Chapter 1

Introduction

In their pioneering "Memoirs monograph" [6], Blok and Pigozzi, based on Czelakowski's [13, 14] and their own previous work [5], made, for the first time, precise the notion of *algebraizable logic*. Many researchers followed in their footsteps and an explosion of research activity ensued, culminating in a beautiful and fascinating body of work that came to be known as *Abstract Algebraic Logic*. Under its roof it houses various directions of study, but its crown jewel, and, perhaps, its best known accomplishment, is the *Leibniz* or *algebraic hierarchy* of classes of logical systems, see, e.g., the surveys [15, 29, 28, 27]. The higher in the hierarchy a class is located, the more intimate the connection of the logics in that class with their algebraic counterparts.

Blok and Pigozzi study *sentential logics* or *deductive systems*. These are pairs $\mathcal{S} = \langle \mathcal{L}, \vdash \rangle$ consisting of a logical (or algebraic, depending on the point of view) language $\mathcal{L}$ and a structural consequence relation

$$\vdash \subseteq \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)) \times \mathrm{Fm}_{\mathcal{L}}(V)$$

on the set $\mathrm{Fm}_{\mathcal{L}}(V)$ of $\mathcal{L}$-formulas constructed using variables in a countably infinite set V. A consequence relation is one that satisfies inflationarity, monotonicity and idempotency. Structurality means that the consequence is invariant under the application of substitutions. Since structural consequence relations turn out to be equivalent to structural closure operators, a logic may be equivalently presented as a pair $\mathcal{S} = \langle \mathcal{L}, C \rangle$, where

$$C : \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V)) \to \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}(V))$$

is the structural closure operator associated with $\vdash$, given, for all $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$, by

$$C(\Gamma) = \{\varphi \in \mathrm{Fm}_{\mathcal{L}}(V) : \Gamma \vdash \varphi\}.$$

In this formulation, the axioms governing the logic are:

(**Inflationarity**) $\Gamma \subseteq C(\Gamma)$, for all $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$;

(**Monotonicity**) $\Gamma \subseteq \Delta$ implies $C(\Gamma) \subseteq C(\Delta)$, for all $\Gamma, \Delta \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$;

(**Idempotency**) $C(C(\Gamma)) = C(\Gamma)$, for all $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$;

(**Structurality**) $\sigma(C(\Gamma)) \subseteq C(\sigma(\Gamma))$, for all $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$ and all $\sigma : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{Fm}_{\mathcal{L}}(V)$.

Blok and Pigozzi [6] deal exclusively with *finitary* deductive systems, that is, those satisfying the additional axiom that, for all $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

(**Finitarity**) $C(\Gamma) = \bigcup_{\Gamma_0 \subseteq_f \Gamma} C(\Gamma_0)$,

where $\subseteq_f$ denotes the finite subset relation. The majority of their results and almost all their methodology were later generalized and shown to hold for all sentential logics (see. e.g., [33, 34]).

Given two logics $\mathcal{S} = \langle \mathcal{L}, C \rangle$ and $\mathcal{S}' = \langle \mathcal{L}, C' \rangle$, we say that $\mathcal{S}$ is *weaker than* $\mathcal{S}'$ or that $\mathcal{S}'$ is *stronger than* $\mathcal{S}$ and write $\mathcal{S} \leq \mathcal{S}'$ or $C \leq C'$ to signify that $C(\Gamma) \subseteq C'(\Gamma)$, for all $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$.

A *theory* T of a logic $\mathcal{S}$ is a closed set of formulas, that is, such that $C(T) = T$. The set of all theories of $\mathcal{S}$ is denoted $\mathrm{Th}(\mathcal{S})$. Ordered by the subset relation, it forms a complete lattice $\mathbf{Th}(\mathcal{S}) = \langle \mathrm{Th}(\mathcal{S}), \subseteq \rangle$. It turns out that a closure operator is completely specified by its set of theories and, hence, a third equivalent presentation of a logic is as a pair $\mathcal{S} = \langle \mathcal{L}, \mathrm{Th}(\mathcal{S}) \rangle$.

To study the semantics of logics one uses logical matrices. A *logical matrix* $\mathfrak{A} = \langle \mathbf{A}, F \rangle$ consists of an $\mathcal{L}$-algebra $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$ and a set $F \subseteq A$ of designated elements, called the *filter* of the matrix. Given a class M of $\mathcal{L}$-matrices, M induces a logic $\mathcal{S}_{\mathsf{M}} = \langle \mathcal{L}, C_{\mathsf{M}} \rangle$ by setting, for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\varphi \in C_{\mathsf{M}}(\Gamma) \quad \text{iff} \quad \begin{array}{l} \text{for all } \mathfrak{A} = \langle \mathbf{A}, F \rangle \in \mathsf{M}, h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \\ h(\Gamma) \subseteq F \text{ implies } h(\varphi) \in F. \end{array}$$

A matrix $\mathfrak{A}$ is a *matrix model* of $\mathcal{S}$ or an *$\mathcal{S}$-matrix* if

$$C \leq C_{\mathfrak{A}} := C_{\{\mathfrak{A}\}}.$$

In this case F is called an *$\mathcal{S}$-filter*. It is well-known that the collection $\mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ of all $\mathcal{S}$-filters on $\mathbf{A}$, ordered by $\subseteq$, forms a complete lattice $\mathbf{Fi}_{\mathcal{S}}(\mathbf{A}) = \langle \mathrm{Fi}_{\mathcal{S}}(\mathbf{A}), \subseteq \rangle$. Further, one can show that $\mathrm{Fi}_{\mathcal{S}}(\mathbf{Fm}_{\mathcal{L}}(V)) = \mathrm{Th}(\mathcal{S})$. If, for a class M of matrices, we have $C = C_{\mathsf{M}}$, then M is called a *matrix semantics* for $\mathcal{S}$.

One of the goals of algebraization is to connect a deductive system with a class K of algebras. Such a connection is possible when a correspondence may be established between the lattice of theories of the deductive system and the lattice of equational theories, i.e., of congruences, associated with the equational logic induced by the class of algebras. This necessitates the study of equational consequences, which are consequences over the set $\mathrm{Eq}_{\mathcal{L}}(V) = \mathrm{Fm}^2_{\mathcal{L}}(V)$ of *$\mathcal{L}$-equations*. The *equational logic* $\mathcal{S}_{\mathsf{K}} = \langle \mathcal{L}, C_{\mathsf{K}} \rangle$ associated with a class K of $\mathcal{L}$-algebras consists of a language $\mathcal{L}$ and a closure operator

$$C_{\mathsf{K}} : \mathcal{P}(\mathrm{Eq}_{\mathcal{L}}(V)) \to \mathcal{P}(\mathrm{Eq}_{\mathcal{L}}(V)),$$

defined by setting, for all $E \cup \{\varphi \approx \psi\} \subseteq \mathrm{Eq}_{\mathcal{L}}(V)$,

$$\varphi \approx \psi \in C_{\mathsf{K}}(E) \quad \text{iff} \quad \begin{array}{l} \text{for all } \mathbf{A} \in \mathsf{K} \text{ and all } h : \mathbf{Fm}_{\mathcal{L}}(V) \to \mathbf{A}, \\ h(\varepsilon_1) = h(\varepsilon_2), \text{ for all } \varepsilon_1 \approx \varepsilon_2 \in E, \\ \qquad \text{implies } h(\varphi) = h(\psi). \end{array}$$

A *theory* θ of $\mathcal{S}_{\mathsf{K}}$ is a set $\theta \subseteq \mathrm{Eq}_{\mathcal{L}}(V)$, such that $C_{\mathsf{K}}(\theta) = \theta$. Theories coincide with *K-congruences*, i.e., congruences on $\mathbf{Fm}_{\mathcal{L}}(V)$, such that $\mathbf{Fm}_{\mathcal{L}}(V)/\theta \in \mathsf{K}$.

In Abstract Algebraic Logic the association of a deductive system with a class of algebras that serves as its algebraic semantics is done via the Leibniz operator, a tool as important as the hierarchy itself, since it constitutes its building foundation. Given an algebra $\mathbf{A}$ and a subset $F \subseteq A$, i.e., a matrix $\mathfrak{A} = \langle \mathbf{A}, F \rangle$, a congruence θ on $\mathbf{A}$ is *compatible with* F, or is a *congruence of* $\mathfrak{A}$, if, for all $a, b \in A$,

$$\langle a, b \rangle \in \theta \quad \text{and} \quad a \in F \quad \text{imply} \quad b \in F.$$

This is equivalent to saying that F is a union of θ-congruence classes. We denote the collection of all such congruences by $\mathrm{Con}(\mathfrak{A})$. It turns out that it forms a principal ideal of the lattice of all congruences on $\mathbf{A}$, ordered by $\subseteq$, denoted $\mathbf{Con}(\mathfrak{A}) = \langle \mathrm{Con}(\mathfrak{A}), \subseteq \rangle$. Its generator, i.e., the largest congruence on $\mathbf{A}$ compatible with F, is called the *Leibniz congruence* of F on $\mathbf{A}$, or the *Leibniz congruence* of $\mathfrak{A}$ and is denoted by $\Omega_{\mathbf{A}}(F)$ or $\Omega(\mathfrak{A})$. The mapping $F \mapsto \Omega_{\mathbf{A}}(F)$ on $\mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ is termed the *Leibniz operator* of $\mathcal{S}$ on $\mathbf{A}$ (Definition 1.4 and Theorem 1.5 of [6]). Blok and Pigozzi showed that $\Omega_{\mathbf{A}}(F)$ admits the following characterization:

$$\begin{aligned} \Omega_{\mathbf{A}}(F) \quad = \quad \{\langle a, b \rangle \in A^2 : & \text{ for all } \varphi \in \mathrm{Fm}_{\mathcal{L}}(V) \text{ and all } \overline{c} \in A, \\ & \varphi^{\mathbf{A}}(a, \overline{c}) \in F \text{ iff } \varphi^{\mathbf{A}}(b, \overline{c}) \in F\}. \end{aligned}$$

Close to the bottom of the algebraic hierarchy (see, e.g., [15, 29, 27]) lies the class of *protoalgebraic logics* [5]. They are characterized by the fact that any two sentences which are equivalent modulo the Leibniz congruence of a theory are also interderivable modulo that same theory. This idea is roughly expressed by the motto "indistinguishability implies interderivability". According to Blok and Pigozzi, this class is the widest class of logics on which the powerful methods of algebra can be brought to bear in the study of their properties. Protoalgebraic logics may alternatively be characterized as the ones on whose lattice of theories the Leibniz operator is monotone or, equivalently, meet continuous.

To define algebraizability, Blok and Pigozzi introduced translations and interpretations between formulas and equations (see Definitions 2.2 and 2.8 of [6]). From a slightly more contemporary point of view, a *translation* from formulas to equations is a set

$$\delta(x) \approx \varepsilon(x) = \{\delta_i(x) \approx \varepsilon_i(x) : i \in I\}$$

of equations in a single variable. Dually a *translation* from equations to formulas is a set

$$\Delta(x, y) = \{\Delta_j(x, y) : j \in J\}$$

of formulas in two variables. Blok and Pigozzi considered finitary deductive systems only and so took translations to be finite, i.e., both sets of indices I

and J were taken to be finite. Let $\mathcal{S} = \langle \mathcal{L}, C\rangle$ be a logic and K be a class of $\mathcal{L}$-algebras. A translation $\delta \approx \varepsilon$ from formulas to equations is an *interpretation* from $\mathcal{S}$ to $\mathcal{S}_{\mathsf{K}}$, written $\delta \approx \varepsilon : \mathcal{S} \to \mathcal{S}_{\mathsf{K}}$, if, for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$,

$$\varphi \in C(\Gamma) \quad \text{iff} \quad \delta(\varphi) \approx \varepsilon(\varphi) \subseteq C_{\mathsf{K}}(\delta(\Gamma) \approx \varepsilon(\Gamma)),$$

where we adopt the relatively intuitive notation

$$\delta(\varphi) \approx \varepsilon(\varphi) := \{\delta_i(\varphi) \approx \varepsilon_i(\varphi) : i \in I\}$$

and

$$\delta(\Gamma) \approx \varepsilon(\Gamma) := \bigcup_{\gamma \in \Gamma} \delta(\gamma) \approx \varepsilon(\gamma).$$

Dually, a translation Δ from equations to formulas is an *interpretation* from $\mathcal{S}_{\mathsf{K}}$ to $\mathcal{S}$, written $\Delta : \mathcal{S}_{\mathsf{K}} \to \mathcal{S}$, if, for all $E \cup \{\varphi \approx \psi\} \subseteq \mathrm{Eq}_{\mathcal{L}}(V)$,

$$\varphi \approx \psi \in C_{\mathsf{K}}(E) \quad \text{iff} \quad \Delta(\varphi, \psi) \subseteq C(\Delta(E)),$$

where similar conventions as before are assumed regarding the notation.

Given a deductive system $\mathcal{S} = \langle \mathcal{L}, C\rangle$, a class of $\mathcal{L}$-algebras K is called an *algebraic semantics* for $\mathcal{S}$ (Definition 2.2 of [6]) if there exists an interpretation $\delta \approx \varepsilon : \mathcal{S} \to \mathcal{S}_{\mathsf{K}}$. In this case $\delta \approx \varepsilon$ are called the (system of) *defining equations*. Due to finitarity, it turns out that, if K is an algebraic semantics for $\mathcal{S}$, so is the quasivariety $\mathbb{Q}(\mathsf{K})$ generated by the class K. There is a close relationship between algebraic semantics and matrix semantics. Given a translation $\delta \approx \varepsilon$ from formulas to equations and an algebra $\mathbf{A}$, one may define a filter on $\mathbf{A}$ by setting

$$F_{\mathbf{A}}^{\delta \approx \varepsilon} = \left\{a \in A : \delta^{\mathbf{A}}(a) = \varepsilon^{\mathbf{A}}(a)\right\}.$$

Moreover, given a class K of $\mathcal{L}$-algebras, one may construct an associated class of matrices

$$\mathsf{M} = \left\{\langle \mathbf{A}, F_{\mathbf{A}}^{\delta \approx \varepsilon}\rangle : \mathbf{A} \in \mathsf{K}\right\}.$$

In Theorem 2.4 of [6], Blok and Pigozzi prove that a class K is an algebraic semantics for $\mathcal{S}$ if and only if M is a matrix semantics for $\mathcal{S}$.

The usefulness of possessing an algebraic semantics rests with the fact that the deductive system apparatus of the logic is reflected via the defining equations into the algebraic deductive apparatus induced by the class of algebras. This forces the logic to exhibit some of the characteristics of equational consequences. E.g., in Theorem 2.7 of [6], it is shown that, if $\mathcal{S}$ has an algebraic semantics with defining equations $\delta \approx \varepsilon$, then $\mathcal{S}$ must satisfy the deduction

$$\varepsilon_i(x) \in C(x, \delta_i(x)), \quad i \in I.$$

This is a consequence of the fact that, regardless of the class K, one has equationally

$$\delta(\varepsilon_i(x)) \approx \varepsilon(\varepsilon_i(x)) \subseteq C_{\mathsf{K}}(\delta(x) \approx \varepsilon(x), \delta(\delta_i(x)) \approx \varepsilon(\delta_i(x))).$$

Let $\mathcal{S} = \langle \mathcal{L}, C \rangle$ be a logic and K an algebraic semantics with defining equations $\delta \approx \varepsilon$. K is said to be an *equivalent algebraic semantics* for $\mathcal{S}$ and $\mathcal{S}$ is then called *algebraizable* if there exists also a translation $\Delta : \mathcal{S}_\mathsf{K} \to \mathcal{S}$, such that, for all $\varphi, \psi \in \mathrm{Fm}_\mathcal{L}(V)$,

$$C_\mathsf{K}(\delta(\Delta(\varphi,\psi)) \approx \varepsilon(\Delta(\varphi,\psi))) = C_\mathsf{K}(\varphi \approx \psi).$$

In Corollary 2.9 of [6], it is shown that the roles of $\delta \approx \varepsilon$ and Δ are completely symmetric, in the sense that K is an equivalent algebraic semantics of $\mathcal{S}$ if and only if $\Delta : \mathcal{S}_\mathsf{K} \to \mathcal{S}$ is an interpretation and, for all $\varphi \in \mathrm{Fm}_\mathcal{L}(V)$,

$$C(\Delta(\delta(\varphi), \varepsilon(\varphi))) = C(\varphi).$$

If any of these equivalent pairs of conditions hold (and, therefore, all four), then $\delta \approx \varepsilon$ and Δ are said to be *inverses* of one another. Blok and Pigozzi proceed to show (Theorem 2.15 of [6]) that, if a deductive system is algebraizable, it is algebraizable in an essentially unique way, in the sense that all translations used (in either direction) are interderivable and the equational consequences generated by the classes of algebras used are identical.

Besides a plenitude of examples to support the usefulness of the definition and the accompanying theory (see Chapter 5 of [6]), Blok and Pigozzi provide alternative characterizations of algebraizability, which further attest to the naturalness of the notion. The first characterization assumes that a class of algebras K is given and it is to be tested for whether it forms an equivalent algebraic semantics of the logic $\mathcal{S}$ at hand. Because K is external information (to the logic) this characterization is a, so called, *extrinsic* characterization. In Theorem 3.7 of [6], it is shown that K is the equivalent algebraic semantics for $\mathcal{S}$ iff there is an isomorphism between the theory lattice of $\mathcal{S}$ and the equational theory lattice of K that commutes with the substitution operators. The other two characterizations they provide are *intrinsic*, in the sense that no data external to the given logic are involved. In Theorem 4.2 of [6] Blok and Pigozzi show that a deductive system $\mathcal{S}$ is algebraizable iff the Leibniz operator is injective and order-preserving on the lattice $\mathbf{Th}(\mathcal{S})$ of theories and (due to finitarity) preserves unions of directed subsets of $\mathrm{Th}(\mathcal{S})$. The second intrinsic characterization is of a more syntactic nature. It asserts (Theorem 4.7 of [6]) that $\mathcal{S}$ is algebraizable if and only if there exist a system Δ of formulas in two variables and a system $\delta \approx \varepsilon$ of equations in a single variable, such that, for all $\varphi, \psi, \chi \in \mathrm{Fm}_\mathcal{L}(V)$, every n-ary λ in $\mathcal{L}$ and all $\varphi_1, \psi_1, \dots, \varphi_n, \psi_n \in \mathrm{Fm}_\mathcal{L}(V)$,

(i) $\Delta(\varphi,\varphi) \subseteq C(\varnothing)$;

(ii) $\Delta(\psi,\varphi) \subseteq C(\Delta(\varphi,\psi))$;

(iii) $\Delta(\varphi,\chi) \subseteq C(\Delta(\varphi,\psi), \Delta(\psi,\chi))$;

(iv) $\Delta(\lambda(\varphi_1,\dots,\varphi_n),\lambda(\psi_1,\dots,\psi_n)) \subseteq C(\Delta(\varphi_1,\psi_1),\dots,\Delta(\varphi_n,\psi_n))$;

(v) $C(\varphi) = C(\Delta(\delta(\varphi),\varepsilon(\varphi)))$.

As we saw, Blok and Pigozzi [6] made for the first time precise the notion of algebraizable logic. Roughly speaking, a logic is algebraizable if its consequence relation is interpretable, in an invertible way, in the equational consequence of a class of algebras of the same type as that of the logic. They characterized algebraizability by showing that a logic is algebraizable if and only if there is an isomorphism from the complete lattice of the theories of the logic onto the complete lattice of the equational theories associated with the algebraizing class of algebras which commutes with substitutions (see Theorem 3.7 of [6]). This characterization is extrinsic in the sense that, apart from the logic, one needs the class of algebras to identify the lattice of equational theories in order to be able to apply it. In what was, perhaps, the most important result in [6], which inspired much of subsequent work, Blok and Pigozzi gave an intrinsic characterization of algebraizability. They showed (Theorem 4.2 of [6]) that a logic is algebraizable if and only if the Leibniz operator, which can be computed only with knowledge of the theories of the logic, is order preserving, injective and commutes with unions of directed sets of theories. Subsequent work focused on introducing and investigating new classes of logics, mostly weaker than the algebraizable ones, that can also be characterized using properties of the Leibniz operator on the lattice of theories. The totality of those classes constitute the algebraic or Leibniz hierarchy of logics (see, e.g., [15, 29, 27]), a cornerstone of Algebraic Logic and one of the most beautiful parts of the theory.

Among the most important contributors to the field have been the members of the Barcelona Group of Algebraic Logic. Their work over many decades has been summarized and explained in a unified, coherent and elegant way in the seminal monograph of Font and Jansana [28]. Our focus in Chapter 3 is on one of their main and most beautiful abstract results, the Isomorphism Theorem 2.30 of [28]. Of course to be able to adapt their theory to our purposes and reformulate and prove a version of the Isomorphism Theorem in the context of graded logics, we have to develop the necessary machinery.

Their starting point is a version of the aforementioned characterization of Blok and Pigozzi asserting that a logic $\mathcal{S}$ is algebraizable if and only if, for every algebra $\mathbf{A}$, the Leibniz operator on $\mathbf{A}$ is an isomorphism between the lattice of $\mathcal{S}$-filters on $\mathbf{A}$ and the lattice of K-congruences on $\mathbf{A}$, where K is the class of algebras serving as the algebraic counterpart of the logic (see Theorem 5.1 of [6]). Font and Jansana recognize, based on previous experience acquired by the Barcelona Group, that a plethora of concrete logics of interest are not algebraizable in the sense of Blok and Pigozzi. Thus, they embark on a quest to generalize the characterization of algebraizable logics in order to obtain an isomorphism theorem between logical objects (such as theories) and congruences that would be valid for arbitrary logics. To achieve this goal, and in quite an ingenious move, they change the types of objects used as models of the logical systems.

In contrast to the theory of Blok and Pigozzi, where the focus is on logical matrices as models of the sentential logics under study, Font and Jansana consider abstract logics or generalized matrices. Whereas logical matrices consist of a single filter on the underlying algebra, abstract logics consist of collections of filters forming a closed set system. This change in focus requires also passing from the Leibniz operator to the Tarski operator. The Tarski operator associates to an abstract logic the largest congruence on its underlying algebra that is compatible with all filters of the abstract logic. As such, it is obtained as the intersection of the Leibniz congruences of the matrices obtained by the abstract logic by considering each of its filters individually. Font and Jansana prove that, for any logic $\mathcal{S}$ (not necessarily algebraizable) and any algebra $\mathbf{A}$, the Tarski operator on $\mathbf{A}$ is an isomorphism from the lattice of full models of $\mathbf{A}$ onto the lattice of $\mathrm{Alg}(\mathcal{S})$-congruences on $\mathbf{A}$ (Theorem 2.30 of [28]).

We look, next, at the influence of the theory to model theoretic investigations. In classical Model Theory (see, e.g., [10, 35]) one establishes results that characterize via operations on classes of models, their definability via syntactic means. Among such results are, e.g., those characterizing elementary classes, universal classes, universal Horn classes and universal atomic classes of structures. In addition, these results encompass some of the signature results in Universal Algebra, such as Birkhoff's result [3] characterizing varieties and Mal'vev's result [36] characterizing (generalized) quasivarieties of algebras. All these results, in their classical form, assume the presence of an equality predicate in the language. On the other hand, the model theory of logical matrices, interpreting sentential logics in the abstract theory of Algebraic Logic [15] is encompassed by the equality free fragment of first order logic with a single unary predicate [8] (see Section 1.3 of [6]). So the accumulation of results in this field during the '80s, with the pioneering work of Blok and Pigozzi, Czelakowski and Font and Jansana, among others, led to a corresponding increase in interest in the equality free aspects of model theory of first order languages. Here the Leibniz congruence of a structure plays the role of equality, since it represents indiscernibility, which is akin to equality in logics defined without equality. As a result, it was only natural that, under the guidance of Pigozzi and Font and Jansana, Elgueta [22] and Dellunde [17] in Barcelona developed the machinery needed for formulating analogs of some of the most important characterization results in Model Theory concerning definability of classes of structures in the context of equality free first order logic.

In Section 1 of [23], Elgueta revisits classical constructions in Model Theory in the context of languages without equality. He introduces substructures, filter extensions and discusses elementarity. He delves into homomorphisms of structures and related constructions of image and preimage structures and discusses some special types of homomorphisms. Further, he defines products, filtered products (and ultraproducts) and subdirect products. Section 2 is dedicated to the introduction of congruences of structures and the definition of Leibniz equality, or indiscernibility relation, in this context, which "stands in" for equality in the absence of an equality predicate. Section 3 recalls quotient structures, formulates an analog of the Homomorphism Theorem and, in addition, asserts the validity of analogs of the remaining homomorphism theorems, including the Correspondence Theorem.

Section 4 introduces operators on classes of structures defined without equality. These form the cornerstone of the characterizations of definability of those classes by various syntactic means. Very briefly, and, mainly, in order to have the relevant notation at hand, the list of operators includes the operator S of taking isomorphic copies of substructures, the operator S_e of taking isomorphic copies of elementary substructures, the operator F of taking isomorphic copies of filter extensions, the operator R of taking isomorphic copies of reductions (images under strict surjective homomorphisms), the operator E of taking isomorphic copies of extensions (preimages under strict surjective homomorphisms) and, finally, the operators P, P_f, P_u of taking isomorphic copies of direct products, filtered products and ultraproducts and P_{sd} of taking isomorphic copies of subdirect products. Elgueta studies in a systematic way, reminiscent of the corresponding methodology of Universal Algebra, the properties that hold when these operators are composed. Of equal importance are an analog devised in this context of the Diagram Lemma of Model Theory and the Reduction Operator Lemma. The latter asserts that, given an operator $\mathcal{O} \in \{S, P, P_f, P_u, P_{sd}\}$, $L\mathcal{O} = L\mathcal{O}L$, where L is the class operator of taking Leibniz reductions of structures in a given class. Applying L after an operator $\mathcal{O}$, that is, the composed operator $L\mathcal{O}$ is denoted by $\mathcal{O}^*$. Thus, the assertion above is equivalent to $\mathcal{O}^* = \mathcal{O}^* L$. The second part of the Reduction Operator Lemma asserts that $LS_e = LS_eL = S_eL$.

Section 5 is the main section of [23] and contains the main characterization theorems. In Subsection 5.1, it is shown that K is an elementary class of structures defined without equality if and only if it is closed under E, R, S_e and $\bar{P}_u$, where $\bar{P}_u$ is P_u applied only on nonempty collections of structures. In a companion result for reductions, a class K of reduced structures is a reduced elementary class if and only if it is closed under S_e and $\bar{P}_u^*$. In Subsection 5.2, it is shown that K is a universal class of structures defined without equality if and only if it is closed under E, R, S and $\bar{P}_u$. As for reduced structures, a class K of reduced structures is a reduced universal class if and only if it is closed under S^* and $\bar{P}_u^*$. Subsection 5.3 turns to universal Horn classes of structures defined without equality. It is shown that K is such a class if and only if it is closed under E, R, S and P_f. As for reduced structures, a class K of reduced structures is a reduced universal Horn class if and only if it is closed under S^* and P_f^*. Finally Subsection 5.4 deals with universal atomic classes of structures defined without equality. It is shown that K is a universal atomic class if and only if it is closed under H, E, S and P. Moreover, a class K of reduced structures is a reduced universal atomic class if and only if it is closed under F^*, E, S and P.

In a different direction, and motivated mainly by applied considerations, many scientists have advocated the use of multi-valued logics (and sets) to model various phenomena. The study of such logics (and sets) has given rise to a huge body of work. Let us mention, here, the pioneering works of Zadeh [41], Goguen [32] and Pavelka [39]. Let us, also, point to some, among many available, reference works on the topic which contain material which is closer to the level of abstraction that we are aiming for in this study, e.g., [11], [1], [12]. In these references the reader can find further pointers to the extensive literature available on the topic. Of the last three, Cintula and Noguera's work [12] is very closely related to the material studied in Chapters 2 and 3 (on the logical side), whereas Bělohlávek and Vychodil's work [1] is the one that covers much of the material on the algebraic and the equational logical foundations that is used throughout this work.

In Chapter 2, we present an abstract theory for the algebraization of multi-valued logics (G-logics, as we call them) analogous to the theory of Blok and Pigozzi [6]. For a more detailed summary of the contents by section, see Section 2.1.

In Chapter 3, we present a theory of general algebraic semantics for G-logics analogous to the theory of Font and Jansana [28]. For a more detailed summary of the contents by section, see Section 3.1.

The third part of the work, Chapter 4, develops some model theory by generalizing the structures that are developed in Chapter 3. We follow here the work of Elgueta [23] (see, also, [22] and the work of Dellunde [17] and of Dellunde and Jansana [20]) in developing (up to a point) a model theory of equality free structures for structures in which sentences take truth values in a complete Boolean algebra, rather than being either true or false. For a more detailed summary of the contents by section, see Section 4.1.