcal{S}_{\mathsf{K}}$ if

$$C_{\mathsf{K}}(\Theta) = \Theta.$$

We denote the collection of all G-2-theories of $\mathcal{S}_{\mathsf{K}}$ by $\mathrm{Th}(\mathcal{S}_{\mathsf{K}})$.

The next lemma provides a reassurance that the name G-congruence for G-2-theories is well-deserved, since it does not conflict with previous terminology concerning G-congruences on $\mathcal{L}$-algebras.

Lemma 20 *Let K be a class of G-algebras.*

(a) The G-2-theories of $\mathcal{S}_{\mathsf{K}}$ are G-congruences on $\mathbf{Fm}_{\mathcal{L}}(V)$.

(b) Every G-congruence on $\mathbf{Fm}_{\mathcal{L}}(V)$, such that $\langle \mathbf{Fm}_{\mathcal{L}}(V), \Theta\rangle \in \mathsf{K}$, is a G-2-theory of $\mathcal{S}_{\mathsf{K}}$.

Proof: Let $\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$ and $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$. Then

$$\begin{aligned}
\Theta(\varphi, \varphi) &= C_{\mathsf{K}}(\Theta)(\varphi, \varphi)\\
&= \bigwedge\{E(h(\varphi), h(\varphi)) : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K},\\
&\qquad\qquad h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta \leq E \circ h\}\\
&= \top.
\end{aligned}$$

Let $\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$ and $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$. Then

$$\begin{aligned}
\Theta(\varphi, \psi) &= C_{\mathsf{K}}(\Theta)(\varphi, \psi)\\
&= \bigwedge\{E(h(\varphi), h(\psi)) : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K},\\
&\qquad\qquad h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta \leq E \circ h\}\\
&= \bigwedge\{E(h(\psi), h(\varphi)) : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K},\\
&\qquad\qquad h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta \leq E \circ h\}\\
&= C_{\mathsf{K}}(\Theta)(\psi, \varphi)\\
&= \Theta(\psi, \varphi).
\end{aligned}$$

Let $\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$ and $\varphi, \psi, \chi \in \mathrm{Fm}_{\mathcal{L}}(V)$. Then

$$\begin{aligned}
&\Theta(\varphi,\psi) \wedge \Theta(\psi,\chi)\\
&= C_{\mathsf{K}}(\Theta)(\varphi,\psi) \wedge C_{\mathsf{K}}(\Theta)(\psi,\chi)\\
&= \bigwedge\{E(h(\varphi),h(\psi)) : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta \leq E \circ h\}\\
&\qquad \wedge \bigwedge\{E(h(\psi),h(\chi)) : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta \leq E \circ h\}\\
&\leq \bigwedge\{E(h(\varphi),h(\psi)) \wedge E(h(\psi),h(\chi)) :\\
&\qquad\qquad \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta \leq E \circ h\}\\
&\leq \bigwedge\{E(h(\varphi),h(\chi)) : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta \leq E \circ h\}\\
&= C_{\mathsf{K}}(\Theta)(\varphi,\chi)\\
&= \Theta(\varphi,\chi).
\end{aligned}$$

The Congruence property can be demonstrated similarly.

Suppose, that Θ is a G-congruence on $\mathbf{Fm}_{\mathcal{L}}(V)$, such that $\langle \mathbf{Fm}_{\mathcal{L}}(V), \Theta\rangle \in \mathsf{K}$. Then

$$\begin{aligned}
C_{\mathsf{K}}(\Theta) &= \bigwedge_{\mathcal{A}=\langle \mathbf{A},E\rangle,h}\{E \circ h : \Theta \leq E \circ h\}\\
&\leq \Theta. \quad (\mathcal{A} = \langle \mathbf{Fm}_{\mathcal{L}}(V), \Theta\rangle \in \mathsf{K} \text{ and } h = i_{\mathbf{Fm}_{\mathcal{L}}(V)})
\end{aligned}$$

Hence Θ is a G-2-theory of $\mathcal{S}_{\mathsf{K}}$. ∎

The collection $\mathrm{Th}(\mathcal{S}_{\mathsf{K}})$ of all G-congruences of $\mathcal{S}_{\mathsf{K}}$ forms a complete lattice when ordered by $\leq$. It is denoted by $\mathbf{Th}(\mathcal{S}_{\mathsf{K}}) = \langle \mathrm{Th}(\mathcal{S}_{\mathsf{K}}), \leq\rangle$.

Proposition 21 *Let K be a class of G-algebras. Then $\mathbf{Th}(\mathcal{S}_{\mathsf{K}}) = \langle \mathrm{Th}(\mathcal{S}_{\mathsf{K}}), \leq\rangle$ is a complete lattice.*

Proof: Note, again, that $\top : \mathrm{Eq}_{\mathcal{L}}(V) \to G$, with $\top(\varphi,\psi) = \top$, for all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$, is an $\mathcal{S}_{\mathsf{K}}$-theory. Clearly, it is the largest $\mathcal{S}_{\mathsf{K}}$-theory under $\leq$. We show closure under $\bigwedge$. Let $\{\Theta_i : i \in I\} \subseteq \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$. We have

$$\begin{aligned}
C_{\mathsf{K}}(\bigwedge \Theta_i) &= \bigwedge\{E \circ h : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \bigwedge \Theta_i \leq E \circ h\}\\
&\leq \bigwedge\{E \circ h : \mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \Theta_i \leq E \circ h\}\\
&= C_{\mathsf{K}}(\Theta_i)\\
&= \Theta_i.
\end{aligned}$$

Thus, $C_{\mathsf{K}}(\bigwedge \Theta_i) \leq \bigwedge \Theta_i$. The reverse inclusion always holds. So $C_{\mathsf{K}}(\bigwedge \Theta_i) = \bigwedge \Theta_i$, showing that $\bigwedge \Theta_i \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$. ∎

2.9 Graded Translations and Interpretations

A **translation** from G-formulas to G-equations is a join preserving mapping

$$\mathcal{E} : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}.$$

A **translation** from G-equations to G-formulas is a join preserving mapping

$$\mathcal{F} : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}.$$

Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic and K be a class of G-algebras.
An **interpretation** from $\mathcal{S}$ to $\mathcal{S}_{\mathsf{K}}$ is a translation

$$\mathcal{E} : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)},$$

such that, for all $\Gamma, \Phi : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\Phi \le C(\Gamma) \quad \text{iff} \quad \mathcal{E}(\Phi) \le C_{\mathsf{K}}(\mathcal{E}(\Gamma)).$$

An **interpretation** from $\mathcal{S}_{\mathsf{K}}$ to $\mathcal{S}$ is a translation

$$\mathcal{F} : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)},$$

such that, for all $\Theta, E : \mathrm{Eq}_{\mathcal{L}}(V) \to G$,

$$E \le C_{\mathsf{K}}(\Theta) \quad \text{iff} \quad \mathcal{F}(E) \le C(\mathcal{F}(\Theta)).$$

The following propositions provide useful characterizations of interpretations.

Proposition 22 *Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic and K be a class of G-algebras. Suppose $\mathcal{E}$ is a translation from G-formulas to G-equations. Then $\mathcal{E}$ is an interpretation $\mathcal{E} : \mathcal{S} \to \mathcal{S}_{\mathsf{K}}$ if and only if, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,*

$$C_{\mathsf{K}}(\mathcal{E}(\Gamma)) = C_{\mathsf{K}}(\mathcal{E}(C(\Gamma))$$

and

$$C(\Gamma) = \bigwedge\{C(\Phi) : \mathcal{E}(\Gamma) \le C_{\mathsf{K}}(\mathcal{E}(\Phi))\}.$$

Proof: Suppose that $\mathcal{E}$ is an interpretation. Then, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$, we have

$$\begin{array}{rll} \Gamma \le C(C(\Gamma)) & \text{implies} & \mathcal{E}(\Gamma) \le C_{\mathsf{K}}(\mathcal{E}(C(\Gamma))) \\ & \text{implies} & C_{\mathsf{K}}(\mathcal{E}(\Gamma)) \le C_{\mathsf{K}}(\mathcal{E}(C(\Gamma))); \\ C(\Gamma) \le C(\Gamma) & \text{implies} & \mathcal{E}(C(\Gamma)) \le C_{\mathsf{K}}(\mathcal{E}(\Gamma)) \\ & \text{implies} & C_{\mathsf{K}}(\mathcal{E}(C(\Gamma))) \le C_{\mathsf{K}}(\mathcal{E}(\Gamma)). \end{array}$$

This proves the first displayed condition. For the second, we have

$$\begin{array}{rcl} C(\Gamma) & = & \bigwedge\{C(\Phi) : \Gamma \le C(\Phi)\} \\ & = & \bigwedge\{C(\Phi) : \mathcal{E}(\Gamma) \le C_{\mathsf{K}}(\mathcal{E}(\Phi))\}. \end{array}$$

Next, we turn to the converse. Suppose that the two displayed conditions hold and let $\Gamma, \Phi : \mathrm{Fm}_{\mathcal{L}}(V) \to G$. Then we have

$$\begin{array}{rll}\Phi \leq C(\Gamma) & \text{implies} & \mathcal{E}(\Phi) \leq \mathcal{E}(C(\Gamma)) \\ & \text{implies} & \mathcal{E}(\Phi) \leq C_{\mathsf{K}}(\mathcal{E}(C(\Gamma))) \\ & \text{implies} & \mathcal{E}(\Phi) \leq C_{\mathsf{K}}(\mathcal{E}(\Gamma)).\end{array}$$

Further, if $\mathcal{E}(\Phi) \leq C_{\mathsf{K}}(\mathcal{E}(\Gamma))$, then

$$\begin{array}{rcl}C(\Phi) & = & \bigwedge\{C(\Phi') : \mathcal{E}(\Phi) \leq C_{\mathsf{K}}(\mathcal{E}(\Phi'))\} \\ & \leq & \bigwedge\{C(\Phi') : \mathcal{E}(\Gamma) \leq C_{\mathsf{K}}(\mathcal{E}(\Phi'))\} \\ & = & C(\Gamma).\end{array}$$

This shows that $\Phi \leq C(\Gamma)$. Hence, the two displayed conditions are necessary and sufficient for a translation $\mathcal{E}$ to be an interpretation. ■

The dual statement is formalized as

Proposition 23 *Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic and K be a class of G-algebras. Suppose $\mathcal{F}$ is a translation from G-equations to G-formulas. Then $\mathcal{F}$ is an interpretation $\mathcal{F} : \mathcal{S}_{\mathsf{K}} \to \mathcal{S}$ if and only if, for all $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$,*

$$C(\mathcal{F}(\Theta)) = C(\mathcal{F}(C_{\mathsf{K}}(\Theta))$$

and

$$C_{\mathsf{K}}(\Theta) = \bigwedge\{C_{\mathsf{K}}(E) : \mathcal{F}(\Theta) \leq C(\mathcal{F}(E))\}.$$

Proof: The proof is dual to that of Proposition 22. ■

2.10 Slicing

Given a G-set of formulas $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and formula $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$, we define the G-set $\Gamma^\varphi : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ by setting, for all $\psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Gamma^\varphi(\psi) = \begin{cases} \Gamma(\varphi), & \text{if } \psi = \varphi, \\ \bot, & \text{otherwise.} \end{cases}$$

Γ^φ is called the **instantiation of Γ to φ**. Similarly, given a G-set of equations $E : \mathrm{Eq}_{\mathcal{L}}(V) \to G$ and an equation $\varphi \approx \psi \in \mathrm{Eq}_{\mathcal{L}}(V)$, we define the G-set $E^{\varphi,\psi} : \mathrm{Eq}_{\mathcal{L}}(V) \to G$ by setting, for all $\delta \approx \varepsilon \in \mathrm{Eq}_{\mathcal{L}}(V)$,

$$E^{\varphi,\psi}(\delta, \varepsilon) = \begin{cases} E(\varphi, \psi), & \text{if } \langle \delta, \varepsilon \rangle = \langle \varphi, \psi \rangle, \\ \bot, & \text{otherwise.} \end{cases}$$

$E^{\varphi,\psi}$ is called the **instantiation of E to $\varphi \approx \psi$**.

Related to this notation, we also have the following.

Given a function $\Gamma : X \to G$, where $X \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$, define $\widehat{\Gamma} : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ by setting, for all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\widehat{\Gamma}(\varphi) = \begin{cases} \Gamma(\varphi), & \text{if } \varphi \in X, \\ \bot, & \text{otherwise.} \end{cases}$$

$\widehat{\Gamma}$ is called the **lifting of** Γ.

Given a function $E : Y \to G$, where $Y \subseteq \mathrm{Eq}_{\mathcal{L}}(V)$, define $\widehat{E} : \mathrm{Eq}_{\mathcal{L}}(V) \to G$ by setting, for all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\widehat{E}(\varphi,\psi) = \begin{cases} E(\varphi,\psi), & \text{if } \langle \varphi,\psi\rangle \in Y, \\ \bot, & \text{otherwise.} \end{cases}$$

$\widehat{E}$ is called the **lifting of** E.

E.g., identifying the function $\Gamma = \{\langle \varphi, g\rangle\}$ with the pair it contains, we have, for all $\psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\widehat{\langle \varphi, g\rangle}(\psi) = \begin{cases} g, & \text{if } \psi = \varphi, \\ \bot, & \text{otherwise.} \end{cases}$$

A **hybrid translation** from formulas to equations is a function

$$E : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \times \mathrm{Fm}_{\mathcal{L}}(V) \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$$

that satisfies the following conditions, for all $\Gamma, \Gamma_i \in G$ and all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

(**Bottom**) $E(\bot, \varphi) = \bot$;

(**Slicing**) $E(\Gamma, \varphi) = E(\Gamma^{\varphi}, \varphi)$;

(**Join Continuity**) $E(\bigvee_{i\in I} \Gamma_i, \varphi) = \bigvee_{i\in I} E(\Gamma_i, \varphi)$.

A **hybrid translation** from equations to formulas is a function

$$F : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \times \mathrm{Eq}_{\mathcal{L}}(V) \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$$

that satisfies the following conditions, for all $E, E_i \in G^{\mathrm{Eq}_{\mathcal{L}}(V)}$ and all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

(**Bottom**) $F(\bot, \langle \varphi, \psi\rangle) = \bot$;

(**Slicing**) $F(E, \langle \varphi, \psi\rangle) = F(E^{\varphi,\psi}, \langle \varphi, \psi\rangle)$;

(**Join Continuity**) $F(\bigvee_{i\in I} E_i, \langle \varphi, \psi\rangle) = \bigvee_{i\in I} F(E_i, \langle \varphi, \psi\rangle)$.

There is a close connection between translations and hybrid translations, which we now describe. We do this in full detail for translations from formulas to equations. The treatment for translations from equations to formulas, which is dual, is then only briefly described.

First, consider a translation $\mathcal{E} : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$. We define a function

$$\mathcal{E}^h : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \times \mathrm{Fm}_{\mathcal{L}}(V) \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$$

by setting, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\mathcal{E}^h(\Gamma,\varphi) = \mathcal{E}(\Gamma^\varphi).$$

We show that $\mathcal{E}^h$ is a hybrid translation. First, for all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\begin{array}{rcll} \mathcal{E}^h(\bot,\varphi) & = & \mathcal{E}(\bot^\varphi) & (\text{Definition of } \mathcal{E}^h)\\ & = & \mathcal{E}(\bot) & (\bot^\varphi = \bot)\\ & = & \bot. & (\mathcal{E} \text{ join continuous}) \end{array}$$

Second, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\begin{array}{rcll} \mathcal{E}^h(\Gamma,\varphi) & = & \mathcal{E}(\Gamma^\varphi) & (\text{Definition of } \mathcal{E}^h)\\ & = & \mathcal{E}((\Gamma^\varphi)^\varphi) & ((\Gamma^\varphi)^\varphi = \Gamma^\varphi)\\ & = & \mathcal{E}(\Gamma^\varphi,\varphi). & (\text{Definition of } \mathcal{E}^h) \end{array}$$

Finally, for all $\Gamma_i : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\begin{array}{rcll} \mathcal{E}^h(\bigvee_{i\in I}\Gamma_i,\varphi) & = & \mathcal{E}((\bigvee_{i\in I}\Gamma_i)^\varphi) & (\text{Definition of } \mathcal{E}^h)\\ & = & \mathcal{E}(\bigvee_{i\in I}\Gamma_i^\varphi) & ((\bigvee_{i\in I}\Gamma_i)^\varphi = \bigvee_{i\in I}\Gamma_i^\varphi)\\ & = & \bigvee_{i\in I}\mathcal{E}(\Gamma_i^\varphi) & (\mathcal{E} \text{ join continuous})\\ & = & \bigvee_{i\in I}\mathcal{E}^h(\Gamma_i,\varphi). & (\text{Definition of } \mathcal{E}^h) \end{array}$$

Conversely, consider a hybrid translation $E : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \times \mathrm{Fm}_{\mathcal{L}}(V) \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$. Define a function

$$E^t : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$$

by setting, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$E^t(\Gamma) = \bigvee\{E(\Gamma^\varphi,\varphi) : \varphi \in \mathrm{Fm}_{\mathcal{L}}(V)\}.$$

We show that E^t is join continuous and, therefore, a translation. We have, for all $\Gamma_i : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\begin{array}{rcll} E^t(\bigvee_{i\in I}\Gamma_i) & = & \bigvee_\varphi E((\bigvee_{i\in I}\Gamma_i)^\varphi,\varphi) & (\text{Definition of } E^t)\\ & = & \bigvee_\varphi E(\bigvee_{i\in I}\Gamma_i^\varphi,\varphi) & ((\bigvee_{i\in I}\Gamma_i)^\varphi = \bigvee_{i\in I}\Gamma_i^\varphi)\\ & = & \bigvee_\varphi \bigvee_{i\in I} E(\Gamma_i^\varphi,\varphi) & (E \text{ join continuous})\\ & = & \bigvee_{i\in I}\bigvee_\varphi E(\Gamma_i^\varphi,\varphi) & (\text{Identical joins})\\ & = & \bigvee_{i\in I} E^t(\Gamma_i). & (\text{Definition of } E^t) \end{array}$$

Finally, we show that the two processes are inverses of each other. We have, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\begin{array}{rcll}\mathcal{E}^{ht}(\Gamma) & = & \bigvee_\psi \mathcal{E}^h(\Gamma^\psi, \psi) & \text{(Definition of } \mathcal{E}^{ht}) \\ & = & \bigvee_\psi \mathcal{E}((\Gamma^\psi)^\psi) & \text{(Definition of } \mathcal{E}^h) \\ & = & \bigvee_\psi \mathcal{E}(\Gamma^\psi) & ((\Gamma^\psi)^\psi = \Gamma^\psi) \\ & = & \mathcal{E}(\bigvee_\psi \Gamma^\psi) & (\mathcal{E} \text{ join continuous}) \\ & = & \mathcal{E}(\Gamma). & (\bigvee_\psi \Gamma^\psi = \Gamma)\end{array}$$

Finally, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\begin{array}{rcll}E^{th}(\Gamma, \varphi) & = & E^t(\Gamma^\varphi) & \text{(Definition of } E^{th}) \\ & = & \bigvee_\psi E((\Gamma^\varphi)^\psi, \psi) & \text{(Definition of } E^t) \\ & = & E(\Gamma^\varphi, \varphi) & \text{(Cases and Bottom)} \\ & = & E(\Gamma, \varphi). & \text{(Slicing)}\end{array}$$

We close the section by describing briefly the dual scenario concerning translations from equations to formulas.

Suppose we are given a translation $\mathcal{F} : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$ from G-equations to G-formulas. Then we define a hybrid translation

$$\mathcal{F}^h : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \times \mathrm{Eq}_{\mathcal{L}}(V) \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$$

by setting, for all $E : \mathrm{Eq}_{\mathcal{L}}(V) \to G$ and $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\mathcal{F}^h(E, \langle \varphi, \psi \rangle) = \mathcal{F}(E^{\varphi,\psi}).$$

Conversely, given a hybrid translation $F : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \times \mathrm{Eq}_{\mathcal{L}}(V) \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$, we define a translation

$$F^t : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$$

by setting, for all $E : \mathrm{Eq}_{\mathcal{L}}(V) \to G$,

$$F^t(E) = \bigvee_{\varphi,\psi} F(E^{\varphi,\psi}, \langle \varphi, \psi \rangle).$$

Then, as before, $\mathcal{F}^h$ is a hybrid translation, F^t is a translation and

$$\mathcal{F}^{ht} = \mathcal{F} \quad \text{and} \quad F^{th} = F,$$

that is, the two processes are inverses of one another.

We conclude that considering translations or their corresponding hybrid versions is only a matter of convenience (and/or taste) depending on context, since they are interchangeable.

2.11 Structural Translations and Slicing

A translation $\mathcal{E} : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$ from G-formulas to G-equations is said to be **structural** if, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$, all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$ and all $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$,

$$\mathcal{E}(\Gamma^{\sigma(\varphi)}) = \mathcal{E}(\Gamma^{\varphi}) \circ \sigma^2.$$

A translation $\mathcal{F} : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$ from G-equations to G-formulas is said to be **structural** if, for all $E : \mathrm{Eq}_{\mathcal{L}}(V) \to G$, all $\varphi \approx \psi \in \mathrm{Eq}_{\mathcal{L}}(V)$ and all $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$,

$$\mathcal{F}(E^{\sigma(\varphi),\sigma(\psi)}) = \mathcal{F}(E^{\varphi,\psi}) \circ \sigma.$$

A hybrid translation from formulas to equations

$$E : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \times \mathrm{Fm}_{\mathcal{L}}(V) \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$$

is **structural** if, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$, all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$ and all $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$,

(**Structurality**) $E(\Gamma^{\sigma(\varphi)}, \sigma(\varphi)) = E(\Gamma^{\varphi}, \varphi) \circ \sigma^2.$

A hybrid translation from equations to formulas

$$F : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \times \mathrm{Eq}_{\mathcal{L}}(V) \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$$

is **structural** if, for all $E : \mathrm{Eq}_{\mathcal{L}}(V) \to G$, all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$ and all $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$,

(**Structurality**) $F(E^{\sigma(\varphi),\sigma(\psi)}, \langle\sigma(\varphi), \sigma(\psi)\rangle) = F(E^{\varphi,\psi}, \langle\varphi, \psi\rangle) \circ \sigma.$

Let us note, here, that in the preceding definitions, one could relax structurality to surjective structurality without losing any power. Surjective structurality stipulates that the structurality condition hold for all surjective substitutions $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$.

We want to show that the correspondence established in Section 2.10 between translations and hybrid translations extends to one between structural translations and structural hybrid translations. We deal again only with translations from formulas to equations, since translations from equations to formulas are handled dually. Moreover, we only check structurality, since all other properties were checked in detail in Section 2.10.

Consider a structural translation $\mathcal{E} : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$. We show that the hybrid translation

$$\mathcal{E}^h : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \times \mathrm{Fm}_{\mathcal{L}}(V) \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$$

defined, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$, by

$$\mathcal{E}^h(\Gamma, \varphi) = \mathcal{E}(\Gamma^\varphi),$$

is also structural. We have, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$, all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$ and all $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$,

$$\begin{array}{rcll}
\mathcal{E}^h(\Gamma^{\sigma(\varphi)}, \sigma(\varphi)) & = & \mathcal{E}((\Gamma^{\sigma(\varphi)})^{\sigma(\varphi)}) & \text{(Definition of } \mathcal{E}^h) \\
& = & \mathcal{E}(\Gamma^{\sigma(\varphi)}) & ((\Gamma^{\sigma(\varphi)})^{\sigma(\varphi)} = \Gamma^{\sigma(\varphi)}) \\
& = & \mathcal{E}(\Gamma^\varphi) \circ \sigma^2 & (\mathcal{E} \text{ structural}) \\
& = & \mathcal{E}((\Gamma^\varphi)^\varphi) \circ \sigma^2 & ((\Gamma^\varphi)^\varphi = \Gamma^\varphi) \\
& = & \mathcal{E}^h(\Gamma^\varphi, \varphi) \circ \sigma^2. & \text{(Definition of } \mathcal{E}^h)
\end{array}$$

Consider, next, a structural hybrid translation $E : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \times \mathrm{Fm}_{\mathcal{L}}(V) \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$. We show that the translation

$$E^t : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$$

defined, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$, by

$$E^t(\Gamma) = \bigvee \{E(\Gamma^\varphi, \varphi) : \varphi \in \mathrm{Fm}_{\mathcal{L}}(V)\}$$

is structural. We have, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$, all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$ and all $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$,

$$\begin{array}{rcll}
E^t(\Gamma^\varphi) \circ \sigma^2 & = & \bigvee_\psi E((\Gamma^\varphi)^\psi, \psi) \circ \sigma^2 & \text{(Definition of } E^t) \\
& = & E(\Gamma^\varphi, \varphi) \circ \sigma^2 & \text{(Cases and Bottom)} \\
& = & E(\Gamma^{\sigma(\varphi)}, \sigma(\varphi)) & (E \text{ structural}) \\
& = & \bigvee_\psi E((\Gamma^{\sigma(\varphi)})^\psi, \psi) & \text{(Cases and Bottom)} \\
& = & E^t(\Gamma^{\sigma(\varphi)}). & \text{(Definition of } E^t)
\end{array}$$

Since the definitions of ^{h} and of ^{t} are the same as in the preceding section, we know, based on our work there, that the two processes are inverses of each other.

2.12 Algebraic and Matrix Semantics

Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic and K be a class of G-algebras.

We say that K is a G-**algebraic semantics** for $\mathcal{S}$ if there exists an interpretation $\mathcal{E}$ from $\mathcal{S}$ to $\mathcal{S}_{\mathsf{K}}$.

We say that $\mathcal{S}$ is a G-**logical semantics** for K if there exists an interpretation $\mathcal{F}$ from $\mathcal{S}_{\mathsf{K}}$ to $\mathcal{S}$.

For specific types of translations, these notions are connected with some concepts encountered earlier.

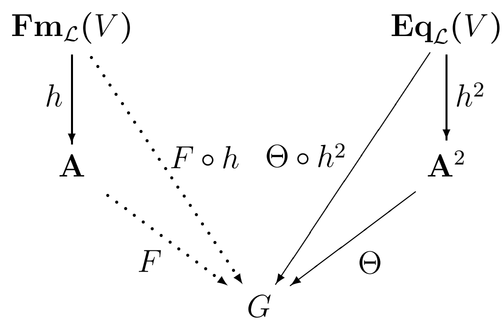
We say that a translation $\mathcal{E} : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$ from G-formulas to G-equations is **order reflecting** if, for all $\Gamma, \Gamma' : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\mathcal{E}(\Gamma) \leq \mathcal{E}(\Gamma') \quad \text{implies} \quad \Gamma \leq \Gamma'.$$

Since, by definition, $\mathcal{E}$ is join continuous, it is, a fortiori, order preserving, whence this condition may be equivalently expressed as a biconditional, i.e., for all $\Gamma, \Gamma' : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\Gamma \leq \Gamma' \quad \text{iff} \quad \mathcal{E}(\Gamma) \leq \mathcal{E}(\Gamma').$$

Further, the G-translation $\mathcal{E}$ is called **reflectively structural** if, for every algebra $\mathbf{A}$ and every G-congruence $\Theta : A^2 \to G$ on $\mathbf{A}$, there exists a G-filter $F : A \to G$ on $\mathbf{A}$, such that, for all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$,



$$\mathcal{E}(F \circ h) = \Theta \circ h^2.$$

This property, when present, allows one to construct, given a G-congruence on an algebra $\mathbf{A}$, a "corresponding" G-filter F on $\mathbf{A}$, where "corresponding" here means that they are connected via the displayed equation. Exploiting this, given a reflectively structural translation $\mathcal{E}$ and a G-algebra $\mathcal{A} = \langle \mathbf{A}, E \rangle$, we define the G-matrix

$$\mathcal{A}^{\mathcal{E}} = \langle \mathbf{A}, E^{\mathcal{E}} \rangle,$$

where $E^{\mathcal{E}}$ is the G-filter on $\mathbf{A}$, such that, for all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$,

$$\mathcal{E}(E^{\mathcal{E}} \circ h) = E \circ h^2.$$

Then, given a class K of G-algebras, we define the class $\mathsf{K}^{\mathcal{E}}$ of G-matrices by

$$\mathsf{K}^{\mathcal{E}} = \{\mathcal{A}^{\mathcal{E}} : \mathcal{A} \in \mathsf{K}\}.$$

Theorem 24 *Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic, K be a class of G-algebras and $\mathcal{E} : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$ a reflectively structural, order reflecting G-translation. K is a G-algebraic semantics for $\mathcal{S}$ via $\mathcal{E}$ if and only if $\mathsf{K}^{\mathcal{E}}$ is a G-matrix semantics for $\mathcal{S}$.*

Proof: We have the following equivalences, for all $\Gamma, \Phi : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\begin{array}{lll} \mathcal{E}(\Phi) \leq C_{\mathsf{K}}(\mathcal{E}(\Gamma)) & \text{iff} & \mathcal{E}(\Phi) \leq \bigwedge_{\mathcal{A},h}\{E \circ h^2 : \mathcal{E}(\Gamma) \leq E \circ h^2\} \\ & & \text{(Definition of } C_{\mathsf{K}}) \\ & \text{iff} & \mathcal{E}(\Phi) \leq \bigwedge_{\mathcal{A},h}\{\mathcal{E}(E^{\mathcal{E}} \circ h) : \mathcal{E}(\Gamma) \leq \mathcal{E}(E^{\mathcal{E}} \circ h)\} \\ & & \text{(Refl. Struct. and Definition of } E^{\mathcal{E}}) \\ & \text{iff} & \Phi \leq \bigwedge_{\mathcal{A},h}\{E^{\mathcal{E}} \circ h : \Gamma \leq E^{\mathcal{E}} \circ h\} \\ & & \text{(Order Reflectivity)} \\ & \text{iff} & \Phi \leq C_{\mathsf{K}^{\mathcal{E}}}(\Gamma). \quad \text{(Definition of } C_{\mathsf{K}^{\mathcal{E}}}) \end{array}$$

We now obtain that $\mathsf{K}^{\mathcal{E}}$ is a G-matrix semantics for $\mathcal{S}$ if and only if $C = C_{\mathsf{K}^{\mathcal{E}}}$ if and only if, using the preceding equalities and the definition, K is a G-algebraic semantics for $\mathcal{S}$ via $\mathcal{E}$. ∎

Of course, a dual treatment leads to a dual theorem. Let us briefly recount the basic conditions and steps for the sake of completeness.

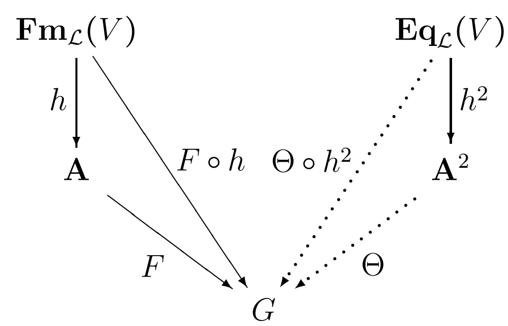
We say that a translation $\mathcal{F} : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$ from G-equations to G-formulas is **order reflecting** if, for all $E, E' : \mathrm{Eq}_{\mathcal{L}}(V) \to G$,

$$\mathcal{F}(E) \leq \mathcal{F}(E') \quad \text{implies} \quad E \leq E'.$$

Again, taking into account the join continuity of $\mathcal{F}$, we may equivalently write, for all $\Gamma, \Gamma' : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$E \leq E' \quad \text{iff} \quad \mathcal{F}(E) \leq \mathcal{F}(E').$$

Further, the translation $\mathcal{F}$ is called **reflectively structural** if, for every algebra $\mathbf{A}$ and every G-filter $F : A \to G$ on $\mathbf{A}$, there exists a G-congruence $\Theta : A^2 \to G$ on $\mathbf{A}$, such that, for all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$,



$$\mathcal{F}(\Theta \circ h^2) = F \circ h.$$

If $\mathcal{F}$ is reflectively structural, given a G-filter on an algebra $\mathbf{A}$, there exists a "corresponding" G-congruence Θ on $\mathbf{A}$ satisfying the displayed equation. Exploiting this, given a reflectively structural translation $\mathcal{F}$ and a G-matrix $\mathfrak{A} = \langle \mathbf{A}, F \rangle$, we define the G-algebra

$$\mathfrak{A}^{\mathcal{F}} = \langle \mathbf{A}, F^{\mathcal{F}} \rangle,$$

where $F^{\mathcal{F}}$ is the G-congruence on $\mathbf{A}$, such that, for all $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$,

$$\mathcal{F}(F^{\mathcal{F}} \circ h^2) = F \circ h.$$

Then, given a class M of G-matrices, we define the class $\mathsf{M}^{\mathcal{F}}$ of G-algebras by

$$\mathsf{M}^{\mathcal{F}} = \{\mathfrak{A}^{\mathcal{F}} : \mathfrak{A} \in \mathsf{M}\}.$$

Theorem 25 *Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic, K be a class of G-algebras and $\mathcal{F} : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$ a reflectively structural, order reflecting translation. C_{M} is a G-logical semantics for K via $\mathcal{F}$ if and only if $\mathsf{M}^{\mathcal{F}}$ is a G-2-matrix semantics for $\mathcal{S}_{\mathsf{K}}$.*

Proof: Dual to the proof of Theorem 24. ∎

It can be seen that, as in the classical framework, if the G-logic has a G-algebraic semantics, then $\mathcal{S}$ must exhibit some special characteristics inherited from the G-2-logic $\mathcal{S}_{\mathsf{K}}$ of the G-algebraic semantics K. In the classical framework of Blok and Pigozzi, this property is exploited, e.g., in Theorem 2.7 of [6]. After some deliberation, its statement and proof may be seen to be a special case of the following result.

Theorem 26 *Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic and K be a G-algebraic semantics for $\mathcal{S}$ via the interpretation $\mathcal{E}$. Then, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,*

$$\bigvee\{\Phi : \mathcal{E}(\Phi) \le C_{\mathsf{K}}(\mathcal{E}(\Gamma))\} \le C(\Gamma).$$

Proof: Suppose $\mathcal{S} = \langle \mathcal{L}, C\rangle$ is a G-logic and K a G-algebraic semantics for $\mathcal{S}$ via the interpretation $\mathcal{E}$. Fix $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and let $\Phi : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ be arbitrary, such that $\mathcal{E}(\Phi) \le C_{\mathsf{K}}(\mathcal{E}(\Gamma))$. Since K is a G-algebraic semantics for $\mathcal{S}$ via $\mathcal{E}$, we get $\Phi \le C(\Gamma)$. Hence, taking joins over all such Φ, we get

$$\bigvee\{\Phi : \mathcal{E}(\Phi) \le C_{\mathsf{K}}(\mathcal{E}(\Gamma))\} \le C(\Gamma),$$

which is the inequality in the statement. ∎

2.13 Equivalent Graded Algebraic Semantics

Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic and K a G-algebraic semantics for $\mathcal{S}$, via the interpretation $\mathcal{E}$, that is, we have, for all $\Gamma, \Phi : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\Phi \le C(\Gamma) \quad \text{iff} \quad \mathcal{E}(\Phi) \le C_{\mathsf{K}}(\mathcal{E}(\Gamma)).$$

K is said to be **equivalent to** $\mathcal{S}$ if there exists a translation $\mathcal{F}$ from G-equations to G-fomulas, such that, for all $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$,

$$C_{\mathsf{K}}(\Theta) = C_{\mathsf{K}}(\mathcal{E}(\mathcal{F}(\Theta))).$$

In this case, $\mathcal{F}$ is said to be **inverse** to $\mathcal{E}$.

We have the following characterization of an equivalent G-algebraic semantics, paralleling the one proved in the classical case in Corollary 2.9 of [6]. Among other things, it shows that the roles of the interpretation $\mathcal{E}$ and the translation $\mathcal{F}$, establishing the equivalence, are completely symmetric.

Proposition 27 *Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic and K an equivalent G-algebraic semantics for $\mathcal{S}$, via an interpretation $\mathcal{E}$ and a translation $\mathcal{F}$. Then, for all $\Theta, E : \mathrm{Eq}_{\mathcal{L}}(V) \to G$, and all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,*

(i) $E \leq C_{\mathsf{K}}(\Theta)$ *iff* $\mathcal{F}(E) \leq C(\mathcal{F}(\Theta))$;

(ii) $C(\Gamma) = C(\mathcal{F}(\mathcal{E}(\Gamma))$.

Conversely, if there exist translations $\mathcal{E}$ and $\mathcal{F}$ satisfying Conditions (i) and (ii), then K is equivalent to $\mathcal{S}$ via the interpretations $\mathcal{E}$ and $\mathcal{F}$.

Proof: Suppose, first, that K is equivalent to $\mathcal{S}$ via an interpretation $\mathcal{E}$ and a translation $\mathcal{F}$. Then, for all $\Theta, E : \mathrm{Eq}_{\mathcal{L}}(V) \to G$, we have

$$\begin{array}{rl} E \leq C_{\mathsf{K}}(\Theta) & \text{iff} \quad \mathcal{E}(\mathcal{F}(E)) \leq C_{\mathsf{K}}(\mathcal{E}(\mathcal{F}(\Theta))) \\ & \text{iff} \quad \mathcal{F}(E) \leq C(\mathcal{F}(\Theta)). \end{array}$$

Moreover, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$, we have

$$\begin{array}{rl} C_{\mathsf{K}}(\mathcal{E}(\Gamma)) = C_{\mathsf{K}}(\mathcal{E}(\Gamma)) & \text{iff} \quad C_{\mathsf{K}}(\mathcal{E}(\Gamma)) = C_{\mathsf{K}}(\mathcal{E}(\mathcal{F}(\mathcal{E}(\Gamma)))) \\ & \text{iff} \quad C(\Gamma) = C(\mathcal{F}(\mathcal{E}(\Gamma))). \end{array}$$

Conversely, if Conditions (i) and (ii) hold, then one can prove similarly that, for all $\Gamma, \Phi : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\Phi \in C(\Gamma) \quad \text{iff} \quad \mathcal{E}(\Phi) \leq C_{\mathsf{K}}(\mathcal{E}(\Gamma))$$

and, for all $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$,

$$C_{\mathsf{K}}(\Theta) = C_{\mathsf{K}}(\mathcal{E}(\mathcal{F}(\Theta))).$$

Thus, K is equivalent to $\mathcal{S}$ via $\mathcal{E}$ and $\mathcal{F}$ iff Conditions (i) and (ii) hold. ■

We say that a G-logic $\mathcal{S} = \langle \mathcal{L}, C \rangle$ is **algebraizable** if it has an equivalent G-algebraic semantics.

Lemma 2.13 of [6] shows how one may take advantage of equivalence to translate properties of a class of algebras into properties of the logic. Its abstraction to the graded setting yields a kind of dual result to Theorem 26.

Lemma 28 *Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic and K an equivalent G-algebraic semantics of $\mathcal{S}$ via the interpretations $\mathcal{E}$ and $\mathcal{F}$. Then, for every G-congruence Θ of $\mathcal{S}_{\mathsf{K}}$,*

$$\bigvee\{E : \mathcal{F}(E) \leq C(\mathcal{F}(\Theta))\} = \Theta.$$

Proof: Let Θ be a G-2-theory of $\mathcal{S}_{\mathsf{K}}$. Then, for any $E : \mathrm{Eq}_{\mathcal{L}}(V) \to G$, such that $\mathcal{F}(E) \leq C(\mathcal{F}(\Theta))$, we have $E \leq C_K(\Theta) = \Theta$. Therefore,

$$\bigvee\{E : \mathcal{F}(E) \leq C(\mathcal{F}(\Theta))\} \leq \Theta.$$

On the other hand, observe that Θ is a member of the set on the left, since $\mathcal{F}(\Theta) \leq C(\mathcal{F}(\Theta)$. Thus, the displayed equality holds. ∎

We now specialize this result to get an analog of Lemma 2.13 of [6].

Corollary 29 *Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic and K an equivalent G-algebraic semantics via interpretations $\mathcal{E}$ and $\mathcal{F}$. Then, for every G-congruence Θ on $\mathbf{Fm}_{\mathcal{L}}(V)$, all $\varphi, \psi, \chi \in \mathrm{Fm}_{\mathcal{L}}(V)$ and all $\xi(x) \in \mathrm{Fm}_{\mathcal{L}}(V)$:*

(a) $\mathcal{F}(\Theta^{\varphi,\varphi}) \leq C(\bot)$;

(b) $\mathcal{F}(\Theta^{\psi,\varphi}) \leq C(\mathcal{F}(\Theta^{\varphi,\psi}))$;

(c) $\mathcal{F}(\Theta^{\varphi,\chi}) \leq C(\mathcal{F}(\Theta^{\varphi,\psi}) \vee \mathcal{F}(\Theta^{\psi,\chi}))$;

(d) $\mathcal{F}(\Theta^{\xi(\varphi),\xi(\psi)}) \leq C(\mathcal{F}(\Theta^{\varphi,\psi}))$.

Proof: Let $\mathsf{A}_{\mathcal{L}}$ be the class of all G-algebras of type $\mathcal{L}$ and consider $\mathcal{S}_{\mathsf{A}_{\mathcal{L}}} = \langle \mathcal{L}, C_{\mathsf{A}_{\mathcal{L}}} \rangle$. Then, it can be shown that, for all $\varphi, \psi, \chi \in \mathrm{Fm}_{\mathcal{L}}(V)$ and all $\xi(x) \in \mathrm{Fm}_{\mathcal{L}}(V)$:

(a) $\Theta^{\varphi,\varphi} \leq C_{\mathsf{A}_{\mathcal{L}}}(\bot)$;

(b) $\Theta^{\psi,\varphi} \leq C_{\mathsf{A}_{\mathcal{L}}}(\Theta^{\varphi,\psi})$;

(c) $\Theta^{\varphi,\chi} \leq C_{\mathsf{A}_{\mathcal{L}}}(\Theta^{\varphi,\psi} \vee \Theta^{\psi,\chi})$;

(d) $\Theta^{\xi(\varphi),\xi(\psi)} \leq C_{\mathsf{A}_{\mathcal{L}}}(\Theta^{\varphi,\psi})$.

Taking, first, into account that $C_{\mathsf{A}_{\mathcal{L}}} \leq C_{\mathsf{K}}$ and then the fact that $\mathcal{F}$ is a G-interpretation, we obtain the conclusions in the statement. ∎

For the next two lemmas, we must specialize to specific types of interpretations that mimic the ones employed in the deductive system framework by Blok and Pigozzi [6].

Let us call an interpretation $\mathcal{E} : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$ **standard** if, there exists a set

$$\delta(x) \approx \varepsilon(x) = \{\delta_i(x) \approx \varepsilon_i(x) : i \in I\}$$

of equations in a single variable x, such that, for all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$ and all $g \in G$,

$$\mathcal{E}(\widehat{\langle\varphi, g\rangle}) = \widehat{\langle\delta(\varphi) \approx \varepsilon(\varphi), g\rangle},$$

where

$$\widehat{\langle\delta(\varphi) \approx \varepsilon(\varphi), g\rangle} := \widehat{\{\langle\langle\delta_i(\varphi), \varepsilon_i(\varphi)\rangle, g\rangle : i \in I\}}.$$

Similarly, a G-interpretation $\mathcal{F} : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$ is called **standard** if, there exists a set

$$\Delta(x, y) = \{\Delta_j(x, y) : j \in J\}$$

of formulas in two variables x and y, such that, for all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$ and all $g \in G$,

$$\mathcal{F}(\widehat{\langle\varphi \approx \psi, g\rangle}) = \widehat{\langle\Delta(\varphi, \psi), g\rangle},$$

where

$$\widehat{\langle\Delta(\varphi, \psi), g\rangle} := \widehat{\{\langle\Delta_j(\varphi, \psi), g\rangle : j \in J\}}.$$

Lemma 30 *Let $\mathcal{S} = \langle\mathcal{L}, C\rangle$ be a G-logic and K an equivalent G-algebraic semantics via standard interpretations $\mathcal{E}$ and $\mathcal{F}$. Then, for all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$ and all $g, g' \in G$,*

$$\widehat{\langle\psi, g \wedge g'\rangle} \leq C(\widehat{\langle\Delta(\varphi, \psi), g\rangle, \langle\varphi, g'\rangle}).$$

Proof: First, note that by properties of equational consequence, we get

$$\widehat{\langle\delta(\psi) \approx \varepsilon(\psi), g \wedge g'\rangle} \leq C_{\mathsf{K}}(\widehat{\langle\varphi \approx \psi, g\rangle, \langle\delta(\varphi) \approx \varepsilon(\varphi), g'\rangle}).$$

Therefore, interpreting through the standard $\mathcal{F}$,

$$\widehat{\langle\Delta(\delta(\psi), \varepsilon(\psi)), g \wedge g'\rangle} \leq C(\widehat{\langle\Delta(\varphi, \psi), g\rangle, \langle\Delta(\delta(\varphi), \varepsilon(\varphi)), g'\rangle}).$$

Hence, by the equivalence of the semantics,

$$\widehat{\langle\psi, g \wedge g'\rangle} \leq C(\widehat{\langle\Delta(\varphi, \psi), g\rangle, \langle\varphi, g'\rangle}).$$

This proves the displayed equation of the statement. ■

Lemma 31 *Let $\mathcal{S} = \langle\mathcal{L}, C\rangle$ be a G-logic. Suppose K is an equivalent G-algebraic semantics for $\mathcal{S}$ via standard interpretations $\mathcal{E}$ and $\mathcal{F}$, and K' is an equivalent G-algebraic semantics for $\mathcal{S}$ via standard interpretations $\mathcal{E}'$ and $\mathcal{F}'$. Then, all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$ and all $g \in G$,*

$$C(\mathcal{F}(\widehat{\langle\varphi \approx \psi, g\rangle})) = C(\mathcal{F}'(\widehat{\langle\varphi \approx \psi, g\rangle})).$$

Proof: Using Part (d) of Corollary 29, we have

$$\widehat{\langle\Delta'(\varphi,\varphi)\approx\Delta'(\varphi,\psi),g\rangle} \leq C_{\mathsf{K}}(\widehat{\langle\varphi\approx\psi,g\rangle}).$$

By applying the standard interpretation $\mathcal{F}$, we get

$$\widehat{\langle\Delta(\Delta'(\varphi,\varphi),\Delta'(\varphi,\psi)),g\rangle} \leq C(\widehat{\langle\Delta(\varphi,\psi),g\rangle}).$$

By Lemma 30,

$$\widehat{\langle\Delta'(\varphi,\psi),g\rangle} \leq C(\widehat{\langle\Delta(\Delta'(\varphi,\varphi),\Delta'(\varphi,\psi)),g\rangle,\langle\Delta'(\varphi,\varphi),\top\rangle}).$$

By Part (a) of Corollary 29,

$$\widehat{\langle\Delta'(\varphi,\varphi),\top\rangle} \leq C(\bot).$$

Putting all these pieces together, we get

$$\begin{array}{rcl}\widehat{\langle\langle\Delta'(\varphi,\psi),g\rangle\rangle} & \leq & C(\widehat{\langle\Delta(\Delta'(\varphi,\varphi),\Delta'(\varphi,\psi)),g\rangle,\langle\Delta'(\varphi,\varphi),\top\rangle}) \\ & \leq & C(\widehat{\langle\Delta(\Delta'(\varphi,\varphi),\Delta'(\varphi,\psi)),g\rangle}) \\ & \leq & C(\widehat{\langle\Delta(\varphi,\psi),g\rangle}).\end{array}$$

Translating in the language of interpretations, we get that

$$\mathcal{F}'(\widehat{\langle\varphi\approx\psi,g\rangle}) \leq C(\mathcal{F}(\widehat{\langle\varphi\approx\psi,g\rangle}).$$

By symmetry, we obtain the required equality. ∎

It turns out, that, for any G-logic $\mathcal{S}$ for which any two G-equivalent algebraic semantics K via interpretations $\mathcal{E}$, $\mathcal{F}$ and K' via interpretations $\mathcal{E}'$, $\mathcal{F}'$ satisfy, for all $\Theta:\mathrm{Eq}_{\mathcal{L}}(V)\to G$,

$$C(\mathcal{F}(\Theta)) = C(\mathcal{F}'(\Theta)),$$

can be algebraized in an essentially unique way. It is conjectured that unique algebraization without proviso, as proved by Blok and Pigozzi in Theorem 2.15 of [6] for ordinary deductive systems, does not extend to arbitrary algebraizable G-logics.

Proposition 32 *Let $\mathcal{S}=\langle\mathcal{L},C\rangle$ be a G-logic. Suppose K is an equivalent G-algebraic semantics of $\mathcal{S}$, via interpretations $\mathcal{E}$ and $\mathcal{F}$, and K' is an equivalent G-algebraic semantics of $\mathcal{S}$, via interpretations $\mathcal{E}'$ and $\mathcal{F}'$, such that, for all $\Theta:\mathrm{Eq}_{\mathcal{L}}(V)\to G$, and all $\varphi,\psi\in\mathrm{Fm}_{\mathcal{L}}(V)$,*

$$C(\mathcal{F}(\Theta^{\varphi,\psi})) = C(\mathcal{F}'(\Theta^{\varphi,\psi})).$$

Then the following hold:

(i) $C(\mathcal{F}(\Theta)) = C(\mathcal{F}'(\Theta))$, *for all* $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$;

(ii) $\mathcal{S}_{\mathsf{K}} = \mathcal{S}_{\mathsf{K}'}$;

(iii) $C_{\mathsf{K}}(\mathcal{E}(\Gamma)) = C_{\mathsf{K}}(\mathcal{E}'(\Gamma))$, *for all* $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$.

Proof: First, let $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$. Then we obtain

$$\begin{array}{rcll} C(\mathcal{F}(\Theta)) & = & C(\mathcal{F}(\bigvee_{\varphi,\psi} \Theta^{\varphi,\psi})) & (\Theta = \bigvee_{\varphi,\psi} \Theta^{\varphi,\psi}) \\ & = & C(\bigvee_{\varphi,\psi}(\mathcal{F}(\Theta^{\varphi,\psi}))) & (\mathcal{F} \text{ join continuous}) \\ & = & \bigvee^{\mathbf{Th}(\mathcal{S})}_{\varphi,\psi} C(\mathcal{F}(\Theta^{\varphi,\psi})) & (\text{Property of theories}) \\ & = & \bigvee^{\mathbf{Th}(\mathcal{S})}_{\varphi,\psi} C(\mathcal{F}'(\Theta^{\varphi,\psi})) & (\text{Hypothesis}) \\ & = & C(\bigvee_{\varphi,\psi}(\mathcal{F}'(\Theta^{\varphi,\psi}))) & (\text{Property of theories}) \\ & = & C(\mathcal{F}'(\bigvee_{\varphi,\psi} \Theta^{\varphi,\psi})) & (\mathcal{F}' \text{ join continuous}) \\ & = & C(\mathcal{F}'(\Theta)). & (\Theta = \bigvee_{\varphi,\psi} \Theta^{\varphi,\psi}) \end{array}$$

Now, let $\Theta, E : \mathrm{Eq}_{\mathcal{L}}(V) \to G$. Then, we have

$$\begin{array}{rll} E \leq C_{\mathsf{K}}(\Theta) & \text{iff} & \mathcal{F}(E) \leq C(\mathcal{F}(\Theta)) \quad (\mathcal{F} \text{ an interpretation}) \\ & \text{iff} & \mathcal{F}'(E) \leq C(\mathcal{F}'(\Theta)) \quad (\text{Part (i)}) \\ & \text{iff} & E \leq C_{\mathsf{K}'}(\Theta). \quad (\mathcal{F}' \text{ an interpretation}) \end{array}$$

Next, suppose $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$. Then

$$\begin{array}{rll} C(\Gamma) = C(\Gamma) & \text{iff} & C(\mathcal{F}(\mathcal{E}(\Gamma))) = C(\mathcal{F}'(\mathcal{E}'(\Gamma))) \\ & & (\mathcal{E}, \mathcal{F} \text{ and } \mathcal{E}', \mathcal{F}' \text{ inverse interpretations}) \\ & \text{iff} & C(\mathcal{F}(\mathcal{E}(\Gamma))) = C(\mathcal{F}(\mathcal{E}'(\Gamma))) \quad (\text{Part (i)}) \\ & \text{iff} & C_{\mathsf{K}}(\mathcal{E}(\Gamma)) = C_{\mathsf{K}}(\mathcal{E}'(\Gamma)). \quad (\mathcal{F} \text{ an interpretation}) \end{array}$$

■

Lemma 31 and Proposition 32 ensure that, if a *G*-logic is algebraizable solely via standard interpretations, then it is algebraizable in an essentially unique way.

Theorem 33 (Special Uniqueness) *Let* $\mathcal{S} = \langle \mathcal{L}, C \rangle$ *be a* G*-logic,* K *an equivalent* G*-algebraic semantics for* $\mathcal{S}$ *via standard interpretations* $\mathcal{E}, \mathcal{F}$ *and* K' *an equivalent* G*-algebraic semantics for* $\mathcal{S}$ *via standard interpretations* $\mathcal{E}', \mathcal{F}'$*. Then the following hold:*

(i) $C(\mathcal{F}(\Theta)) = C(\mathcal{F}'(\Theta))$, *for all* $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$;

(ii) $\mathcal{S}_{\mathsf{K}} = \mathcal{S}_{\mathsf{K}'}$;

(iii) $C_{\mathsf{K}}(\mathcal{E}(\Gamma)) = C_{\mathsf{K}}(\mathcal{E}'(\Gamma))$, *for all* $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$.

Proof: Statement (i) is given by Lemma 31. Then, Statements (ii) and (iii) are given by Proposition 32. ■

2.14 The Lattice of Theories

Recall that a given class K of G-algebras induces a G-2-logic $\mathcal{S}_{\mathsf{K}}$. Finitarity is defined by analogy with G-logics, that is, $\mathcal{S}_{\mathsf{K}}$ is **finitary** if, for every $\Theta : \mathrm{Eq}_{\mathcal{L}}(V)$,

$$C_{\mathsf{K}}(\Theta) = \bigvee_{Z \leq_f \Theta} C_{\mathsf{K}}(Z).$$

We also have the following analogs of Lemmas 3 and 4 for the case of a G-2-logic $\mathcal{S}_{\mathsf{K}}$ induced by a class K of G-algebras.

Lemma 34 *Let K be a class of G-algebras. Then, for all $\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$ and all $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$,*

$$\Theta \circ \sigma \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}}).$$

Proof: Using Structurality of $\mathcal{S}_{\mathsf{K}}$ (Proposition 19) and the fact that $\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$, we get

$$C_{\mathsf{K}}(\Theta \circ \sigma) \leq C_{\mathsf{K}}(\Theta) \circ \sigma = \Theta \circ \sigma.$$

Since the reverse inclusion always holds, $C_{\mathsf{K}}(\Theta \circ \sigma) = \Theta \circ \sigma$. Therefore, $\Theta \circ \sigma$ is a theory of $\mathcal{S}_{\mathsf{K}}$. ■

Lemma 35 *Let K be a class of G-algebras. Then, for all $\{\Theta_i : i \in I\} \subseteq \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$ and all $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$,*

$$\bigvee_{i \in I} (\Theta_i \circ \sigma) = \left(\bigvee_{i \in I} \Theta_i \right) \circ \sigma.$$

Proof: This follows directly from the definitions involved. We have, for all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\begin{aligned}
&(\bigvee_{i \in I} (\Theta_i \circ \sigma))(\varphi, \psi) \\
&\quad = \bigvee_{i \in I} ((\Theta_i \circ \sigma)(\varphi, \psi)) \quad \text{(Definition of } \bigvee_{i \in I} (\Theta_i \circ \sigma)) \\
&\quad = \bigvee_{i \in I} \Theta_i(\sigma(\varphi), \sigma(\psi)) \quad \text{(Definition of } \circ) \\
&\quad = (\bigvee_{i \in I} \Theta_i)(\sigma(\varphi), \sigma(\psi)) \quad \text{(Definition of } \bigvee_{i \in I} \Theta_i) \\
&\quad = ((\bigvee_{i \in I} \Theta_i) \circ \sigma)(\varphi, \psi). \quad \text{(Definition of } \circ)
\end{aligned}$$

Therefore, $\bigvee_{i \in I} (\Theta_i \circ \sigma) = (\bigvee_{i \in I} \Theta_i) \circ \sigma$. ■

Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic and K be a G-algebraic semantics for $\mathcal{S}$ via an interpretation $\mathcal{E}$. We define two functions

$$\begin{array}{rccc}
H_{\mathsf{K}} : & \mathrm{Th}(\mathsf{S}_{\mathsf{K}}) & \to & \mathrm{Th}(\mathcal{S}), \\
\Omega_{\mathsf{K}} : & \mathrm{Th}(\mathcal{S}) & \to & \mathrm{Th}(\mathcal{S}_{\mathsf{K}}).
\end{array}$$

First, regarding H_{K}, we define, for all $\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$,

$$H_{\mathsf{K}}(\Theta) = \bigvee \{\Gamma \in G^{\mathrm{Fm}_{\mathcal{L}}(V)} : \mathcal{E}(\Gamma) \leq \Theta\}.$$

We show that $H_{\mathsf{K}}(\Theta) \in \mathrm{Th}(\mathcal{S})$. Indeed, we have

$$\begin{array}{rcl} C(H_{\mathsf{K}}(\Theta)) & = & \bigvee\{\Gamma : \Gamma \le C(H_{\mathsf{K}}(\Theta))\} \quad \text{(Definition of join)} \\ & = & \bigvee\{\Gamma : \mathcal{E}(\Gamma) \le C_{\mathsf{K}}(\mathcal{E}(H_{\mathsf{K}}(\Theta)))\} \quad (\mathcal{E} \text{ an interpretation}) \\ & \le & \bigvee\{\Gamma : \mathcal{E}(\Gamma) \le C_{\mathsf{K}}(\Theta)\} \quad (\mathcal{E}(H_{\mathsf{K}}(\Theta)) \le \Theta) \\ & = & \bigvee\{\Gamma : \mathcal{E}(\Gamma) \le \Theta\} \quad (\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})) \\ & = & H_{\mathsf{K}}(\Theta). \quad (\text{Definition of } H_{\mathsf{K}}(\Theta)) \end{array}$$

Let us also show that, equivalently, $H_{\mathsf{K}}(\Theta)$ may be defined by

$$H_{\mathsf{K}}(\Theta) = \bigvee^{\mathbf{Th}(\mathcal{S})}\{T \in \mathrm{Th}(\mathcal{S}) : \mathcal{E}(T) \le \Theta\}.$$

To see this, first note that, for all $\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$ and all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\begin{array}{rl} \mathcal{E}(\Gamma) \le \Theta & \text{iff} \quad C_{\mathsf{K}}(\mathcal{E}(\Gamma)) \le \Theta \quad (\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})) \\ & \text{iff} \quad C_{\mathsf{K}}(\mathcal{E}(C(\Gamma))) \le \Theta \quad \text{(Proposition 22)} \\ & \text{iff} \quad \mathcal{E}(C(\Gamma)) \le \Theta. \quad (\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})) \end{array}$$

Therefore, we have

$$\begin{array}{rcl} H_{\mathsf{K}}(\Theta) & = & \bigvee\{\Gamma : \mathcal{E}(\Gamma) \le \Theta\} \quad (\text{Definition of } H_{\mathsf{K}}(\Theta)) \\ & \le & \bigvee\{C(\Gamma) : \mathcal{E}(C(\Gamma)) \le \Theta\} \quad \text{(Inflationarity)} \\ & \le & C(\bigvee\{C(\Gamma) : \mathcal{E}(C(\Gamma)) \le \Theta\}) \quad \text{(Inflationarity)} \\ & \le & C(\bigvee\{\Gamma : \mathcal{E}(\Gamma) \le \Theta\}) \quad \text{(Monotonicity)} \\ & = & H_{\mathsf{K}}(\Theta). \quad (\text{Definition of } H_{\mathsf{K}}(\Theta) \in \mathrm{Th}(\mathcal{S})) \end{array}$$

Thus,

$$H_{\mathsf{K}}(\Theta) = C(\bigvee\{C(\Gamma) : \mathcal{E}(C(\Gamma)) \le \Theta\}) = \bigvee^{\mathbf{Th}(\mathcal{S})}\{T : \mathcal{E}(T) \le \Theta\}.$$

Next, regarding Ω_{K}, define, for all $T \in \mathrm{Th}(\mathcal{S})$,

$$\Omega_{\mathsf{K}}(T) = C_{\mathsf{K}}(\mathcal{E}(T)).$$

We have shown that H_{K} is well defined and it is clear that Ω_{K} is also well defined, that is, they map theories into theories. Moreover, both H_{K} and Ω_{K} depend on $\mathcal{E}$. On the other hand, Theorem 33 partially (but, unfortunately, slightly) alleviates the pain, as it shows that for algebraizability via standard interpretations, all possible standard interpretations result in the same operators H_{K} and Ω_{K}.

Note that, (almost) by definition, each of H_{K} and Ω_{K} is order preserving. We have, for all $\Theta, \Theta' \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$, such that $\Theta \le \Theta'$, and all $T, T' \in \mathrm{Th}(\mathcal{S})$, such that $T \le T'$,

$$\begin{array}{rcl} H_{\mathsf{K}}(\Theta) & = & \bigvee\{\Gamma : \mathcal{E}(\Gamma) \le \Theta\} \quad (\text{Definition of } H_{\mathsf{K}}) \\ & \le & \bigvee\{\Gamma : \mathcal{E}(\Gamma) \le \Theta'\} \quad (\Theta \le \Theta') \\ & = & H_{\mathsf{K}}(\Theta') \quad (\text{Definition of } H_{\mathsf{K}}) \end{array}$$

and, further,

$$\begin{array}{lll}\Omega_{\mathsf{K}}(T) & = & C_{\mathsf{K}}(\mathcal{E}(T)) \quad \text{(Definition of } \Omega_{\mathsf{K}}) \\ & \leq & C_{\mathsf{K}}(\mathcal{E}(T')) \quad (T \leq T', \text{ Join Continuity of } \mathcal{E}, \\ & & \qquad \text{Monotonicity of } C_{\mathsf{K}}) \\ & = & \Omega_{\mathsf{K}}(T'). \quad \text{(Definition of } \Omega_{\mathsf{K}})\end{array}$$

The next lemma gives an alternative way for computing Ω_{K} based on slicing and, in addition, proves a continuity property regarding Ω_{K}.

Lemma 36 *Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic and K a G-algebraic semantics for $\mathcal{S}$ via an interpretation $\mathcal{E}$.*

(a) For every $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\Omega_{\mathsf{K}}(C(\Gamma)) = C_{\mathsf{K}}\left(\bigvee_{\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)} \mathcal{E}(\Gamma^\varphi)\right).$$

(b) Ω_{K} is a join continuous map from $\mathbf{Th}(\mathcal{S})$ into $\mathbf{Th}(\mathcal{S}_{\mathsf{K}})$, i.e., for all $\{T_i : i \in I\} \subseteq \mathrm{Th}(\mathcal{S})$,

$$\Omega_{\mathsf{K}}\left(\bigvee_{i\in I}^{\mathbf{Th}(\mathcal{S})} T_i\right) = \bigvee_{i\in I}^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} \Omega_{\mathsf{K}}(T_i).$$

Proof:

(a) Let $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$. Then we have

$$\begin{array}{lll}\Omega_{\mathsf{K}}(C(\Gamma)) & = & \Omega_{\mathsf{K}}(C(\bigvee_\varphi \Gamma^\varphi)) \quad (\Gamma = \bigvee_\varphi \Gamma^\varphi) \\ & = & C_{\mathsf{K}}(\mathcal{E}(C(\bigvee_\varphi \Gamma^\varphi))) \quad \text{(Definition of } \Omega_{\mathsf{K}}) \\ & = & C_{\mathsf{K}}(\mathcal{E}(\bigvee_\varphi \Gamma^\varphi)) \quad \text{(Proposition 22)} \\ & = & C_{\mathsf{K}}(\bigvee_\varphi \mathcal{E}(\Gamma^\varphi)). \quad (\mathcal{E} \text{ join continuous})\end{array}$$

(b) By the definition of $\bigvee^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})}$, for all $\Gamma_i : \mathrm{Fm}_{\mathcal{L}}(V) \to G$, $i \in I$,

$$C_{\mathsf{K}}\left(\bigvee_i \mathcal{E}(\Gamma_i)\right) = \bigvee_i^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} C_{\mathsf{K}}(\mathcal{E}(\Gamma_i)).$$

The inequality $\leq$ is from the fact that $\bigvee_i \mathcal{E}(\Gamma_i) \leq \bigvee_i^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} C_{\mathsf{K}}(\mathcal{E}(\Gamma_i))$. The reverse inequality follows from $C_{\mathsf{K}}(\mathcal{E}(\Gamma_i)) \leq C_{\mathsf{K}}(\bigvee_i \mathcal{E}(\Gamma_i))$, for all i. Now we have

$$\begin{array}{lll}\Omega_{\mathsf{K}}\left(\bigvee_{i\in I}^{\mathbf{Th}(\mathcal{S})} T_i\right) & = & C_{\mathsf{K}}\left(\mathcal{E}\left(\bigvee_{i\in I}^{\mathbf{Th}(\mathcal{S})} T_i\right)\right) \quad \text{(Definition of } \Omega_{\mathsf{K}}) \\ & = & C_{\mathsf{K}}\left(\mathcal{E}\left(C\left(\bigvee_{i\in I} T_i\right)\right)\right) \quad \text{(Definition of } \bigvee^{\mathbf{Th}(\mathcal{S})}) \\ & = & C_{\mathsf{K}}\left(\mathcal{E}\left(\bigvee_{i\in I} T_i\right)\right) \quad \text{(Proposition 22)} \\ & = & C_{\mathsf{K}}\left(\bigvee_i \mathcal{E}(T_i)\right) \quad (\mathcal{E} \text{ join continuous}) \\ & = & \bigvee_i^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} C_{\mathsf{K}}(\mathcal{E}(T_i)) \quad \text{(Displayed Formula)} \\ & = & \bigvee_i^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} \Omega_{\mathsf{K}}(T_i). \quad \text{(Definition of } \Omega_{\mathsf{K}})\end{array}$$

■

We now prove an analog of Lemma 3.4 of [6], which begins the quest for a characterization of G-algebraic semantics and of equivalent G-algebraic semantics in terms of correspondences between lattices of theories.

Lemma 37 *Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic and K be a G-algebraic semantics for $\mathcal{S}$ via an interpretation $\mathcal{E}$.*

(a) $H_{\mathsf{K}}(\Omega_{\mathsf{K}}(T)) = T$, for every $T \in \mathrm{Th}(\mathcal{S})$;

(b) $\Omega_{\mathsf{K}}(H_{\mathsf{K}}(\Theta)) \leq \Theta$, for all $\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$, and $\Omega_{\mathsf{K}}(H_{\mathsf{K}}(\Theta)) = \Theta$ just in case $\Theta \in \Omega_{\mathsf{K}}(\mathrm{Th}(\mathcal{S}))$, i.e., Θ is the image of some theory of $\mathcal{S}$ under Ω_{K}.

Proof: Let $T \in \mathrm{Th}(\mathcal{S})$. Then

$$\begin{array}{rcl} H_{\mathsf{K}}(\Omega_{\mathsf{K}}(T)) & = & \bigvee\{\Gamma : \mathcal{E}(\Gamma) \leq \Omega_{\mathsf{K}}(T)\} \quad \text{(Definition of } H_{\mathsf{K}}) \\ & = & \bigvee\{\Gamma : \mathcal{E}(\Gamma) \leq C_{\mathsf{K}}(\mathcal{E}(T))\} \quad \text{(Definition of } \Omega_{\mathsf{K}}) \\ & = & \bigvee\{\Gamma : \Gamma \leq C(T)\} \quad (\mathcal{E} \text{ an interpretation}) \\ & = & C(T) \quad \text{(Property of join)} \\ & = & T. \quad (T \in \mathrm{Th}(\mathcal{S})) \end{array}$$

Next, suppose $\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$.

$$\begin{array}{rcl} \Omega_{\mathsf{K}}(H_{\mathsf{K}}(\Theta)) & = & \Omega_{\mathsf{K}}(\bigvee^{\mathbf{Th}(\mathcal{S})}\{T : \mathcal{E}(T) \leq \Theta\}) \quad \text{(Definition of } H_{\mathsf{K}}) \\ & = & \bigvee^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})}\{\Omega_{\mathsf{K}}(T) : \mathcal{E}(T) \leq \Theta\} \quad \text{(Lemma 36)} \\ & = & \bigvee^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})}\{C_{\mathsf{K}}(\mathcal{E}(T)) : \mathcal{E}(T) \leq \Theta\} \quad \text{(Definition of } \Omega_{\mathsf{K}}) \\ & \leq & C_{\mathsf{K}}(\Theta) \quad \text{(Definition of } \bigvee^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} \\ & & \qquad \text{and Monotonicity of } C_{\mathsf{K}}) \\ & = & \Theta. \quad (\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})) \end{array}$$

Finally, assume $\Theta = \Omega_{\mathsf{K}}(T)$, for some $T \in \mathrm{Th}(\mathcal{S})$. Then, we get

$$\Omega_{\mathsf{K}}(H_{\mathsf{K}}(\Theta)) = \Omega_{\mathsf{K}}(H_{\mathsf{K}}(\Omega_{\mathsf{K}}(T))) = \Omega_{\mathsf{K}}(T) = \Theta,$$

the second equality holding by Part (a). ■

By Lemma 37, Ω_{K} is a bijection from $\mathrm{Th}(\mathcal{S})$ to $\Omega_{\mathsf{K}}(\mathrm{Th}(\mathcal{S})) \subseteq \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$. Since Ω_{K} is order preserving, $\Omega_{\mathsf{K}}(\mathrm{Th}(\mathcal{S}))$ forms a complete lattice under the order relation inherited by $\mathbf{Th}(\mathcal{S}_{\mathsf{K}})$. The corresponding lattice is denoted by $\Omega_{\mathsf{K}}(\mathbf{Th}(\mathcal{S}))$. In general, $\Omega_{\mathsf{K}}(\mathbf{Th}(\mathcal{S}))$ may not be a sublattice of $\mathbf{Th}(\mathcal{S}_{\mathsf{K}})$, since $\Omega_{\mathsf{K}}(\mathrm{Th}(\mathcal{S}))$ may fail to be closed under intersections. On the other hand we show that the join operations in the two lattices coincide.

To formulate Part (b) of the following result, concerning equivalence of the G-algebraic semantics, we introduce the notion of an *invertible interpretation* $\mathcal{E}$. We say that an interpretation $\mathcal{E} : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$ is **invertible** if there exists a translation $\mathcal{F} : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$, such that, for all $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$ and all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$H_{\mathsf{K}}(C_{\mathsf{K}}(\Theta^{\varphi,\psi})) = C(\mathcal{F}(\Theta^{\varphi,\psi})).$$

We say that the interpretation $\mathcal{E}$ is **invertible via $\mathcal{F}$** or **$\mathcal{F}$-invertible**.

Lemma 38 *Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic and K be a G-algebraic semantics for $\mathcal{S}$ via an interpretation $\mathcal{E}$.*

(a) Ω_{K} maps $\mathbf{Th}(\mathcal{S})$ isomorphically onto a join-complete subsemilattice of $\mathbf{Th}(\mathcal{S}_{\mathsf{K}})$.

(b) K is equivalent to $\mathcal{S}$ via the interpretation $\mathcal{E}$ if and only if $\Omega_{\mathsf{K}} : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$ is an isomorphism and $\mathcal{E}$ is invertible.

Proof:

(a) By Lemma 37, it suffices to show that, for all $\{\Theta_i : i \in I\} \subseteq \Omega_{\mathsf{K}}(\mathrm{Th}(\mathcal{S}))$,

$$\bigvee\nolimits_{i\in I}^{\Omega_{\mathsf{K}}(\mathbf{Th}(\mathcal{S}))} \Theta_i = \bigvee\nolimits_{i\in I}^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} \Theta_i.$$

For every $i \in I$, there exists $T_i \in \mathrm{Th}(\mathcal{S})$, such that $\Omega_{\mathsf{K}}(T_i) = \Theta_i$. Thus, we have

$$\begin{aligned}
\bigvee\nolimits_{i\in I}^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} \Theta_i &= \bigvee\nolimits_{i\in I}^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} \Omega_{\mathsf{K}}(T_i)\\
&= \Omega_{\mathsf{K}}\left(\bigvee\nolimits_{i\in I}^{\mathbf{Th}(\mathcal{S})} T_i\right) \quad \text{(Lemma 36)}\\
&= \bigvee\nolimits_{i\in I}^{\Omega_{\mathsf{K}}(\mathbf{Th}(\mathcal{S}))} \Omega_{\mathsf{K}}(T_i) \quad \text{(Lemma 36 and definition of } \Omega_{\mathsf{K}}(\mathbf{Th}(\mathcal{S})))\\
&= \bigvee\nolimits_{i\in I}^{\Omega_{\mathsf{K}}(\mathbf{Th}(\mathcal{S}))} \Theta_i.
\end{aligned}$$

(b) Suppose, first, that K is equivalent to $\mathcal{S}$ via interpretations $\mathcal{E}$ and $\mathcal{F}$. First, observe that, for all $\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$,

$$\begin{aligned}
H_{\mathsf{K}}(\Theta) &= \bigvee\{\Gamma : \mathcal{E}(\Gamma) \leq \Theta\} \quad \text{(Definition of } H_{\mathsf{K}})\\
&= \bigvee\{\Gamma : \mathcal{E}(\Gamma) \leq C_{\mathsf{K}}(\mathcal{E}(\mathcal{F}(\Theta)))\} \quad \text{(Equivalence)}\\
&= \bigvee\{\Gamma : \Gamma \leq C(\mathcal{F}(\Theta))\} \quad (\mathcal{E} \text{ an interpretation})\\
&= C(\mathcal{F}(\Theta)). \quad \text{(Property of join)}
\end{aligned}$$

Now we obtain, for all $\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$,

$$\begin{aligned}
\Omega_{\mathsf{K}}(H_{\mathsf{K}}(\Theta)) &= C_{\mathsf{K}}(\mathcal{E}(H_{\mathsf{K}}(\Theta))) \quad \text{(Definition of } \Omega_{\mathsf{K}})\\
&= C_{\mathsf{K}}(\mathcal{E}(C(\mathcal{F}(\Theta)))) \quad \text{(Preceding deduction)}\\
&= C_{\mathsf{K}}(\mathcal{E}(\mathcal{F}(\Theta))) \quad \text{(Proposition 22)}\\
&= C_{\mathsf{K}}(\Theta) \quad \text{(Equivalence)}\\
&= \Theta. \quad (\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}}))
\end{aligned}$$

This shows that $\Omega_{\mathsf{K}} : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$ is an isomorphism. To finish the "only if", we must also show that $\mathcal{E}$ is invertible. In fact, its inverse is $\mathcal{F}$, since, as was shown above, for all $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$,

$$H_{\mathsf{K}}(C_{\mathsf{K}}(\Theta)) = C(\mathcal{F}(C_{\mathsf{K}}(\Theta))) = C(\mathcal{F}(\Theta))).$$

Suppose, conversely, that $\Omega_{\mathsf{K}} : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$ is an isomorphism and that the interpretation $\mathcal{E} : \mathcal{S} \to \mathcal{S}_{\mathsf{K}}$ is invertible via $\mathcal{F}$. By Lemma 37, $\Omega_{\mathsf{K}} : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$ is an isomorphism with inverse H_{K}. To show that K is equivalent to $\mathcal{S}$ via $\mathcal{E}$ and $\mathcal{F}$, let $\Theta : \mathrm{Fm}_{\mathcal{L}}(V) \to G$. We have

$$\begin{array}{rcll} C_{\mathsf{K}}(\mathcal{E}(\mathcal{F}(\Theta))) &=& C_{\mathsf{K}}(\mathcal{E}(C(\mathcal{F}(\Theta)))) & \text{(Proposition 22)} \\ &=& \Omega_{\mathsf{K}}(C(\mathcal{F}(\Theta))) & \text{(Definition of } \Omega_{\mathsf{K}}) \\ &=& \Omega_{\mathsf{K}}(C(\mathcal{F}(\bigvee_{\varphi,\psi}\Theta^{\varphi,\psi}))) & (\Theta = \bigvee_{\varphi,\psi}\Theta^{\varphi,\psi}) \\ &=& \Omega_{\mathsf{K}}(C(\bigvee_{\varphi,\psi}\mathcal{F}(\Theta^{\varphi,\psi}))) & (\mathcal{F} \text{ join continuous}) \\ &=& \Omega_{\mathsf{K}}(\bigvee_{\varphi,\psi}^{\mathbf{Th}(\mathcal{S})} C(\mathcal{F}(\Theta^{\varphi,\psi}))) & \text{(Definition of } \bigvee^{\mathbf{Th}(\mathcal{S})}) \\ &=& \bigvee_{\varphi,\psi}^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} \Omega_{\mathsf{K}}(C(\mathcal{F}(\Theta^{\varphi,\psi}))) & \text{(Hypothesis)} \\ &=& \bigvee_{\varphi,\psi}^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} \Omega_{\mathsf{K}}(H_{\mathsf{K}}(C_{\mathsf{K}}(\Theta^{\varphi,\psi}))) & (\mathcal{F} \text{ inverse of } \mathcal{E}) \\ &=& \bigvee_{\varphi,\psi}^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} C_{\mathsf{K}}(\Theta^{\varphi,\psi}) & (H_{\mathsf{K}} \text{ inverse of } \Omega_{\mathsf{K}}) \\ &=& C_{\mathsf{K}}(\bigvee_{\varphi,\psi}\Theta^{\varphi,\psi}) & \text{(Definition of } \bigvee^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})}) \\ &=& C_{\mathsf{K}}(\Theta). & (\Theta = \bigvee_{\varphi,\psi}\Theta^{\varphi,\psi}) \end{array}$$

Thus, K is equivalent to $\mathcal{S}$ via the invertible interpretation $\mathcal{E}$.

■

Lemma 38 hints at the requirements that one needs to postulate on an isomorphism between theory lattices so that a (equivalent) G-algebraic semantics be obtained for a given G-logic. Clinically, these properties are chosen so that they are both necessary and sufficient. Thus, no unnecessary restrictions on the framework under consideration are imposed. We define the properties precisely, tie them to the property of invertibility of an interpretation introduced earlier and then formulate one of our main theorems.

Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic and K a class of G-algebras. A join complete embedding $\Xi : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$ is called **regular** if there exists a translation $\mathcal{E} : G^{\mathrm{Fm}_{\mathcal{L}}(V)} \to G^{\mathrm{Eq}_{\mathcal{L}}(V)}$, such that, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$ and all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Xi(C(\Gamma^{\varphi})) = C_{\mathsf{K}}(\mathcal{E}(\Gamma^{\varphi})).$$

In this case, we say that Ξ is **regular via $\mathcal{E}$** or **$\mathcal{E}$-regular**. Let us see that the condition above is equivalent to the statement that, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\Xi(C(\Gamma)) = C_{\mathsf{K}}(\mathcal{E}(\Gamma)).$$

We have

$$\begin{array}{rcll} \Xi(C(\Gamma)) &=& \Xi(C(\bigvee_{\varphi}\Gamma^{\varphi})) & (\Gamma = \bigvee_{\varphi}\Gamma^{\varphi}) \\ &=& \Xi(\bigvee_{\varphi}^{\mathbf{Th}(\mathcal{S})} C(\Gamma^{\varphi})) & \text{(Definition of } \bigvee^{\mathbf{Th}(\mathcal{S})}) \\ &=& \bigvee_{\varphi}^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} \Xi(C(\Gamma^{\varphi})) & (\Xi \text{ join continuous}) \\ &=& \bigvee_{\varphi}^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} C_{\mathsf{K}}(\mathcal{E}(\Gamma^{\varphi})) & (\Xi \text{ regular via } \mathcal{E}) \\ &=& C_{\mathsf{K}}(\bigvee_{\varphi}\mathcal{E}(\Gamma^{\varphi})) & \text{(Definition of } \bigvee^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})}) \\ &=& C_{\mathsf{K}}(\mathcal{E}(\bigvee_{\varphi}\Gamma^{\varphi})) & (\mathcal{E} \text{ join continuous}) \\ &=& C_{\mathsf{K}}(\mathcal{E}(\Gamma)). & (\Gamma = \bigvee_{\varphi}\Gamma^{\varphi}) \end{array}$$

We show, first, that, if Ξ is regular via $\mathcal{E}$, then $\mathcal{E}$ is an interpretation from $\mathcal{S}$ into $\mathcal{S}_{\mathsf{K}}$.

Lemma 39 *Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic, K a class of G-algebras and $\Xi : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$ a join complete embedding. If Ξ is regular via $\mathcal{E}$, then $\mathcal{E} : \mathcal{S} \to \mathcal{S}_{\mathsf{K}}$ is an interpretation.*

Proof: We have, for all $\Gamma, \Phi : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\begin{array}{lll} \Phi \leq C(\Gamma) & \text{iff} & C(\Phi) \leq C(\Gamma) \quad \text{(Property of } C) \\ & \text{iff} & \Xi(C(\Phi)) \leq \Xi(C(\Gamma)) \quad (\Xi \text{ join complete embedding}) \\ & \text{iff} & C_{\mathsf{K}}(\mathcal{E}(\Phi)) \leq C_{\mathsf{K}}(\mathcal{E}(\Gamma)) \quad (\Xi \text{ regular via } \mathcal{E}) \\ & \text{iff} & \mathcal{E}(\Phi) \leq C_{\mathsf{K}}(\mathcal{E}(\Gamma)). \quad \text{(Property of } C_{\mathsf{K}}) \end{array}$$

This shows that $\mathcal{E}$ is an interpretation. ■

Naturally, if $\Xi : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$ is an isomorphism, then we say that $\Xi^{-1} : \mathbf{Th}(\mathcal{S}_{\mathsf{K}}) \to \mathbf{Th}(\mathcal{S})$ is **regular** if there exists a translation $\mathcal{F} : G^{\mathrm{Eq}_{\mathcal{L}}(V)} \to G^{\mathrm{Fm}_{\mathcal{L}}(V)}$, such that, for all $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$ and all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Xi^{-1}(C_{\mathsf{K}}(\Theta^{\varphi,\psi})) = C(\mathcal{F}(\Theta^{\varphi,\psi})).$$

We also use the term **regular via** $\mathcal{F}$ or $\mathcal{F}$-**regular** if this situation obtains. In the same way as above, it can be seen that this is equivalent to declaring that, for all $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$,

$$\Xi^{-1}(C_{\mathsf{K}}(\Theta)) = C(\mathcal{F}(\Theta)).$$

Further, if Ξ^{-1} is regular via $\mathcal{F}$, then, as in Lemma 39, it may be shown that $\mathcal{F} : \mathcal{S}_{\mathsf{K}} \to \mathcal{S}$ is an interpretation.

Lemma 40 *Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic, K a class of G-algebras and $\Xi : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$ an isomorphism. If Ξ^{-1} is regular via $\mathcal{F}$, then $\mathcal{F} : \mathcal{S}_{\mathsf{K}} \to \mathcal{S}$ is an interpretation.*

Proof: Dual to the proof of Lemma 39. ■

Recall that an interpretation $\mathcal{E}$ is said to be *invertible* in case there exists an interpretation $\mathcal{F}$, such that

$$H_{\mathsf{K}}(C_{\mathsf{K}}(\Theta^{\varphi,\psi})) = C(\mathcal{F}(\Theta^{\varphi,\psi})).$$

If $\Xi : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$ is an isomorphism that is regular via $\mathcal{E}$, it turns out that the regularity via $\mathcal{F}$ of Ξ^{-1} is closely related to the invertibility of $\mathcal{E}$.

Lemma 41 *Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic, K a class of G-algebras and $\Xi : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$ an $\mathcal{E}$-regular order isomorphism. $\mathcal{E}$ is invertible via $\mathcal{F}$ if and only if $\Xi^{-1} : \mathbf{Th}(\mathcal{S}_{\mathsf{K}}) \to \mathbf{Th}(\mathcal{S})$ is $\mathcal{F}$-regular.*

Proof: By hypothesis, for all $\Gamma : \mathrm{Fm}_{\mathcal{L}}(V) \to G$,

$$\begin{array}{rcll}\Xi(C(\Gamma)) & = & C_{\mathsf{K}}(\mathcal{E}(\Gamma)) & (\Xi \text{ regular via } \mathcal{E}) \\ & = & C_{\mathsf{K}}(\mathcal{E}(C(\Gamma))) & (\text{Lemma 39} \\ & & & \text{and Proposition 22}) \\ & = & \Omega_{\mathsf{K}}(C(\Gamma)). & (\text{Definition of } \Omega_{\mathsf{K}})\end{array}$$

Thus, since Ξ is invertible, by Lemma 37, $\Xi^{-1} = H_{\mathsf{K}}$. Thus, for all $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$ and all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$H_{\mathsf{K}}(C_{\mathsf{K}}(\Theta^{\varphi,\psi})) = C(\mathcal{F}(\Theta^{\varphi,\psi})) \quad \text{iff} \quad \Xi^{-1}(C_{\mathsf{K}}(\Theta^{\varphi,\psi})) = C(\mathcal{F}(\Theta^{\varphi,\psi})).$$

This proves that $\mathcal{E}$ is invertible via $\mathcal{F}$ iff Ξ^{-1} is regular via F. ■

Now we present the main characterization theorem, which we view, in the present context, as an analog of the celebrated Characterization Theorem 3.7 of [6], which has triggered several generalizations in various directions, e.g., [40],[4] and [31], with [30] being the most definitive among them, encompassing all its predecessors.

Theorem 42 *Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic and K a class of G-algebras.*

(a) K is a G-algebraic semantics for $\mathcal{S}$ if and only if there exists an $\mathcal{E}$-regular isomorphism $\Xi : \mathbf{Th}(\mathcal{S}) \to \Xi(\mathbf{Th}(\mathcal{S}))$, where $\Xi(\mathbf{Th}(\mathcal{S}))$ is a join-complete subsemilattice of $\mathbf{Th}(\mathcal{S}_{\mathsf{K}})$.

(b) K is equivalent to $\mathcal{S}$ if and only if there exists an $\mathcal{E}$-regular isomorphism $\Xi : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$, with $\mathcal{E}$ invertible.

Proof:

(a) Part (a) of Lemma 38 proves necessity. For sufficiency, suppose

$$\Xi : \mathbf{Th}(\mathcal{S}) \to \Xi(\mathbf{Th}(\mathcal{S}))$$

is an $\mathcal{E}$-regular isomorphism of $\mathbf{Th}(\mathcal{S})$ onto a join-complete subsemilattice of $\mathbf{Th}(\mathcal{S}_{\mathsf{K}})$. By Lemma 39, $\mathcal{E} : \mathcal{S} \to \mathcal{S}_{\mathsf{K}}$ is a G-interpretation. Hence, K is a G-algebraic semantics for $\mathcal{S}$.

(b) If K is equivalent to $\mathcal{S}$, then by Part (b) of Lemma 38, there exists an $\mathcal{E}$-regular isomorphism $\Omega_{\mathsf{K}} : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$, with $\mathcal{E}$ invertible.

Assume, conversely, that $\Xi : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$ is an $\mathcal{E}$-regular isomorphism, with $\mathcal{E}$ $\mathcal{F}$-invertible. Then, by Lemma 41, Ξ^{-1} is $\mathcal{F}$-regular. Therefore, for all $\Theta : \mathrm{Eq}_{\mathcal{L}}(V) \to G$,

$$\begin{array}{rcll}C_{\mathsf{K}}(\mathcal{E}(\mathcal{F}(\Theta))) & = & \Xi(C(\mathcal{F}(\Theta)) & (\Xi \text{ regular via } \mathcal{E}) \\ & = & \Xi(\Xi^{-1}(C_{\mathsf{K}}(\Theta))) & (\Xi^{-1} \text{ regular via } \mathcal{F}) \\ & = & C_{\mathsf{K}}(\Theta). & \end{array}$$

Thus, K is an equivalent G-algebraic semantics for $\mathcal{S}$.

■

If $\mathcal{S}$ is algebraizable and K is its equivalent G-algebraic semantics, then Ω_{K} has a simple characterization in terms of the interpretation $\mathcal{F} : \mathcal{S}_{\mathsf{K}} \to \mathcal{S}$. Note that this makes Ω_{K} appear completely analogous to H_{K}, since H_{K} was defined exactly in the dual way in terms of $\mathcal{E}$.

Lemma 43 *Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic and K an equivalent G-algebraic semantics for $\mathcal{S}$ via interpretations $\mathcal{E}$ and $\mathcal{F}$. Then, for every $T \in \mathrm{Th}(\mathcal{S})$,*

$$\Omega_{\mathsf{K}}(T) = \bigvee\{\Theta \in G^{\mathrm{Eq}_{\mathcal{L}}(V)} : \mathcal{F}(\Theta) \leq T\}.$$

Proof: We have, for all $T \in \mathrm{Th}(\mathcal{S})$,

$$\begin{array}{rcl}\Omega_{\mathsf{K}}(T) & = & C_{\mathsf{K}}(\mathcal{E}(T)) \quad \text{(Definition of } \Omega_{\mathsf{K}})\\ & = & \bigvee\{\Theta : \Theta \leq C_{\mathsf{K}}(\mathcal{E}(T))\} \quad \text{(Property of join)}\\ & = & \bigvee\{\Theta : \mathcal{F}(\Theta) \leq C(\mathcal{F}(\mathcal{E}(T)))\} \quad (\mathcal{F} \text{ an interpretation})\\ & = & \bigvee\{\Theta : \mathcal{F}(\Theta) \leq C(T)\} \quad \text{(Equivalence)}\\ & = & \bigvee\{\Theta : \mathcal{F}(\Theta) \leq T\}. \quad (T \in \mathrm{Th}(\mathcal{S}))\end{array}$$

This proves the statement. ■

2.15 The Leibniz Operator

In this section we start working with algebraization in the standard sense. In other words, we assume that the interpretations involved are standard (see Section 2.13). In this case, Theorem 33 ensures that the equational G-2-logic $\mathcal{S}_{\mathsf{K}}$ is unique and that the interpretations involved are essentially unique, i.e., they are interderivable modulo the equational and logical entailments. The following result asserts that the isomorphism $\Omega_{\mathsf{K}} : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$, induced by the standard interpretation $\mathcal{E} : \mathcal{S} \to \mathcal{S}_{\mathsf{K}}$, coincides with the Leibniz operator Ω on $\mathbf{Th}(\mathcal{S})$. This forms an analog of Theorem 4.1 of Blok and Pigozzi [6].

Theorem 44 *Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be an algebraizable G-logic and K an equivalent G-algebraic semantics via standard interpretations $\mathcal{E}$ and $\mathcal{F}$. Then $\Omega_{\mathsf{K}} : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_{\mathsf{K}})$ coincides with the Leibniz operator, i.e.,*

$$\Omega_{\mathsf{K}}(T) = \Omega(T), \ \textit{for all } T \in \mathrm{Th}(\mathcal{S}).$$

Proof: Let $T \in \mathrm{Th}(\mathcal{S})$. First, by definition, $\Omega_{\mathsf{K}}(T) \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$. By Lemma 20, $\Omega_{\mathsf{K}}(T)$ is a G-congruence. By Lemma 43 and the fact that $\mathcal{E}$ and $\mathcal{F}$ are standard, for all $T \in \mathrm{Th}(\mathcal{S})$ and all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Omega_{\mathsf{K}}(T)(\varphi, \psi) = \bigwedge_{j \in J} T(\Delta_j(\varphi, \psi)).$$

Further, by Lemma 30, for all $T \in \mathrm{Th}(\mathcal{S})$ and all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\bigwedge_{j \in J} T(\Delta_j(\varphi, \psi)) \wedge T(\varphi) \leq T(\psi).$$

Thus, we have

$$\Omega_{\mathsf{K}}(T)(\varphi, \psi) \wedge T(\varphi) = \bigwedge_{j \in J} T(\Delta_j(\varphi, \psi)) \wedge T(\varphi) \leq T(\psi).$$

We conclude that $\Omega_{\mathsf{K}}(T)$ is a G-congruence compatible with T. But $\Omega(T)$ is the largest G-congruence on $\mathbf{Fm}_{\mathcal{L}}(V)$ compatible with T, whence $\Omega_{\mathsf{K}}(T) \leq \Omega(T)$.

Conversely, by the fact that $\Omega(T)$ is a G-congruence compatible with T, we get that, for all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Omega(T)(\varphi, \psi) \leq T(\Delta_j(\varphi, \varphi)) \leftrightarrow T(\Delta_j(\varphi, \psi)), \quad j \in J.$$

As, for all $j \in J$, $T(\Delta_j(\varphi, \varphi)) = \top$, this gives that

$$\Omega(T)(\varphi, \psi) \leq \bigwedge_{j \in J} T(\Delta_j(\varphi, \psi)) = \Omega_{\mathsf{K}}(T)(\varphi, \psi).$$

We conclude that, for all $T \in \mathrm{Th}(\mathcal{S})$, $\Omega_{\mathsf{K}}(T) = \Omega(T)$. ■

In Lemma 46, we show that, if Ω is order preserving, then it is meet continuous and surjectively structural. For the latter property, we must have available a technical lemma on the behavior of surjective substitutions.

Lemma 45 *Suppose $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \twoheadrightarrow \mathbf{Fm}_{\mathcal{L}}(V)$ is a surjective substitution. Then, for all $\vartheta \in \mathrm{Fm}_{\mathcal{L}}(V)$ and every variable x occurring in ϑ, there exists $\vartheta' \in \mathrm{Fm}_{\mathcal{L}}(V)$ and a variable y, such that, for all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$,*

$$\sigma(\vartheta'(\varphi/y)) = \vartheta(\sigma(\varphi)/x).$$

Proof: An inverse image of a variable under any substitution must also be a variable. Thus, since σ is surjective, there exists, for each variable z, another variable z', such that $\sigma(z') = z$. Let ϑ' be obtained from ϑ by simultaneously replacing each variable z different from x by z' and x by any variable y different from all the z'. Then, we can see that, for all $\varphi \in \mathrm{Fm}_{\mathcal{L}}(V)$, $\sigma(\vartheta'(\varphi/y)) = \vartheta(\sigma(\varphi)/x)$. ■

Lemma 46 *Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a G-logic and suppose Ω is order preserving on $\mathbf{Th}(\mathcal{S})$.*

(a) For all $\{T_i : i \in I\} \subseteq \mathrm{Th}(\mathcal{S})$,

$$\Omega\left(\bigwedge_{i \in I} T_i\right) = \bigwedge_{i \in I} \Omega(T_i),$$

Hence, $\Omega(\mathrm{Th}(\mathcal{S}))$ is closed under meets.

(b) For all $T \in \mathrm{Th}(\mathcal{S})$ *and all* $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \twoheadrightarrow \mathbf{Fm}_{\mathcal{L}}(V)$,

$$\Omega(T) \circ \sigma = \Omega(T \circ \sigma).$$

Hence, $\Omega(\mathrm{Th}(\mathcal{S}))$ *is "surjectively structural", a property akin to closure under inverse surjective substitutions.*

Proof:

(a) By hypothesis, Ω is order preserving. Thus, for all $i \in I$, $\Omega(\bigwedge_{i\in I} T_i) \leq \Omega(T_i)$. Therefore, $\Omega(\bigwedge_{i\in I} T_i) \leq \bigwedge_{i\in I} \Omega(T_i)$. Conversely, note that, for all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\bigwedge_{i\in I} \Omega(T_i)(\varphi,\psi) \wedge \bigwedge_{i\in I} T_i(\varphi) \leq \Omega(T_i)(\varphi,\psi) \wedge T_i(\varphi) \leq T_i(\psi).$$

Thus,

$$\bigwedge_{i\in I} \Omega(T_i)(\varphi,\psi) \wedge \bigwedge_{i\in I} T_i(\varphi) \leq \bigwedge_{i\in I} T_i(\psi).$$

This shows that $\bigwedge_{i\in I} \Omega(T_i)$ is a G-congruence on $\mathbf{Fm}_{\mathcal{L}}(V)$ compatible with $\bigwedge_{i\in I} T_i$ and, hence, by the maximality property of the Leibniz G-congruence, $\bigwedge_{i\in I} \Omega(T_i) \leq \Omega(\bigwedge_{i\in I} T_i)$.

(b) First, by the compatibility of $\Omega(T)$ with T, for all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Omega(T)(\sigma(\varphi),\sigma(\psi)) \wedge T(\sigma(\varphi)) \leq T(\sigma(\psi)).$$

Hence, $\Omega(T) \circ \sigma$ is compatible with $T \circ \sigma$. Thus, by the maximality property of the Leibniz G-congruence, $\Omega(T) \circ \sigma \leq \Omega(T \circ \sigma)$. For the reverse inequality, suppose for the sake of obtaining a contradiction, that $\Omega(T \circ \sigma) \nleq \Omega(T) \circ \sigma$. Thus, there exist $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$, such that

$$\Omega(T \circ \sigma)(\varphi,\psi) \nleq \Omega(T)(\sigma(\varphi),\sigma(\psi)).$$

Recalling Theorem 16, there exists $\vartheta(x) \in \mathrm{Fm}_{\mathcal{L}}(V)$, such that

$$\Omega(T \circ \sigma)(\varphi,\psi) \nleq T(\vartheta(\sigma(\varphi))) \leftrightarrow T(\vartheta(\sigma(\psi))).$$

So, by Lemma 45, there exists ϑ', such that

$$\Omega(T \circ \sigma)(\varphi,\psi) \nleq T(\sigma(\vartheta'(\varphi))) \leftrightarrow T(\sigma(\vartheta'(\psi))).$$

This contradicts the compatibility of $\Omega(T \circ \sigma)$ with $T \circ \sigma$.

■

Since $\Omega(\mathrm{Th}(\mathcal{S}))$ is closed under arbitrary intersections, it forms a complete lattice which is denoted by $\Omega(\mathbf{Th}(\mathcal{S}))$. If Ω is injective, then it is an isomorphism from $\mathbf{Th}(\mathcal{S})$ onto $\Omega(\mathbf{Th}(\mathcal{S}))$. Our goal is to be able to apply

Theorem 42. This requires showing that $\Omega(\mathbf{Th}(\mathcal{S}))$ coincides with $\mathbf{Th}(\mathcal{S}_{\mathsf{K}})$ for some class K of G-algebras. More precisely, we aim to show that $\mathbf{Th}(\mathcal{S}_{\mathsf{K}})$ and $\Omega(\mathbf{Th}(\mathcal{S}))$ are isomorphic under the identity mapping.

For a G-congruence Θ on $\mathbf{Fm}_{\mathcal{L}}(V)$, we define a G-algebra

$$\mathcal{F}^{\Theta} = \langle \mathbf{Fm}_{\mathcal{L}}(V), \Theta \rangle.$$

Given any G-algebra $\mathcal{A} = \langle \mathbf{A}, E \rangle$ and a homomorphism $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}$, we define the G-**kernel** $\Theta_{\mathcal{A},h} : \mathrm{Eq}_{\mathcal{L}}(V) \to G$ by setting

$$\Theta_{\mathcal{A},h} = E \circ h^2,$$

$$\begin{array}{ccc} \mathrm{Eq}_{\mathcal{L}}(V) & \xrightarrow{\quad h^2 \quad} & A^2 \\ & \Theta_{\mathcal{A},h} \searrow \quad \swarrow E & \\ & G & \end{array}$$

i.e., we have, for all $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\Theta_{\mathcal{A},h}(\varphi, \psi) = E(h(\varphi), h(\psi)).$$

$\Theta_{\mathcal{A},h}$ is a G-congruence. Moreover, note that, given a G-congruence Θ,

$$\Theta_{\mathcal{F}^{\Theta},i} = \Theta,$$

where $i : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$ is the identity homomorphism.

Now let K be a class of G-algebras and suppose that $\mathcal{A} = \langle \mathbf{A}, E \rangle \in \mathsf{K}$. Then $\Theta_{\mathcal{A},h} \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$. To see this, it suffices to show that $C_{\mathsf{K}}(\Theta_{\mathcal{A},h}) \le \Theta_{\mathcal{A},h}$. We have

$$\begin{array}{rcl} C_{\mathsf{K}}(\Theta_{\mathcal{A},h}) & = & \bigwedge_{\substack{\mathcal{B}=\langle \mathbf{B},F\rangle \in \mathsf{K} \\ g:\mathbf{Fm}_{\mathcal{L}}(V)\to \mathbf{B}}} \{F \circ g^2 : \Theta_{\mathcal{A},h} \le F \circ g^2\} \\ & & \text{(Definition of } C_{\mathsf{K}}) \\ & = & \bigwedge_{\substack{\mathcal{B}=\langle \mathbf{B},F\rangle \in \mathsf{K} \\ g:\mathbf{Fm}_{\mathcal{L}}(V)\to \mathbf{B}}} \{F \circ g^2 : E \circ h^2 \le F \circ g^2\} \\ & & \text{(Definition of } \Theta_{\mathcal{A},h}) \\ & \le & E \circ h^2 \quad (\mathcal{A} = \langle \mathbf{A}, E\rangle \in \mathsf{K} \text{ and } h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}) \\ & = & \Theta_{\mathcal{A},h}. \quad \text{(Definition of } \Theta_{\mathcal{A},h}) \end{array}$$

More generally, by definition, given a G-set of equations $Z : \mathrm{Eq}_{\mathcal{L}}(V) \to G$, the G-congruence in $\mathrm{Th}(\mathcal{S}_{\mathsf{K}})$ generated by Z is given by

$$C_{\mathsf{K}}(Z) = \bigwedge_{\substack{\mathcal{A}=\langle \mathbf{A},E\rangle \in \mathsf{K} \\ h:\mathbf{Fm}_{\mathcal{L}}(V)\to \mathbf{A}}} \{E \circ h^2 : Z \le E \circ h^2\}.$$

Our next goal is to show that, if the Leibniz operator Ω is join continuous on the theories of a G-logic, one can construct a class K of G-algebras, such that the image $\Omega(\mathrm{Th}(\mathcal{S}))$ coincides with $\mathrm{Th}(\mathcal{S}_{\mathsf{K}})$. However, we precede this by a technical result which is needed for the proof of the main lemma.

Lemma 47 *Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic and $h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$ a homomorphism, with the property that each formula is the image under h of infinitely many variables. Then, for all $T \in \mathrm{Th}(\mathcal{S})$, there exists a surjective $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \twoheadrightarrow \mathbf{Fm}_{\mathcal{L}}(V)$, such that*

$$\Theta_{\mathcal{F}^{\Omega(T)},h} = \Omega(T) \circ \sigma.$$

Proof: Let σ be any substitution, such that $\sigma(v_i) = h(v_i)$, for $i = 1, 2, \ldots$, and furthermore, such that each v_i is the image under σ of some v_j. Such a σ exists because of the assumption that each formula is the image under h of an infinite number of variables. Then σ is surjective and we have $h(v_i) = \sigma(v_i)$, for all $i = 1, 2, \ldots$. Now, using the definition of the G-kernel $\Theta_{\mathcal{F}^{\Omega(T)},h}$, we get

$$\Theta_{\mathcal{F}^{\Omega(T)},h} = \Omega(T) \circ h = \Omega(T) \circ \sigma.$$

So the conclusion holds. ∎

Lemma 48 *Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic and set*

$$\mathsf{K} = \{\mathcal{F}^{\Theta} : \Theta \in \Omega(\mathrm{Th}(\mathcal{S}))\}.$$

If Ω is join continuous on $\mathbf{Th}(\mathcal{S})$ then, we have $\Omega(\mathrm{Th}(\mathcal{S})) = \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$.

Proof: We begin with some observations. Since, by hypothesis, for every $\{T_i :\in I\} \subseteq \mathrm{Th}(\mathcal{S})$,

$$\Omega\Big(\bigvee\nolimits_{i\in I}^{\mathbf{Th}(\mathcal{S})} T_i\Big) = \bigvee\nolimits_{i\in I}^{\mathbf{Th}(\mathcal{S}_{\mathsf{K}})} \Omega(T_i),$$

we infer that $\Omega(\mathrm{Th}(\mathcal{S}))$ is closed under joins and, also, that Ω is order preserving. Thus, by Lemma 46, it is also closed under meets and it is surjectively structural.

We look, first, at the inclusion $\Omega(\mathrm{Th}(\mathcal{S})) \subseteq \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$. Suppose $T \in \mathrm{Th}(\mathcal{S})$ and let $\Theta = \Omega(T)$. Then, by the definition of K, we get $\mathcal{F}^{\Theta} \in \mathsf{K}$. Thus, $\Omega(T) = \Theta = \Theta_{\mathcal{F}^{\Theta},i} \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$ (see work preceding Lemma 47).

Next, we turn to the reverse inclusion. Suppose $\Theta \in \mathrm{Th}(\mathcal{S}_{\mathsf{K}})$. Consider, first, the case where Θ is finitely generated. Then, there exists finite $Q : \mathrm{Eq}_{\mathcal{L}}(V) \to G$, such that $\Theta = C_{\mathsf{K}}(Q)$. We have

$$\begin{aligned}\Theta &= \bigwedge\nolimits_{\substack{\mathcal{A}=\langle \mathbf{A},E\rangle\in\mathsf{K}\\ h:\mathbf{Fm}_{\mathcal{L}}(V)\to\mathbf{A}}} \{E \circ h^2 : Q \le F \circ h^2\}\\ &= \bigwedge\nolimits_{\substack{T\in\mathrm{Th}(\mathcal{S})\\ h:\mathbf{Fm}_{\mathcal{L}}(V)\to\mathbf{Fm}_{\mathcal{L}}(V)}} \{\Omega(T) \circ h^2 : Q \le \Omega(T) \circ h^2\},\end{aligned}$$

the last equality following by the definition of K. Now note that $Q(\varphi, \psi) \neq \bot$ only for finitely many formulas. So all h's can be taken to satisfy the hypothesis of Lemma 47. Thus, we obtain

$$\begin{aligned}\Theta &= \bigwedge\nolimits_{\substack{T\in\mathrm{Th}(\mathcal{S})\\ \sigma \text{ surjective}}} \{\Omega(T) \circ \sigma^2 : Q \le \Omega(T) \circ \sigma^2\}\\ &= \bigwedge\nolimits_{\substack{T\in\mathrm{Th}(\mathcal{S})\\ \sigma \text{ surjective}}} \{\Omega(T \circ \sigma) : Q \le \Omega(T \circ \sigma)\} \quad \text{(Lemma 46(b))}\\ &\in \Omega(\mathrm{Th}(\mathcal{S})). \quad \text{(Lemma 46(a))}\end{aligned}$$

Finally, we turn to the general case. Consider an arbitrary $\Theta \in \mathrm{Th}(\mathcal{S}_\mathsf{K})$. Then, we have

$$\Theta = \bigvee^{\mathbf{Th}(\mathcal{S}_\mathsf{K})}\{C_\mathsf{K}(\Gamma) : \Gamma \leq \Theta \text{ finite}\}.$$

By what was shown above, for all finite $\Gamma \leq \Theta$, $C_\mathsf{K}(\Gamma) \in \Omega(\mathrm{Th}(\mathcal{S}))$. By hypothesis, $\Omega(\mathrm{Th}(\mathcal{S}))$ is closed under joins. Thus $\Theta \in \Omega(\mathrm{Th}(\mathcal{S}))$. Thus, we conclude that $\mathrm{Th}(\mathcal{S}_\mathsf{K}) \subseteq \Omega(\mathrm{Th}(\mathcal{S}))$. ■

Theorem 49 *Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a G-logic. $\mathcal{S}$ is algebraizable via standard interpretations if and only if the Leibniz operator is $\mathcal{E}$-regular, with $\mathcal{E}$ standard and standardly invertible, injective and join continuous on $\mathbf{Th}(\mathcal{S})$.*

Proof: By Theorem 44, $\Omega_\mathsf{K} = \Omega$. By Lemma 38, Ω is an $\mathcal{E}$-regular, with $\mathcal{E}$ standard and standardly invertible, isomorphism. Thus, in particular, it is injective and join continuous.

Suppose, conversely, that Ω is $\mathcal{E}$-regular, with $\mathcal{E}$ standard and standardly invertible, injective and join continuous. Then, by Lemma 48, it is an isomorphism $\Omega : \mathbf{Th}(\mathcal{S}) \to \mathbf{Th}(\mathcal{S}_\mathsf{K})$, where

$$\mathsf{K} = \{\langle \mathbf{Fm}_\mathcal{L}(V), \Omega(T)\rangle : T \in \mathrm{Th}(\mathcal{S})\}.$$

By Theorem 42, Part (b), K is equivalent to $\mathcal{S}$ and $\mathcal{S}$ is algebraizable. ■